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The Time-dependent Response of a Deep Rigid Anchor in a Viscoelastic Medium

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The problem of axially symmetric loading of a spheroidal rigid inclusion embedded in bonded contact with an isotropic linear viscoelastic infinite medium is investigated. This particular problem is of interest in connection with the analysis of the time-dependent anchorage efficiency of deep ground anchors and rock anchors embedded in soil and rock media susceptible to creep. The solution to the spheroidal rigid anchor problem pertaining to a linear viscoelastic medium is facilitated by the application of the Laplace transform-based correspondence principle. The transformed viscoelasticity problem is then solved by the use of Boussinesq's three-function approach developed for the solution of three-dimensional problems in classical elasticity. For the purposes of illustration, the particular viscoelastic material behavior is restricted to one which exhibits a dilatational response which is elastic and a deviatoric response which is that of a standard linear solid. Explicit analytical results are presented for anchors of both prolate and oblate spheroidal shapes which are subjected to either loads or displacements which vary as step functions of time. Numerical results are presented for cases where the rigid anchor embedded in typical geological materials is subjected to a step function of displacement. These results illustrate the manner in which the tension in the anchor rod can be influenced by the creep of the geological material in the vicinity of the anchor region.

INTRODUCTION

The use of ground and rock anchors as an effective technique for providing temporary or permanent support in geotechnical problems associated with excavations, retaining walls, foundations subjected to uplift loads and in rock mechanics problems associated with *in-situ* testing and strata control is now well recognized. References to some examples of such applications are given by Lang [1], Coates [2], Stagg and Zienkiewicz [3] and Selvadurai [4,5]. In this respect, the currently available techniques for the analysis and design of both ground and rock anchors largely concentrate upon the estimation of the ultimate bearing capacity of shallow and deep anchors, taking into account their individual or group action. Progress has also been made in the investigation of the elastic stress distribution in the vicinity of the anchor and the estimation of its elastic load–deflection characteristics (see for example [4–7]). Relatively little consideration, however, has been given to the examination of time-dependent effects in the anchor problem. Such time-dependent effects are of fundamental importance to the determination of the anchor efficiency under long term conditions (see for example [8–9]), and in calculation of creep deformations of anchorages under sustained loads. This paper is primarily concerned with the analytical treatment of a deep-anchor region embedded in a soil or rock medium exhibiting creep properties. Investigations relating to the creep behaviour of soil and rock media have long been the subject of extensive research. Comprehensive accounts of both experimental and theoretical results on the subject are to be found in many recent studies [10–18]. Creep effects in soil and rock media may manifest under natural conditions associated with long-term geological processes or as a result of interaction with engineering structures such as excavations, mining operations, structural foundations or ground anchors. Laboratory creep measurements together with field observations, involving closure of underground openings and oil well boreholes, variation of contact stresses in tunnel linings etc., indicate that many geologic materials (such as cohesive soils, frozen soils, rocksalts, shale, coal, limestone, granite, etc.) encountered in civil engineering practice exhibit creep behaviour to varying degrees. The creep behaviour of these materials which is generally postulated as a result of experimental observations made under conditions of

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uniaxial testing, indicate the familiar traits of primary, secondary and tertiary creep phenomena. A complete description of the creep process is, of course, a difficult if not impossible task as it embodies a combination of nonlinear and irreversible processes associated with elastic, viscous and plastic phenomena. A variety of phenomenological models and constitutive relations have been developed to describe the creep behaviour of soil and rock media under a variety of environmental conditions and references to these may be found in the literature already cited.

In the analytical work described here we restrict our attention to the class of soil and rock materials whose behaviour can be adequately described by a linear viscoelasticity theory. The linear theory of viscoelasticity, like its elastic counterpart, provides only an approximate theoretical basis for the analysis of creep phenomena in soil and rock media. It has been effectively employed by several investigators (see for example [11, 12, 19–28]), to provide informative and analytical solutions to several problems of engineering interest.

In the present treatment of the rigid anchor problem, the anchor itself is represented by a rigid spheroidal region of a prolate or oblate shape which is in bonded contact with an initially unstressed homogeneous isotropic linear viscoelastic infinite medium. The spheroidal anchor is subjected either to a resultant force or a rigid-body displacement which is directed along the axis of symmetry. The time dependence of the resultant force and the rigid displacement is assumed to be of the form of a Heaviside step function. The anchor problem as posed can be conveniently analyzed by employing the correspondence principle. Briefly, the correspondence principle establishes the fact that under appropriate conditions, the stress analysis of a problem in the linear theory of viscoelasticity can be reduced to the stress analysis of an analogous linear elastic problem having the same geometry and boundary conditions. In the case of a viscoelastic problem for which the geometry does not change and the boundary conditions remain of the same type during the loading process, the application of the Laplace transform removes the time dependence and transforms the problem to an associated elastic problem. The viscoelastic solution can thus be obtained by an inversion of the transformed solution. Using this technique in conjunction with Boussinesq's three-function method, applicable for the solution of three-dimensional problems in the theory of elasticity, it is possible to develop the solutions to the deep anchor problem for a range of linear viscoelastic phenomena. Explicit solutions are presented for the spheroidal anchor problem in which the dilatational behaviour of the viscoelastic material exhibits elastic characteristics and the deviatoric behaviour is represented by a three-parameter solid. In principle, this analysis can be extended to other types of viscoelastic response; however, such analysis entails an inordinate amount of algebra. The viscoelastic behaviour chosen here is quite a simple model consistent with creep observations in materials which exhibit an instantaneous response and a delayed creep, a limiting case of which corresponds to a Maxwell-type deviatoric response. The analysis of the anchor problem can be considerably simplified if, at the outset, we assume the viscoelastic material to exhibit incompressible behaviour. In this case the load–relaxation behaviour of an anchor subjected to a step function of displacement, for example, can be directly obtained by the transform inversion of the relaxation function for purely deviatoric response. Numerical results are presented to illustrate the load–relaxation behaviour of a deep rigid anchor which is embedded in typical geological materials with viscoelastic characteristics and subjected to a step function of displacement. In particular the geological materials have a linear viscoelastic response which is elastic in its dilatational behaviour and of a three-parameter solid type in its distortional behaviour. These numerical results illustrate the manner in which the loss of load in an anchor is influenced by the geometric shape of the anchor region and the shear relaxation behaviour of the viscoelastic material. An extension of the present problem to incorporate experimental creep and relaxation functions for both dilatational and distortional behaviour, needs further investigation.

LINEAR VISCOELASTICITY THEORY

The linear theory of viscoelasticity has been developed in considerable detail by numerous authors. A full account of these developments can be found in a recent review article by Leitman and Fisher [29]. However, for completeness as well as for future reference, we shall present here a brief summary of the basic results. Consider a fixed region R of infinite extent with boundary B which is occupied by a homogeneous isotropic viscoelastic material. Let u_i, ϵ_{ij} and σ_{ij}, each of which is to be regarded as a function of position x_i and non-negative time, t, denote the Cartesian components of the displacement vector, strain tensor and the stress tensor, respectively. Employing the usual indicial notation, the linearized displacement–strain relations appear as

$$2\epsilon_{ij} = \frac{\partial}{\partial x_j} u_i(\mathbf{x}, t) + \frac{\partial}{\partial x_i} u_j(\mathbf{x}, t). \tag{1}$$

Here, as in the sequel, the single argument $\mathbf{x}$ stands for the triplet of coordinates x_i ($i = 1, 2, 3$). In the absence of body and inertia forces, the equations of motion reduce to the equations of equilibrium

$$\frac{\partial}{\partial x_j} \sigma_{ij}(\mathbf{x}, t) = 0; \sigma_{ij} = \sigma_{ji}. \tag{2}$$

For the purposes of defining the constitutive relationships for an isotropic linear viscoelastic material we introduce deviatoric components of the stress and strain tensors through the relationships

$$S_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk};\ e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\delta_{ij}\epsilon_{kk}. \tag{3}$$

respectively. We adopt here constitutive laws in integral form for the relaxation behaviour of the material and cite the *relaxation integral law*

$$S_{ij}(\mathbf{x},t) = \int_{-\infty}^{t} J_1(t-\tau)\frac{\partial}{\partial\tau} e_{ij}(\mathbf{x},\tau)\,d\tau$$
$$\sigma_{kk}(\mathbf{x},t) = \int_{-\infty}^{t} J_2(t-\tau)\frac{\partial}{\partial\tau} \epsilon_{kk}(\mathbf{x},\tau)\,d\tau. \quad (4)$$

In equations (4) $J_1(t)$ and $J_2(t)$ are relaxation moduli in shear and isotropic compression, respectively. In this connection we also stipulate that

$$J_i(t) = 0; \qquad (i = 1, 2) \text{ for } -\infty < t < 0. \quad (5)$$

The integral representations of the linear viscoelastic constitutive equations (4) are generally applicable for instances in which the material exhibits an infinite spectrum of relaxation or retardation times. Alternatively, for a medium with a finite and discrete spectrum of relaxation or retardation times the linear viscoelastic constitutive relationship admits the differential operator representations

$$R_1(D)\, s_{ij}(\mathbf{x},t) = Q_1(D)\, e_{ij}(\mathbf{x},t)$$
$$R_2(D)\, \sigma_{kk}(\mathbf{x},t) = Q_2(D)\, \epsilon_{kk}(\mathbf{x},t) \quad (6)$$

provided

$$D = \frac{\partial}{\partial t};\ R_k(D) = \sum_{n=0}^{N_k} r_k^{(n)} D^n;\ Q_k(D) = \sum_{n=0}^{M_k} q_k^{(n)} D^n. \quad (7)$$

In addition to the foregoing field equations certain boundary conditions must be met. These boundary conditions, which prescribe displacements and tractions respectively, on complementary subsets $B_1(t)$ and $B_2(t)$ of the boundary B for all time, take the form

$$u_i(\mathbf{x},t) = V_i(\mathbf{x},t), (\mathbf{x},t) \text{ on } B_1(t)$$
$$\sigma_{ij}(\mathbf{x},t) n_j(\mathbf{x}) = T_i(\mathbf{x},t), (\mathbf{x},t) \text{ on } B_2(t) \quad (8)$$

where $n_j(\mathbf{x})$ are the components of the outward unit normal to B, and $V_i(\mathbf{x}, t)$ and $T_i(\mathbf{x}, t)$ are given functions. Again, as in the classical theory of elasticity, it is possible to demand certain combinations of traction components and displacement components to be prescribed on B. In equation (8) however, it is explicitly assumed that for all $t > 0 B_1(t) \cup B_2(t) = B; B_1(t) \cap B_2(t) = 0$.

THE CORRESPONDENCE PRINCIPLE

The classical method of solving boundary value problems in the linear quasistatic theory of viscoelasticity is based on the application of an integral transform with respect to time, to the time-dependent field equations and boundary conditions [equations (1–8)]. These transformed field equations are formally similar to the field equations of the classical theory of elasticity. If a solution of these, which is compatible with the transformed boundary conditions, can be found, then the solution to the original viscoelasticity problem is essentially reduced to transform inversion. This method of solving the stress analysis problem in linear viscoelasticity theory is referred to as the "correspondence principle". Comprehensive accounts of the method are given by Lee [30, 31], Bland [32] and Christensen [33]. In these works, the correspondence principle is formulated with the explicit assumption that B_1 and B_2 are time independent. The correspondence between the linear viscoelastic problem and the associated linear elastic problem is achieved with the aid of the Laplace transform. Suppose $f(\mathbf{x}, t)$ is a function of position and time; then we adopt the notation

$$\bar{f}(\mathbf{x},s) = \mathscr{L}\{f(\mathbf{x},t); s\} = \int_0^{\infty} f(\mathbf{x},t)e^{-st}\,dt \quad (9)$$

whence $\bar{f}(\mathbf{x}, s)$ is the Laplace transform (with respect to time) of $f(\mathbf{x}, t)$, s being the transform parameter. Referring to the general linear viscoelasticity problem stated earlier, since the regions B_1 and B_2, over which different boundary conditions are prescribed, do not vary or interact with time, the application of the Laplace transform to the field equations (1), (2), (4) and (6), and the boundary conditions (8) give the set of equations

$$2\bar{\epsilon}_{ij}(\mathbf{x},s) = \bar{u}_{i,j}(\mathbf{x},s) + \bar{u}_{j,i}(\mathbf{x},s) \quad (10)$$

$$\bar{\sigma}_{ij,j}(\mathbf{x},s) = 0 \quad (11)$$

$$\bar{S}_{ij}(\mathbf{x},s) = s\bar{J}_1(s)\bar{e}_{ij}(\mathbf{x},s);\ \bar{\sigma}_{kk}(\mathbf{x},s) = s\bar{J}_2(s)\bar{\epsilon}_{kk}(\mathbf{x},s) \quad (12)$$

$$\bar{R}_1(s)\bar{S}_{ij}(\mathbf{x},s) = \bar{Q}_1(s)\bar{\epsilon}_{ij}(\mathbf{x},s); \quad (13)$$
$$\bar{R}_2(s)\bar{\sigma}_{kk}(\mathbf{x},s) = \bar{Q}_2(s)\bar{\epsilon}_{kk}(\mathbf{x},s)$$

$$\bar{u}_i(\mathbf{x},s) = \bar{V}_i(\mathbf{x},s) \text{ on } B_1 \quad (14)$$

$$\bar{\sigma}_{ij}(\mathbf{x},s)n_j(\mathbf{x}) = \bar{T}_i(\mathbf{x},s) \text{ on } B_2. \quad (15)$$

The equations (10)–(15) now determine an elastic stress analysis problem in which the elastic constants and boundary conditions are functions of the parameter s. We note that the development of the transformed constitutive relations (12) is facilitated by the application of the convolution theorem [32]. For future reference we shall establish here the interrelationships between the two types of linear viscoelastic constitutive relationships (4) and (6) and their transformed counterparts equations (12) and (13), respectively.

The equivalence of equations (4) and (6) is assured provided

$$\bar{J}_k(s) = \frac{\bar{Q}_k(s)}{s\bar{R}_k(s)}, \quad (k = 1, 2). \quad (16)$$

By making use of equation (12) to analyze homogeneous states of stress or strain, it is possible to define the viscoelastic counterparts of the uniaxial relaxation modulus $E(t)$; the relaxation shear modulus $G(t)$; and a Poisson's ratio $\nu(t)$. The Laplace transforms of these, namely $E(s)$, $\bar{G}(s)$ and $\bar{\nu}(s)$, can be expressed, for example, in terms of $\bar{J}_\beta(s)$, $(\beta = 1, 2)$ as follows [34]:

$$\bar{E}(s) = \frac{3\bar{J}_1(s)\bar{J}_2(s)}{\bar{J}_1(s) + 2\bar{J}_2(s)};\ \bar{G}(s) = 2s\bar{J}_1(s);$$
$$\bar{\nu}(s) = \frac{\bar{J}_2(s) - \bar{J}_1(s)}{\bar{J}_1(s) + 2\bar{J}_2(s)}. \quad (17)$$

TRANSFORMED EQUATIONS OF LINEAR VISCOELASTICITY IN SPHEROIDAL COORDINATES

For the solution of the rigid spheroidal anchor problem associated with a viscoelastic medium it is convenient to employ the three-function approach proposed by Boussinesq [see 34]. Extending this technique to the transformed viscoelastic formulation, it can be shown that the general axisymmetric solution of the transformed displacement equations of equilibrium [obtained by eliminating $\bar{S}_{ij}(s)$ and $\bar{\sigma}_{kk}(s)$, given by equation (12), between equations (10) and (11)] can be represented as the sum of the displacement fields generated by two harmonic stress functions $\bar{\Phi}(\mathbf{x}, s)$ and $\bar{\Psi}(\mathbf{x}, s)$. We shall adopt here the formulation of the linear elastic problem in generalized curvilinear coordinates given by Sadowsky and Sternberg [35] and Sternberg *et al.* [36], to generate the transformed viscoelastic equations. The results are initially considered for a system of prolate spheroidal coordinates (α, β, γ). We define the prolate spheroidal coordinate system by the transformation

$$x = c_p \sinh\alpha \sin\beta \cos\gamma; \quad y = c_p \sinh\alpha \sin\beta \sin\gamma; \quad z = c_p \cosh\alpha \cos\beta \tag{18}$$

in which c_p is a positive constant. The parametric surfaces α = const, say α_0; $\beta = \beta_0$; $\gamma = \gamma_0$ form a triple orthogonal confocal family of prolate spheroids, hyperboloids of two sheets and meridional half planes, respectively (Fig. 1). For deformations which are symmetric about the z-axis, the displacements and stress fields in the viscoelastic medium are independent of the longitude γ. The transformed Boussinesq potentials $\bar{\Phi}(\alpha, \beta, s)$ and $\bar{\Psi}(\alpha, \beta, s)$ referred to the spheroidal coordinate system, satisfy the differential equation

$$\nabla^2\bar{\Phi}(\alpha, \beta, s) = 0; \qquad \nabla^2\bar{\Psi}(\alpha, \beta, s) = 0 \tag{19}$$

where

$$\nabla^2 = h^2\left\{\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + \coth\alpha\frac{\partial}{\partial\alpha} + \cot\beta\frac{\partial}{\partial\beta}\right\} \tag{20}$$

is Laplace's operator in prolate spheroidal coordinates, and $h^2 = c_p[\sinh^2\alpha + \sin^2\beta]$. We denote the transformed curvilinear components of the displacement vector by $\bar{u}_\alpha(\alpha, \beta, s)$ and $\bar{u}_\beta(\alpha, \beta, s)$ and the curvilinear components of the Cauchy stress tensor $\bar{\sigma}_{ij}(\alpha, \beta, s)$ are given by

$$\bar{\sigma}_{ij} = \begin{bmatrix} \bar{\sigma}_{\alpha\alpha} & 0 & \bar{\sigma}_{\alpha\beta} \\ 0 & \bar{\sigma}_{\gamma\gamma} & 0 \\ \bar{\sigma}_{\alpha\beta} & 0 & \bar{\sigma}_{\beta\beta} \end{bmatrix}. \tag{21}$$

The transformed displacement fields corresponding to the Boussinesq solutions $\bar{\Phi}(\alpha, \beta, s)$ and $\bar{\Psi}(\alpha, \beta, s)$ are given by

$$[\bar{u}_\alpha; \bar{u}_\beta] = \frac{h}{2\bar{G}(s)}[\bar{\Phi}_{,\alpha}; \bar{\Phi}_{,\beta}] \tag{22a}$$

$$[\bar{u}_\alpha; \bar{u}_\beta] = \frac{h}{2\bar{G}(s)}[\{g\bar{\Psi}_{,\alpha} - (3 - 4\bar{\nu}(s))g_{,\alpha}\bar{\Psi}\}; \{g\bar{\Psi}_{,\beta} - (3 - 4\bar{\nu}(s))g_{,\beta}\bar{\Psi}\}] \tag{22b}$$

where $\bar{G}(s)$ and $\bar{\nu}(s)$ are the transformed values of the linear elastic shear modulus and Poisson's ratio, respectively, and $g = c_p \cosh\alpha \cos\beta$. The commas, i.e. $\bar{\Phi}_{,\alpha}$, $\bar{\Phi}_{,\beta}$ ... etc. denote partial differentiation with respect to the appropriate spatial coordinate. Similarly, the transformed stress components derived from the Boussinesq stress functions take the form

$$\bar{\sigma}_{\alpha\alpha}(\alpha, \beta, s) = h^2\left[\bar{\Phi}_{,\alpha\alpha} + \frac{h_{,\alpha}}{h}\bar{\Phi}_{,\alpha} - \frac{h_{,\beta}}{h}\bar{\Phi}_{,\beta}\right]$$

$$\bar{\sigma}_{\beta\beta}(\alpha, \beta, s) = h^2\left[\bar{\Phi}_{,\beta\beta} + \frac{h_{,\beta}}{h}\bar{\Phi}_{,\beta} - \frac{h_{,\alpha}}{h}\bar{\Phi}_{,\alpha}\right] \tag{23a}$$

$$\bar{\sigma}_{\gamma\gamma}(\alpha, \beta, s) = h^2\left[\frac{f_{,\alpha}}{f}\bar{\Phi}_{,\alpha} + \frac{f_{,\beta}}{f}\bar{\Phi}_{,\beta}\right]$$

$$\bar{\sigma}_{\alpha\beta}(\alpha, \beta, s) = h^2\left[\bar{\Phi}_{,\alpha\beta} + \frac{h_{,\beta}}{h}\bar{\Phi}_{,\alpha} + \frac{h_{,\alpha}}{h}\bar{\Phi}_{,\beta}\right] \tag{23b}$$

and

$$\bar{\sigma}_{\alpha\alpha}(\alpha, \beta, s) = h^2\left[g\bar{\Psi}_{,\alpha\alpha} + \left(\frac{g_h}{h_{,\alpha}} - 2g_{,\alpha}\right)\bar{\Psi}_{,\alpha} - \frac{g_h}{h_{,\beta}}\bar{\Psi}_{,\beta} + 2\bar{\nu}(s)(g_{,\alpha}\bar{\Psi}_{,\alpha} - g_{,\beta}\bar{\Psi}_{,\beta})\right]$$

$$\bar{\sigma}_{\beta\beta}(\alpha, \beta, s) = h^2\left[g\bar{\Psi}_{,\beta\beta} + \left(\frac{g_h}{h_{,\beta}} - 2g_{,\beta}\right)\bar{\Psi}_{,\beta} - \frac{g_h}{h_{,\alpha}}\bar{\Psi}_{,\alpha} + 2\bar{\nu}(s)(g_{,\beta}\bar{\Psi}_{,\beta} - g_{,\alpha}\bar{\Psi}_{,\alpha})\right]$$

$$\bar{\sigma}_{\gamma\gamma}(\alpha, \beta, s) = h^2\left[\frac{g}{f}(f_{,\alpha}\bar{\Psi}_{,\alpha} + f_{,\beta}\bar{\Psi}_{,\beta}) - 2\bar{\nu}(s)(g_{,\alpha}\bar{\Psi}_{,\alpha} + g_{,\beta}\bar{\Psi}_{,\beta})\right] \tag{23c}$$

$$\bar{\sigma}_{\alpha\beta}(\alpha, \beta, s) = h^2\left[g\bar{\Psi}_{,\alpha\beta} + \frac{g}{h}(h_{,\alpha}\bar{\Psi}_{,\beta} + h_{,\beta}\bar{\Psi}_{,\alpha}) - (1 - 2\bar{\nu}(s))(g_{,\alpha}\bar{\Psi}_{,\beta} + g_{,\beta}\bar{\Psi}_{,\alpha})\right],$$

respectively, where $f = c_p \sinh\alpha \sin\beta$.

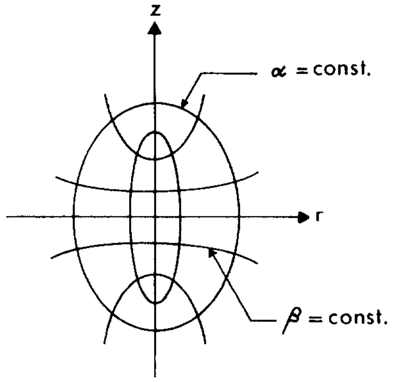

Fig. 1. The spheroidal coordinate system.

THE VISCOELASTIC PROBLEM OF THE SPHEROIDAL ANCHOR

The preceding theory is now applied to the analysis of a prolate spheroidal rigid anchor which is embedded in bonded contact with an infinite medium exhibiting isotropic linear viscoelastic characteristics. The rigid anchor region is represented by the prolate spheroid $\alpha = \alpha_0$. It is further assumed that the anchor is subjected either to a time-dependent displacement $\delta(t)$ or a time-dependent load $P(t)$ which is directed along its axis of symmetry z (see Fig. 2). The first category of boundary conditions is of interest in connection with problems associated with the loss of tension in anchor rods owing to the creep of the medium surrounding the anchor region and the second type of boundary conditions is of importance in the estimation of creep deformations of anchors subjected to sustained loads. We shall denote the transformed values of the anchor displacement or anchor load by $\bar{\delta}(s)$ and $\bar{P}(s)$, respectively. Since perfect bonding at the anchor–viscoelastic medium interface has been assumed, no separation takes place on $\alpha = \alpha_0$; hence, the displacement or load boundary conditions can be assigned transformed values. The viscoelasticity problem is thus reduced essentially to the solution of the analogous linear elastic problem. Briefly, to satisfy the transformed displacement boundary conditions at the interface two independent solutions of equation (19) are required; these in turn should yield transformed displacement and stress fields which are single valued in the domain $\alpha_0 \leq \alpha < \infty$ and $0 \leq \beta \leq \pi$ and which should tend to zero as $\alpha \to \infty$. It can be shown that the appropriate stress functions are of the forms

$$\bar{\Phi}(\alpha, \beta, s) = C_1\left[1 - \frac{\cosh\alpha}{2}\ln\xi\right]\cos\beta$$
$$\bar{\Psi}(\alpha, \beta, s) = C_2[\ln\xi] \tag{24}$$

where $\xi = (\cosh\alpha + 1)/(\cosh\alpha - 1)$, and C_1 and C_2 are arbitrary constants to be determined from the boundary conditions

$$\bar{u}_\alpha(\alpha_0, \beta, s) = \frac{\bar{\delta}(s)\cos\beta}{\Omega_0};\ \bar{u}_\beta(\alpha, \beta, s) = -\frac{\bar{\delta}(s)\coth\alpha_0\sin\beta}{\Omega_0} \tag{25}$$

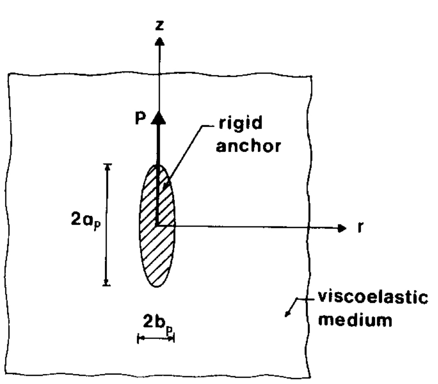

Fig. 2. The prolate spheroidal anchor.

and

$$\Omega_0 = [1 - \sin^2\beta\,\mathrm{csch}^2\alpha_0]^{1/2}.$$

Using equations (22) and (24) and the boundary condition (25) it can be shown that the values of C_1 and C_2 are given by

$$[C_1; C_2] = \frac{2c_p\bar{\delta}(s)\bar{G}(s)}{\chi_0}[\coth\alpha_0; \mathrm{csch}\alpha_0\,\mathrm{sech}\alpha_0] \tag{26}$$

where

$$\chi_0 = -\mathrm{csch}\alpha_0\mathrm{sech}\alpha_0\{\tfrac{1}{2}(\cosh^2\alpha_0 + 1)\ln\xi_0 - \cosh\alpha_0\} \tag{27}$$

and $\xi_0 = \xi(\alpha_0)$.

A relationship between the transformed value of the anchor displacement $\bar{\delta}(s)$ and the transformed value of the anchor load $\bar{P}(s)$ can now be established by considering the resultant of tractions in the z-direction acting on any closed surface $\alpha = \text{const}$. By considering this surface to be $\alpha = \alpha_0$, it can be shown that

$$\bar{P}(s) = 2\pi c_p^2\int_0^\pi \Theta_0^{1/2}[\bar{\sigma}_{\alpha\alpha}n_z - \bar{\sigma}_{\alpha\beta}n_r]\sinh\alpha_0\sin\beta\,d\beta \tag{28}$$

where $\Theta_0 = (\sinh^2\alpha_0 + \sin^2\beta)$, and n_r and n_z are the direction cosines of the unit outward normal to $\alpha = \alpha_0$ given by

$$[n_r; n_z] = \frac{1}{\Theta_0^{1/2}}[\cosh\alpha_0\sin\beta; \sinh\alpha\cos\beta]. \tag{29}$$

Using the displacement components given by equations (22) and (24), in equation (23) the stress components can be evaluated in terms of $\bar{\delta}(s)$. Substituting the resulting equations in the integral expression for $\bar{P}(s)$ and performing the integration a relationship between $\bar{P}(s)$ and $\bar{\delta}(s)$ is obtained for the prolate spheroidal anchor; this in turn can be expressed as

$$\bar{P}(s) = \frac{16\pi a_p(1 - \lambda^2)^{1/2}\bar{\delta}(s)\bar{G}(s)\{1 - \bar{v}(s)\}}{\left[\frac{1}{2}\left\{3 - 4\bar{v}(s) + \frac{1}{(1 - \lambda^2)}\right\} \times \ln\left(\frac{1 + \sqrt{1 - \lambda^2}}{1 - \sqrt{1 - \lambda^2}}\right) - \frac{1}{(1 - \lambda^2)^{1/2}}\right]} \tag{30}$$

where $\lambda (= b_p/a_p) < 1$ is the aspect ratio of the prolate spheroidal anchor.

The preceding analytical procedure could also be extended to the case of an oblate spheroidal anchor (Fig. 3), by considering at the outset the appropriate oblate spheroidal coordinate system. Omitting details of analysis it can be shown that for the analogous problem of an oblate spheroidal anchor, the relationship between the transformed variables $\bar{P}(s)$ and $\bar{\delta}(s)$ is of the form

$$\bar{P}(s) = \frac{16\pi b_0(1 - \mu^2)^{1/2}\bar{\delta}(s)\bar{G}(s)\{1 - \bar{v}(s)\}}{\left[\frac{\mu}{(1 - \mu^2)^{1/2}} + \left\{3 - 4\bar{v}(s) - \frac{\mu^2}{(1 - \mu^2)}\right\}\cot^{-1}\left\{\frac{\mu}{(1 - \mu^2)^{1/2}}\right\}\right]} \tag{31}$$

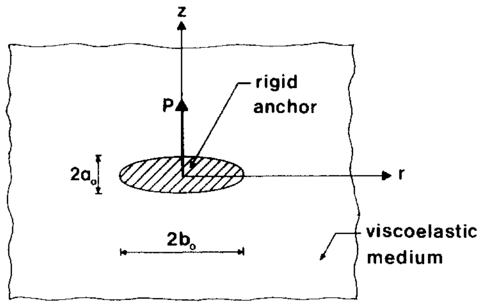

Fig. 3. The oblate spheroidal anchor.

where $\mu\ (= a_0/b_0) < 1$ is the aspect ratio of the oblate spheroidal anchor.

As mentioned earlier, for the purposes of illustration we shall restrict our attention to the particular problem of a spheroidal anchor embedded in a linear viscoelastic medium which behaves as (i) an elastic solid in its dilatational response, and (ii) as a three-parameter solid (see for example Flugge [37]), in its distortional response. The constitutive equations (6) for the distortional and dilatational responses have the forms

$$\left\{D + \frac{G_0(1+\alpha)}{\eta}\right\} S_{ij} = \left\{G_0 D + \frac{G_0^2\tilde{\alpha}}{\eta}\right\} e_{ij} \qquad (32a)$$

and

$$\sigma_{kk} = 3K\epsilon_{kk}, \qquad (32b)$$

respectively; where G_0 is related to the instantaneous linear elastic shear modulus G as $G_0 = 2G$; η is the shear viscosity and $\tilde{\alpha}G_0$ is the delayed value of the shear modulus; and K is the instantaneous bulk modulus. Here, as $\tilde{\alpha} \to 0$ the material exhibits a Maxwell type of deviatoric response; $\tilde{\alpha} \to \infty$ the retardation time $\tau\ (= \eta/\tilde{\alpha}G_0)$ tends to zero and the material behaves as an elastic solid in its deviatoric response. Although this particular choice of viscoelastic response is chosen somewhat arbitrarily, it has been used quite successfully to approximately describe the creep behaviour of soils and other geologic materials such as granite, sandstone, limestone, rocksalt, coal, etc. (see for example Jaeger and Cook [12]; Robertson [15]; Gnirk & Johnson [20]; Booker & Poulos [27]). The transformed values of the relaxation functions $J_1(t)$ and $J_2(t)$ corresponding to the prescribed viscoelastic behaviour are given by

$$\bar{J}_1(s) = \frac{G_0[s + (\tilde{\alpha}G_0/\eta)]}{s[s + (1+\tilde{\alpha})(G_0/\eta)]};\ \bar{J}_2(s) = \frac{3K}{s}. \qquad (33)$$

Using the transformed relaxation functions equations (33) and (17), it is possible to obtain expressions for $\bar{G}(s)$ and $\bar{\nu}(s)$. To solve the anchor problem it is also necessary to specify, in addition to the material parameters $\bar{J}_1(s)$ and $\bar{J}_2(s)$, the loading or displacement conditions prescribed by $\bar{P}(s)$ or $\bar{\delta}(s)$.

(a) *Relaxation of load in a pre-tensioned anchor rod*

We shall first consider the problem of a rigid spheroidal anchor which is subjected to the imposed displacement

$$\delta(t) = \delta_0 H(t), \qquad (34)$$

where $H(t)$ is the Heaviside step function. Using the expressions for $\bar{G}(s)$, $\bar{\nu}(s)$ [obtained from equations (33) and (17)] together with $\bar{\delta}(s)\ (= \delta_0/s)$ in equation (30), we obtain the following expression for the transformed anchor load in the case of the prolate spheroidal anchor: we have

$$\frac{\bar{P}(s)}{(16\pi a_p\delta_0G_0/2)} = \frac{\omega_1\kappa_3}{(\omega_3+\kappa_3)}\left|\frac{(\tau s+\zeta_1)(\tau s+\kappa_1)}{s(\tau s+\zeta_2)(\tau s+\zeta_3)}\right| \qquad (35)$$

where

$$\omega_1 = \frac{1}{2}\frac{(1-\lambda^2)^{1/2}}{\ln\xi_0};\ \omega_2 = \frac{1}{4}\left\{\frac{\lambda^2}{(1-\lambda^2)} - \frac{2}{\ln\xi_0(1-\lambda^2)^{1/2}}\right\}$$

$$\xi_0 = \left[\frac{1+\sqrt{1-\lambda^2}}{1-\sqrt{1-\lambda^2}}\right];\ \kappa_1 = \left\{\frac{3\phi(1+\tilde{\alpha})+2\tilde{\alpha}}{\tilde{\alpha}(3\phi+2)}\right\};$$

$$\kappa_3 = \left\{\frac{3\phi+2}{6\phi+1}\right\} \qquad (36)$$

$$\zeta_1 = 1;\ \zeta_2 = \frac{1+\tilde{\alpha}}{\tilde{\alpha}};$$

$$\zeta_3 = \frac{1}{\alpha}\left[\frac{\omega_2\{6\phi(1+\tilde{\alpha})+\tilde{\alpha}\} + 3\phi(1+\tilde{\alpha})+2\tilde{\alpha}}{\omega_2\{6\phi+1\}+3\phi+2}\right]$$

$$\phi = \frac{K}{G_0}$$

and $\tau = \eta/\tilde{\alpha}G_0$ is the retardation time for the viscoelastic medium, in simple shear. Obtaining the inverse transform of the result, equation (35), we can now determine the load relaxation behaviour of the deep rigid anchor with a *prolate spheroidal shape*; we have

$$\frac{P(t)}{(16\pi a_p\delta_0G_0/2)} = \frac{\omega_1\kappa_3}{(\omega_2+\kappa_3)}\left|\frac{\zeta_1\kappa_1}{\zeta_2\zeta_3}\{1 - e^{-\zeta_2 t/\tau}\} + \left\{\frac{\zeta_3(\zeta_1-\zeta_2) - \kappa_1(\zeta_1-\zeta_3)}{\zeta_3(\zeta_3-\zeta_2)}\right\}e^{-\zeta_2 t/\tau} + \left\{\frac{(\zeta_1-\zeta_3)(\kappa_1-\zeta_3)}{\zeta_3(\zeta_3-\zeta_2)}\right\}e^{-\zeta_3 t/\tau}\right|. \qquad (37)$$

Similarly, it can be shown that the load–relaxation behaviour for the rigid anchor with an *oblate spheroidal shape* is given by

$$\frac{P(t)}{(16\pi b_0\delta_0G_0/2)} = \frac{\omega_1^*\kappa_3}{(\omega_2^*+\kappa_3)}\left|\frac{\zeta_1^*\kappa_1}{\zeta_2^*\zeta_3^*}\{1 - e^{-\zeta_2^* t/\tau}\} + \left\{\frac{\zeta_3^*(\zeta_1^*-\zeta_2^*) - \kappa_1(\zeta_1^*-\zeta_3^*)}{\zeta_3^*(\zeta_3^*-\zeta_2^*)}\right\}e^{-\zeta_2^* t/\tau} + \left\{\frac{(\zeta_1^*-\zeta_3^*)(\kappa_1-\zeta_3^*)}{\zeta_3^*(\zeta_3^*-\zeta_2^*)}\right\}e^{-\zeta_3^* t/\tau}\right| \qquad (38)$$

where

$$\omega_1^* = \frac{(1-\mu^2)^{1/2}}{4\cot^{-1}(\Gamma)};\ \omega_2^* = \frac{1}{4}\left|\frac{\Gamma}{\cot^{-1}(\Gamma)} - \frac{1}{(1-\mu^2)}\right|$$

$$\Gamma = \frac{\mu}{(1-\mu^2)^{1/2}};\ \zeta_1^* = \zeta_1;\ \zeta_2^* = \zeta_2 \qquad (39)$$

and ξ_3^* may be obtained by replacing ω_2 in the expression for ξ_3, by ω_2^*.

(b) *Displacement of the rigid anchor under a sustained load*

We next consider the problem of a rigid spheroidal anchor which is subjected to a time-dependent load which is of the form

$$P(t) = P_0 H(t). \tag{40}$$

Using the expressions for $\bar{G}(s)$ and $\bar{v}(s)$ and equation (40) in equations (30) and (31) we obtain an equation for the transformed anchor displacement $\bar{\delta}(s)$ as follows

$$\bar{\delta}(s) = \frac{P_0}{8\pi\rho G_0}\left[\chi_1\left\{\frac{\tau}{(\tau s + \zeta_1)} + \frac{\zeta_2}{s(\tau s + \zeta_1)}\right\} + \frac{\chi_2}{\kappa_3}\left\{\frac{\tau^2 s}{(\tau s + \zeta_1)(\tau s + \kappa_1)} + \frac{\tau(\zeta_2 + \kappa_2)}{(\tau s + \zeta_1)(\tau s + \kappa_1)} + \frac{\zeta_2\kappa_2}{s(\tau s + \zeta_1)(\tau s + \kappa_2)}\right\}\right] \tag{41}$$

where, for the

(i) prolate spheroidal anchor

$$\rho = a_p(1 - \lambda^2)^{1/2};\ \chi_1 = 2\ln\xi_0;$$
$$\chi_2 = \left\{\frac{\lambda^2 \ln\xi_0}{2(1 - \lambda^2)} - \frac{1}{(1 - \lambda^2)^{1/2}}\right\}, \tag{42a}$$

(ii) oblate spheroidal anchor

$$\rho = b_0(1 - \mu^2)^{1/2};\ \chi_1 = 4\cot^{-1}(\Gamma);$$
$$\chi_2 = \left\{\frac{\mu}{(1 - \mu^2)^{1/2}} - \frac{\cot^{-1}(\Gamma)}{(1 - \mu^2)}\right\}, \tag{42b}$$

and constants, such as τ, ζ_n, κ_n $(n = 1, 2, 3)$ etc., are defined in equation (36). By transform inversion we obtain the expression for the time-dependent creep displacement of a rigid spheroidal anchor of both prolate and oblate shapes, as follows

$$\delta(t) = \frac{P_0}{8\pi\rho G_0 \zeta_1}\left[\chi_1\{\zeta_2 + (\zeta_1 - \zeta_2)e^{-\zeta_1 t/\tau}\} + \frac{\chi_2}{\kappa_1\kappa_3(\zeta_1 - \kappa_1)}\{\kappa_1(\zeta_1 - \zeta_2)(\zeta_1 - \kappa_2)e^{-\zeta_1 t/\tau} + \zeta_1(\kappa_1 - \kappa_2)(\zeta_2 - \kappa_1)e^{-\kappa_1 t/\tau} + \zeta_2\kappa_2(\zeta_1 - \kappa_1)\}\right]. \tag{43}$$

NUMERICAL RESULTS

In order to illustrate the influence of viscoelastic material properties and the anchor aspect ratio on the relaxation of load with time in a deep rigid anchor of spheroidal shape, the equations (37) and (38) have been evaluated for a range of viscoelastic material parameters representative of certain typical geological materials. Ideally, to evaluate the three-dimensional response of a linear viscoelastic material it is necessary to conduct either creep or relaxation tests which would investigate the dilatational and distortional responses of the material, separately. Only a few such investigations have been reported in the literature. However, results derived from simpler tests such as uniaxial creep tests or torsion creep tests can be utilized to compute material parameters governing the linear viscoelastic response. Experimental results for the viscoelastic response of various geological materials such as granite, sandstone, limestone, mudstone, shale, rocksalt and potash have been reported [15,20,26,38–42]. The typical material parameters shown in Table 1 have been estimated from the experimental values given in the above literature. The viscoelastic material properties for the unreinforced concrete given in Table 1 are due to Neville [43]. The aspect ratios for the spheroidal anchors are assigned the following values; $\lambda = 0.1, 0.5, 0.95$ (the value of $\lambda = 0.1$ approximately resembles a needle-shaped anchor, and $\lambda = 0.95$ approximates a spherical shape) and $\mu = 0.1, 0.5, 0.95$ (the value of $\mu = 0.1$ resembles a flat disc-shaped anchor and $\mu = 0.95$ again approximates a spherical shape).

The load–relaxation behaviour for the anchors with a prolate spheroidal shape is shown in Figs. 4 and 5; similar results for the oblate spheroidal anchor are

TABLE 1. TYPICAL VISCOELASTIC PARAMETERS FOR GEOLOGICAL MATERIALS [15,20,26,38,39,40,41,42,43]

Material	G_0(MN/m²)	η(MNh/m²)	ϕ	$\bar{\alpha}$	τ(h)
Granite	7.0×10^4	1.0×10^8	0.69	12.0	119.0
Sandstone	6.0×10^4	1.5×10^7	0.50	9.0	27.8
Limestone	5.6×10^4	4.4×10^7	0.90	6.0	130.9
Mudstone	3.5×10^4	6.0×10^6	0.57	2.0	85.7
Concrete	2.9×10^4	2.2×10^6	0.80	0.6	126.4
Shale	1.0×10^4	2.7×10^6	2.00	0.2	1350.0
Rocksalt	7.0×10^3	2.0×10^4	0.85	0.3	9.5
Potash	3.8×10^3	6.4×10^5	1.31	0.5	336.8

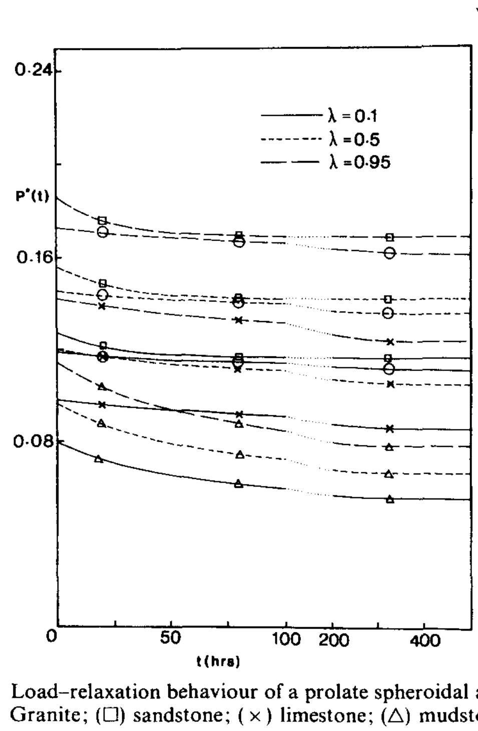

Fig. 4. Load–relaxation behaviour of a prolate spheroidal anchor. (○) Granite; (□) sandstone; (×) limestone; (△) mudstone.

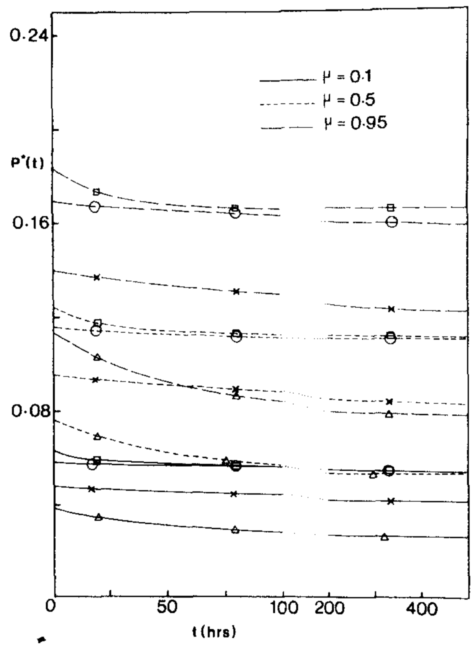

Fig. 5. Load-relaxation behaviour of an oblate spheroidal anchor.

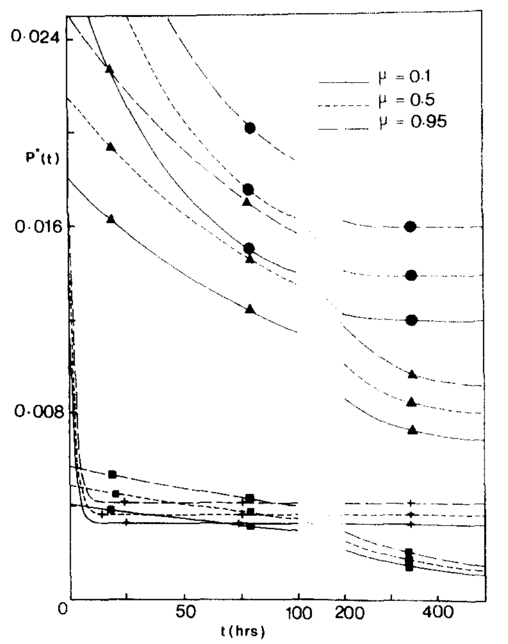

Fig. 7. Load–relaxation behaviour of an oblate spheroidal anchor.

shown in Figs. 6 and 7. The non-dimensional anchor load $P^*(t) = P(t)/8\pi\delta_0\rho G_0$ where for the prolate spheroidal anchor $\rho = a_p$ and for the oblate spheroidal anchor $\rho = b_0$. In summary, these results generally indicate that the deviatoric response of the viscoelastic medium has a significant influence on the magnitude of the load relaxation. From the theoretical results, equations (37) and (38), we observe that the rate of relaxation of the anchor load is influenced by its geometric aspect ratio. (The last terms in the right-hand sides of equations (37) and (38) have their exponents proportional to ζ_3 and ζ_3^* which are in turn functions of λ and μ, respectively.) The influence of the anchor aspect ratio on the rate of relaxation of the anchor load $P(t)$ appears to be of some significance only in the case of anchors with a prolate spheroidal shape embedded in soft viscoelastic media.

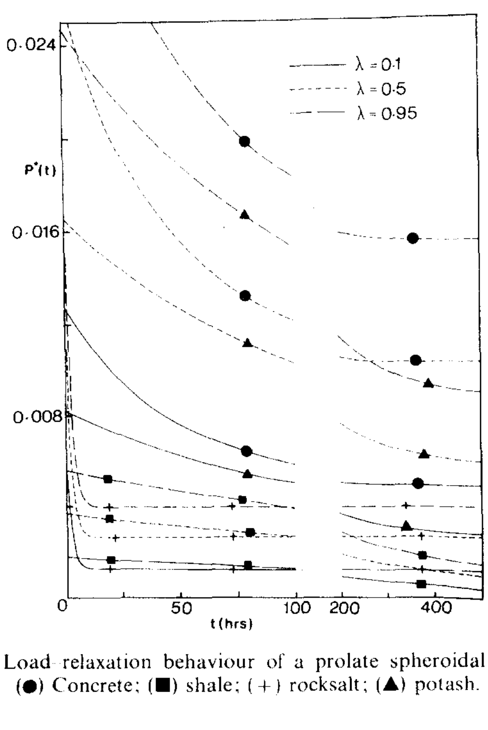

Fig. 6. Load–relaxation behaviour of a prolate spheroidal anchor. (●) Concrete; (■) shale; (+) rocksalt; (▲) potash.

As indicated earlier, considerable simplifications arise in the particular instance where the anchor is embedded in a linear viscoelastic medium which exhibits incompressible behaviour for all $t > 0$. In this case the transformed load–deflection relationships become directly proportional to the transformed shear modulus $\bar{G}(s)$; the constant of proportionality will be a function of the anchor aspect ratio for the particular anchor. In these circumstances, the load–relaxation curve can then be obtained by a direct transform inversion of $\bar{G}(s)/s$. With the simplifications offered by the assumption of incompressibility, the treatment of the anchor problem can be extended to include experimentally determined creep or relaxation functions.

CONCLUSIONS

This paper presents an analytical treatment of the time-dependent behaviour of a rigid anchor with a spheroidal shape, embedded in an infinite medium exhibiting linear viscoelastic characteristics. In particular, the viscoelastic medium chosen exhibits elastic dilatational behaviour and the deviatoric behaviour is represented by a standard linear solid. The numerical solutions are presented for the case of a spheroidal anchor which is subjected to a step function of displacement. These results indicate the manner in which relaxation of load in an anchor tie rod occurs as a result of creep in the geological material surrounding the anchor region.

The deep anchor problem as treated in this paper admittedly neglects other factors, such as the anchor elasticity, interaction effects of neighbouring anchors, influence of finite depth of embedment, presence of non-contiguous boundary conditions at the anchor-soil or rock interface and other phenomena associated with non-linear material properties. Clearly the analysis of all such effects is beyond the scope of any single theoretical investigation. The main purpose of this investigation is, however, to outline a fairly elementary treatment of the anchor problem with a view to illustrating some trends observed in its time-dependent behaviour. To the author's knowledge this paper appears to be the first to consider an investigation of the time-dependent effects associated with anchor problems. The analytical results presented here can be used as a basis for the approximate treatment of deep anchors embedded in soil and rock media. Alternatively, they can be employed to verify the accuracy of other numerical methods of time-dependent stress analysis (such as finite element or finite difference techniques) which may then be used in the treatment of more complex anchor problems.

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