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The indentation of a precompressed penny-shaped crack

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Abstract

The paper examines the axisymmetric problem related to the indentation of the plane surface of a penny-shaped crack by a smooth rigid disc inclusion. The crack is also subjected to a far-field compressive stress field which induces closure over a part of the crack. The paper presents the Hankel integral transform development of the governing mixed boundary value problem and its reduction to a single Fredholm integral equation of the second kind and an appropriate consistency condition which considers the stress state at the boundary of the crack closure zone. A numerical solution of this integral equation is used to develop results for the axial stiffness of the inclusion and for the stress intensity factors at the tip of the penny-shaped crack. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

The elastostatic problem which deals with a penny-shaped crack, which is located in an isotropic elastic solid of infinite extent, was examined by Sneddon [1] and Sack [2] using dual integral equation and spheroidal function techniques, respectively. These studies which have been extended by a number of researchers to cover effects such as anisotropy, non-homogeneity, localized loading, effects of localized plastic flow, dynamic loading, etc., present useful techniques for the examination of fracture, crack propagation, etc., in solids with predominantly brittle elastic behaviour. Detailed accounts of recent developments in fracture mechanics are given by Liebowitz [3], Sih [4] and Atkinson [5]. The category of problems in which the plane faces of the penny-shaped crack are subjected to displacement constraints has several useful engineering applications. Bridging of penny-shaped cracks by fibres of reinforced materials, proppant-induced hydraulic fracture of oil and gas bearing geological formations, creation of cracks in rocks due to expansive

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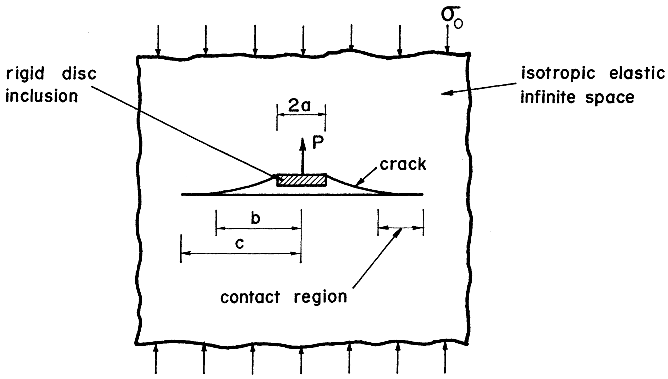

Fig. 1. The indentation of a precompressed penny-shaped crack.

cements used for anchoring devices, fracturing due to thermal mismatch in multiphase composites, etc., are some situations in which the cracked region can be subjected to displacement type boundary conditions at its plane faces [6–9]. In this paper, we consider the problem of a penny-shaped crack which is subjected to both a precompression normal to the plane of the crack and the indentation by a smooth rigid disc over one of its plane faces (Fig. 1). Due to the presence of the precompression the crack experiences closure at the crack tip. For certain combinations of the precompression and the indenting stress the crack remains closed over a finite annular region. In this closed position only a mode II type stress intensity factor occurs at the crack tip. The analysis allows the evaluation of the indenting stress necessary to initiate the flaw opening or mode I stress intensity factors at the crack tip.

2. Basic equations

Owing to the axial symmetry of the problem posed, it is convenient to adopt a formulation based on the strain potential approach proposed by Love [10]. The basic equation governing the strain potential $\phi(r,z)$ is

$$\nabla^2\nabla^2\phi(r,z) = 0, \tag{1}$$

where ∇^2 is the axisymmetric form of Laplace's operator. The displacement and stress fields can be expressed in the forms:

$$2Gu_r = -\phi_{,rz} \tag{2}$$

$$2Gu_z = 2(1-\nu)\nabla^2\phi - \phi_{,zz} \tag{3}$$

and

$$\sigma_{zz} = \left\{(2-\nu)\nabla^2\phi - \phi_{,zz}\right\}_{,z} \tag{4}$$

$$\sigma_{rz} = \left\{(1-\nu)\nabla^2\phi - \phi_{,zz}\right\}_{,r} \tag{5}$$

etc. In (2)–(5), G is the shear modulus; ν is Poisson's ratio and the comma denotes partial differentiation with respect to the appropriate spatial variable.

3. The crack indentation problem

A penny-shaped crack of radius c is located on the plane $z = 0$ of an isotropic elastic solid of infinite extent. The infinite solid is subjected to a state of uniform compression σ_0 normal to the plane of the crack. We assume that the crack experiences closure due to the application of σ_0. A single surface of the crack is now subjected to an indentation (Δ) by a smooth rigid disc inclusion of radius a which is located concentric with the penny-shaped crack. It is assumed that the indentation Δ is such that separation takes place over a finite region ($r \leqslant b$) of the unindented crack surface. For the formulation of the crack indentation problem it is convenient to adopt solutions of (1) which are obtained by a Hankel transform development of the same. Considering the asymmetry of the problem about the plane $z = 0$, we adopt integral solutions of (1) applicable to the halfspace regions $z \geqslant 0$ (suffix 1) and $z \leqslant 0$ (suffix 2). We select solutions of the form

$$\phi_1(r,z) = \int_0^\infty \xi[A(\xi) + B(\xi)z]\mathrm{e}^{-\xi z} J_0(\xi r)\,\mathrm{d}\xi, \tag{6}$$

$$\phi_2(r,z) = \int_0^\infty \xi[C(\xi) + D(\xi)z]\mathrm{e}^{\xi z} J_0(\xi r)\,\mathrm{d}\xi, \tag{7}$$

where $A(\xi)$, $B(\xi)$, etc., are arbitrary functions. The mixed boundary conditions associated with the crack indentation problem take the following forms:

$$u_z^{(1)}(r,0) = \Delta, \quad r \in (0,a), \tag{8}$$

$$\sigma_{zz}^{(1)}(r,0) = \sigma_0, \quad r \in (a,b), \tag{9}$$

$$\sigma_{rz}^{(1)}(r,0) = 0, \quad r \in (0,c), \tag{10}$$

$$\sigma_{zz}^{(2)}(r,0) = \sigma_0, \quad r \in (0,b), \tag{11}$$

$$\sigma_{rz}^{(2)}(r,0) = 0, \quad r \in (0,c), \tag{12}$$

$$\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0) = 0, \quad r \in (b,\infty), \tag{13}$$

$$\sigma_{rz}^{(1)}(r,0) - \sigma_{rz}^{(2)}(r,0) = 0, \quad r \in (c, \infty), \tag{14}$$

$$u_z^{(1)}(r,0) - u_z^{(2)}(r,0) = 0, \quad r \in (b, \infty), \tag{15}$$

$$u_r^{(1)}(r,0) - u_r^{(2)}(r,0) = 0, \quad r \in (c, \infty). \tag{16}$$

Considering the general solutions for $\phi_i(r,z)$ $(i = 1, 2)$ the mixed boundary conditions (8)–(16) can be written in the form:

$$\int_0^\infty E(\xi) J_0(\xi r)\, \mathrm{d}\xi = -2G\Delta, \quad 0 < r < a, \tag{17}$$

$$\int_0^\infty \xi[E(\xi) + (1-2\nu)F(\xi)] J_0(\xi r)\, \mathrm{d}\xi = 2\sigma_0(1-\nu), \quad a < r < b, \tag{18}$$

$$\int_0^\infty \xi F(\xi) J_1(\xi r)\, \mathrm{d}\xi = 0, \quad 0 < r < c, \tag{19}$$

$$\int_0^\infty \xi M(\xi) J_0(\xi r)\, \mathrm{d}\xi = \sigma_0, \quad 0 < r < b, \tag{20}$$

$$\int_0^\infty \xi[(1-2\nu)M(\xi) - N(\xi)] J_1(\xi r)\, \mathrm{d}\xi = 0, \quad 0 < r < c, \tag{21}$$

$$\int_0^\infty \xi[E(\xi) + (1-2\nu)F(\xi)] J_0(\xi r)\, \mathrm{d}\xi = 2(1-\nu)\int_0^\infty \xi M(\xi) J_0(\xi r)\, \mathrm{d}\xi, \quad b < r < \infty, \tag{22}$$

$$\int_0^\infty E(\xi) J_0(\xi r)\, \mathrm{d}\xi = \frac{1}{2(1-\nu)}\int_0^\infty [-(1-2\nu)N(\xi) - (3-4\nu)M(\xi)] J_0(\xi r)\, \mathrm{d}\xi, \quad b < r < \infty, \tag{23}$$

$$\int_0^\infty \xi F(\xi) J_1(\xi r)\, \mathrm{d}\xi + \frac{1}{2(1-\nu)}\int_0^\infty \xi[(1-2\nu)M(\xi) - N(\xi)] J_1(\xi r)\, \mathrm{d}\xi = 0, \quad c < r < \infty, \tag{24}$$

$$\int_0^\infty [(1-2\nu)E(\xi) - (3-4\nu)F(\xi)] J_1(\xi r)\, \mathrm{d}\xi = 2(1-\nu)\int_0^\infty N(\xi) J_1(\xi r)\, \mathrm{d}\xi, \quad c < r < \infty, \tag{25}$$

where

$$A(\xi) = \frac{1}{\xi^2(1-v)}[vE(\xi) + (1-2v)F(\xi)], \tag{26}$$

$$B(\xi) = \frac{1}{2\xi(1-v)}[E(\xi) - F(\xi)], \tag{27}$$

$$C(\xi) = \frac{-1}{2\xi^2(1-v)}[M(\xi) - (1-2v)N(\xi)], \tag{28}$$

$$D(\xi) = \frac{1}{2\xi(1-v)}[M(\xi) + N(\xi)]. \tag{29}$$

Considering the Hankel inversion theorem and Eq. (20) we have

$$M(\xi) = \frac{\sigma_0 b J_1(\xi b)}{\xi} + \int_b^\infty r\rho_1(r) J_0(\xi r)\,\mathrm{d}r. \tag{30}$$

where $\rho_1(r)$ is defined on the interval $b < r < \infty$.

Similarly by assuming that

$$\int_0^\infty \xi[(1-2v)M(\xi) - N(\xi)]J_1(\xi r)\,\mathrm{d}\xi = \rho_2(r), \quad c < r < \infty \tag{31}$$

we have

$$(1-2v)M(\xi) - N(\xi) = \int_c^\infty r\rho_2(r) J_1(\xi r)\,\mathrm{d}r. \tag{32}$$

Using (32), (19) and the Hankel inversion theorem we can show that

$$(1-2v)M(\xi) - N(\xi) = -2(1-v)F(\xi). \tag{33}$$

Considering (33), Eq. (25) can be written in the form

$$\int_0^\infty [(1-2v)E(\xi) - \left(7 - 12v + 4v^2\right)F(\xi)]J_1(\xi r)\,\mathrm{d}\xi$$
$$= 2(1-v)(1-2v)\int_0^\infty M(\xi) J_1(\xi r)\,\mathrm{d}\xi, \quad c < r < \infty. \tag{34}$$

Let

$$E(\xi) + (1-2v)F(\xi) = p(\xi), \tag{35}$$

we can rewrite (34) in the form

$$\int_0^\infty F(\xi)J_1(\xi r)\,\mathrm{d}\xi = -\frac{(1-2v)}{8(1-v)^2}\int_0^\infty [2(1-v)M(\xi)-p(\xi)]J_1(\xi r)\,\mathrm{d}\xi, \quad c<r<\infty. \tag{36}$$

Utilizing the method proposed by Noble [12], the solution of the dual integral equations (19) and (36) can be written in the following form:

$$F(\xi) = \sqrt{\frac{2\xi}{\pi}}\int_c^\infty x^{3/2}J_{3/2}(\xi x)G_1(x)\,\mathrm{d}x, \tag{37}$$

where

$$G(x) = \frac{(1-2v)}{8(1-v)^2}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty \frac{\mathrm{d}x}{\sqrt{r^2-x^2}}\int_0^\infty [2(1-v)M(\zeta)-p(\zeta)]J_1(\zeta r)\,\mathrm{d}\zeta. \tag{38}$$

Considering the integral

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^\infty \frac{J_1(r\zeta)\,\mathrm{d}x}{\sqrt{r^2-x^2}}\mathrm{d}x = \frac{x\zeta\cos(\zeta x)-\sin(\zeta x)}{\zeta x^2}. \tag{39}$$

Eq. (37) can be rewritten in the form

$$F(\xi) = \frac{(1-2v)}{8(1-v)^2}\sqrt{\frac{2\xi}{\pi}}\int_0^\infty [2(1-v)M(\zeta)-p(\zeta)]K(\xi,\zeta)\,\mathrm{d}\zeta, \quad 0<\xi<\infty, \tag{40}$$

where

$$K(\xi,\zeta) = \frac{1}{\zeta}\int_c^\infty \frac{J_{3/2}(\xi x)[x\zeta\cos(\zeta x)-\sin(\zeta x)]}{\sqrt{x}}\,\mathrm{d}x. \tag{41}$$

Eqs. (17), (18) and (23) can be written in the form

$$\int_0^\infty p(\xi)J_0(\xi r)\,\mathrm{d}\xi = -2G\Delta + (1-2v)\int_0^\infty F(\xi)J_0(\xi r)\,\mathrm{d}\xi, \quad 0<r<a, \tag{42}$$

$$\int_0^\infty p(\xi)J_0(\xi r)\,\mathrm{d}\xi = 2(1-v)\sigma_0, \quad a<r<b, \tag{43}$$

$$\int_0^\infty p(\xi)J_0(\xi r)\,\mathrm{d}\xi = -2(1-v)\int_0^\infty M(\xi)J_0(\xi r)\,\mathrm{d}\xi, \quad b<r<\infty. \tag{44}$$

Let

$$\int_0^\infty \xi p(\xi) J_0(\xi r)\,\mathrm{d}\xi = \begin{cases} f_1(r), & 0 < r < a, \\ f_3(r), & b < r < \infty. \end{cases} \tag{45}$$

Using the Hankel inversion theorem we obtain from (43) and (45)

$$p(\xi) = \int_0^a r f_1(r) J_0(\xi r)\,\mathrm{d}r + 2\sigma_0(1-\nu)\int_a^b r J_0(\xi r)\,\mathrm{d}r + \int_b^\infty r f_3(r) J_0(\xi r)\,\mathrm{d}r. \tag{46}$$

Substituting the value of $p(\xi)$ from (46) in Eqs. (42) and (44) and making use of the procedures outlined by Selvadurai and Singh [12] we find that

$$\begin{aligned} F_1(r) + \frac{2}{\pi}\int_b^\infty \frac{sF_3(s)\,\mathrm{d}s}{(s^2 - r^2)} &= -2G\Delta + (1-2\nu)\int_0^\infty F(\xi)\cos(\xi r)\,\mathrm{d}\xi \\ &\quad - 2\sigma_0(1-\nu)\left\{\left(b^2 - r^2\right)^{1/2} - \left(a^2 - r^2\right)^{1/2}\right\}, \quad 0 < r < a, \end{aligned} \tag{47}$$

$$\begin{aligned} F_3(r) + \frac{2r}{\pi}\int_0^a \frac{F_1(s)\,\mathrm{d}s}{(r^2 - s^2)} &= 2\sigma_0(1-\nu)\left\{\left(r^2 - b^2\right)^{1/2} - \left(r^2 - a^2\right)^{1/2}\right\} \\ &\quad - 2(1-\nu)\int_0^\infty M(\xi)\sin(\xi r)\,\mathrm{d}\xi, \quad b < r < \infty, \end{aligned} \tag{48}$$

where

$$F_1(r) = \int_r^a \frac{sf_1(s)}{\left(s^2 - r^2\right)^{1/2}}\,\mathrm{d}s, \quad 0 < r < a, \tag{49}$$

$$F_3(r) = \int_b^r \frac{sf_3(s)\,\mathrm{d}s}{\left(r^2 - s^2\right)^{1/2}}, \quad b < r < \infty. \tag{50}$$

The Abel type integral equations (49) and (50) have the following solutions:

$$sf_1(s) = -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}s}\int_s^a \frac{rF_1(r)\,\mathrm{d}r}{\left(r^2 - s^2\right)^{1/2}}, \quad 0 < s < a, \tag{51}$$

$$sf_3(s) = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}s}\int_b^s \frac{rF_3(r)\,\mathrm{d}r}{\left(s^2 - r^2\right)^{1/2}}, \quad b < s < \infty. \tag{52}$$

Avoiding details of calculations it can be shown that

$$M(\xi) = \frac{\sigma_0 b J_1(\xi b)}{\xi} + \frac{1}{\pi(1-\nu)}\int_b^\infty F_3(s)\sin(\xi s)\,\mathrm{d}s \tag{53}$$

and

$$p(\xi) = \frac{2}{\pi}\left[\int_0^a F_1(s)\cos(\xi s)\,\mathrm{d}s + \pi\sigma_0(1-v)\int_a^b rJ_0(\xi r)\,\mathrm{d}r + \int_b^\infty F_3(s)\sin(\xi s)\,\mathrm{d}s\right]. \tag{54}$$

Consequently, the system of integral equations (47) and (48) gives

$$\begin{aligned} S_1(r) + \frac{2}{\pi}\int_b^\infty \frac{sS_3(s)\,\mathrm{d}s}{(s^2-r^2)} &= -\frac{2G\Delta}{\sigma_0} + (1-2v)\int_0^\infty S(\xi)\cos(r\xi)\,\mathrm{d}\xi \\ &\quad - 2(1-v)\left\{\left(b^2-r^2\right)^{1/2} - \left(a^2-r^2\right)^{1/2}\right\}, \quad 0<r<a, \end{aligned} \tag{55}$$

$$\begin{aligned} S_3(r) + \frac{r}{\pi}\int_0^a \frac{S_1(s)\,\mathrm{d}s}{(r^2-s^2)} &= (1-v)\left\{\left(r^2-b^2\right)^{1/2} - \left(r^2-a^2\right)^{1/2}\right\} \\ &\quad - (1-v)\left\{r - \left(r^2-b^2\right)^{1/2}\right\}, \quad b<r<\infty, \end{aligned} \tag{56}$$

where

$$S(\xi) = \frac{(1-2v)}{8(1-v)^2}\sqrt{\frac{2\xi}{\pi}}\int_0^\infty [2(1-v)M_1(\zeta) - p_1(\zeta)]K(\xi,\zeta)\,\mathrm{d}\zeta \tag{57}$$

and

$$[S_1(\xi),\ S_3(\xi),\ S(\xi),\ p_1(\zeta),\ M_1(\zeta)] = \frac{1}{\sigma_0}[F_1(\xi),\ F_3(\xi),\ F(\xi),\ p(\zeta),\ M(\zeta)]. \tag{58}$$

Finally, the coupled system of integral equations (55) and (56) can be reduced to a single integral equation

$$\begin{aligned} & S_1(r) + \frac{1}{\pi^2}\int_0^a \frac{S_1(x)}{(r^2-x^2)}\left\{r\ln\left|\frac{b-r}{b+r}\right| - x\ln\left|\frac{b-x}{b+x}\right|\right\}\mathrm{d}x \\ &= -\frac{2G\Delta}{\sigma_0} - 2(1-v)\left\{\left(b^2-r^2\right)^{1/2} - \left(a^2-r^2\right)^{1/2}\right\} \\ &\quad - \frac{2(1-v)}{\pi}\int_b^\infty \frac{s\left\{2(s^2-b^2)^{1/2} - s - (s^2-a^2)^{1/2}\right\}}{(s^2-r^2)}\,\mathrm{d}s \\ &\quad + (1-2v)\int_0^\infty S(\xi)\cos(\xi r)\,\mathrm{d}\xi, \quad 0<r<a. \end{aligned} \tag{59}$$

Introducing the substitution

$$\psi(\zeta) = 2(1-v)M_1(\zeta) - p_1(\zeta), \quad 0<\zeta<\infty \tag{60}$$

the result (57) becomes

$$S(\xi) = \frac{(1-2\nu)}{8(1-\nu)^2}\left(\frac{2\xi}{\pi}\right)^{1/2}\int_0^\infty \psi(\zeta)K(\xi,\zeta)\,\mathrm{d}\zeta, \quad 0<\xi<\infty. \tag{61}$$

Using (54) and the "separation condition" (which will be discussed in the ensuing) we have

$$S_3(b) = 0 \tag{62}$$

and

$$\psi(\xi) = 2(1-\nu)\frac{bJ_1(\xi b)}{\xi} - 2(1-\nu)\int_a^b rJ_0(\xi r)\,\mathrm{d}r - \frac{2}{\pi}\int_0^a S_1(s)\cos(\xi s)\,\mathrm{d}s, \quad 0<\xi<\infty. \tag{63}$$

It can also be shown that

$$\int_0^\infty S(\xi)\cos(\xi r)\,\mathrm{d}\xi = -\frac{(1-2\nu)}{16(1-\nu)}\frac{\pi a^2}{c} + \frac{(1-2\nu)}{8(1-\nu)^2 c}\int_0^a S_1(x)\,\mathrm{d}x \tag{64}$$

and that

$$\begin{aligned}\int_b^\infty \frac{s\left\{2(s^2-b^2)^{1/2} - s - (s^2-a^2)^{1/2}\right\}}{(s^2-r^2)}\,\mathrm{d}s &= \frac{\pi}{2}\left(a^2-r^2\right)^{1/2} - \pi\left(b^2-r^2\right)^{1/2} + b \\ &\quad + \left(b^2-a^2\right)^{1/2} + \frac{r}{2}\ln\left|\frac{b-r}{b+r}\right| \\ &\quad - \left(a^2-r^2\right)^{1/2}\tan^{-1}\left(\frac{b^2-a^2}{a^2-r^2}\right)^{1/2}.\end{aligned} \tag{65}$$

Considering these results, the integral equation (59) can be further reduced to the form

$$\begin{aligned}&S_1(r) + \frac{1}{\pi^2}\int_0^a \frac{S_1(x)}{(r^2-x^2)}\left\{r\ln\left|\frac{b-r}{b+r}\right| - x\ln\left|\frac{b-x}{b+x}\right|\right\}\mathrm{d}x - \frac{(1-2\nu)^2}{8(1-\nu)^2}\frac{1}{c}\int_0^a S_1(x)\,\mathrm{d}x \\ &\quad = -\frac{2G\Delta}{\sigma_0} - \frac{(1-2\nu)^2}{16(1-\nu)}\frac{\pi a^2}{c} + (1-\nu)\left(a^2-r^2\right)^{1/2}\left\{1 + \frac{2}{\pi}\tan^{-1}\left(\frac{b^2-a^2}{a^2-r^2}\right)^{1/2}\right\} \\ &\qquad - \frac{2(1-\nu)}{\pi}\left\{b + \left(b^2-a^2\right)^{1/2} + \frac{r}{2}\ln\left|\frac{b-r}{b+r}\right|\right\}, \quad 0<r<a.\end{aligned} \tag{66}$$

The axisymmetric mixed boundary value problem of the partial indentation of a precompressed penny-shaped crack by a smooth, rigid penny-shaped inclusion is reduced to the solution of the single Fredholm integral equation of the second kind, (66), for the unknown function $S_1(r)$. All results of engineering interest can be expressed in terms of the function $S_1(r)$. This integral equation is, however, indeterminate to within the radius of the separation zone $r = b$, which can be derived from the consistency condition (62). The unknown dimension b can be determined by making use of the condition

$$K_{\mathrm{I}}^{b} = \underset{r\to b+}{Lt}\,[2(r-b)]^{1/2}\sigma_{zz}^{(1)}(r,0) = \underset{r\to b+}{Lt}\,[2(r-b)]^{1/2}\sigma_{zz}^{(2)}(r,0) = 0. \tag{67}$$

From the results in the preceding section we have

$$\sigma_{zz}^{(1)}(r,0) = \frac{1}{2(1-\nu)}\int_0^{\infty}\xi p(\xi)J_0(\xi r)\,\mathrm{d}\xi, \quad b<r<\infty, \tag{68}$$

$$\sigma_{zz}^{(2)}(r,0) = \int_0^{\infty}\xi M(\xi)J_0(\xi r)\,\mathrm{d}\xi, \quad b<r<\infty. \tag{69}$$

From (53), (54) and (67)–(69) we obtain the result

$$F_3(b) \equiv 0. \tag{70}$$

Considering integral equation (48) and the result (70) we obtain

$$\int_0^a \frac{S_1(x)\,\mathrm{d}x}{(b^2-x^2)} + \frac{(1-\nu)\pi\left\{b+(b^2-a^2)^{1/2}\right\}}{b} = 0. \tag{71}$$

The coupled systems (66) and (71) can be solved numerically to evaluate $S_1(r)$ and b/a.

4. The load–displacement response

The compliance of the embedded rigid disc inclusion is governed by the extent to which closure is established within the precompressed crack region $(c-b)$. This in turn is influenced by the relative magnitudes of a number of variables including σ_0/G and Poisson's ratio of the elastic solid. The load P inducing the displacement Δ can be obtained from the result

$$P = -2\pi\int_0^a r\sigma_{zz}^{(1)}(r,0)\,\mathrm{d}r. \tag{72}$$

By making use of the results in the preceding section we can show that

$$P = -\frac{2\sigma_0}{(1-\nu)}\int_0^a S_1(r)\,\mathrm{d}r. \tag{73}$$

5. Stress intensity factors

The stress intensity factors at the tip of the penny-shaped crack are also of importance to the evaluation of potential for brittle fracture during the application of the load P. In general, the axisymmetric indentation of the single surface of a penny-shaped crack induces both mode I and mode II stress intensity factors at the crack tip. In situations where the crack tip experiences closure, the flaw opening mode stress intensity factor at the crack tip vanishes. In general, the stress intensity factors at the crack tip ($r = c$) are given by:

$$K_{\mathrm{I}}^{c} = \lim_{r\to c^{+}} [2(r-c)]^{1/2}\left[\sigma_{zz}^{(1)}(r,0)\right], \tag{74}$$

$$K_{\mathrm{II}}^{c} = \lim_{r\to c^{+}} [2(r-c)]^{1/2}\left[\sigma_{rz}^{(1)}(r,0)\right]. \tag{75}$$

Evaluating the results (74) and (75) we have

$$K_{\mathrm{I}}^{c} = \frac{\sigma_0 S_3(c)}{\pi(1-v)\sqrt{c}}, \tag{76}$$

$$K_{\mathrm{II}}^{c} = \frac{2\sigma_0\sqrt{c}\,G_1(c)}{\pi}, \tag{77}$$

where

$$G_1(x) = \frac{(1-2v)}{8(1-v)^2x^2}\int_0^{\infty}[2(1-v)M_1(\zeta) - p_1(\zeta)][x\zeta\cos(\zeta x) - \sin(\zeta x)]\frac{\mathrm{d}\zeta}{\zeta}. \tag{78}$$

Expression (77) for K_{II}^{c} can be expressed in the form

$$\frac{K_{\mathrm{II}}^{c}}{\sigma_0\sqrt{c}} = \frac{(1-2v)}{4\pi(1-v)^2c^2}\left\{-(1-v)\frac{\pi a^2}{2} + \int_0^a S_1(x)\,\mathrm{d}x\right\}. \tag{79}$$

Using the results (76) and (77), the expressions for K_{I}^{c} and K_{II}^{c} can be reduced to the forms

$$\overline{K}_{\mathrm{I}}^{c} = \frac{K_{\mathrm{I}}^{c}}{P/2\pi^2c^{3/2}} = -\frac{2\sigma_0\pi c^2}{(1-v)P}\left\{\int_0^a \frac{S_1(x)\,\mathrm{d}x}{(c^2-x^2)} + \pi(1-v)\frac{\left[c+(c^2-a^2)^{1/2}\right]}{c}\right\}, \tag{80}$$

$$\overline{K}_{\mathrm{II}}^{c} = \frac{K_{\mathrm{II}}^{c}}{P(1-2v)/8\pi(1-v)c^{3/2}} = 1 + \frac{\pi a^2\sigma_0}{P}. \tag{81}$$

6. Numerical results

To evaluate the compliance of the indenting inclusion and the stress intensity factors at the boundary of the penny-shaped crack it is necessary to obtain a solution of the integral Eq. (66)

subject to the constraint (71). The form of the Fredholm type integral Eq. (66) is such that it is not solvable in exact closed form. To develop a numerical solution of (66) we rewrite in the form

$$\chi(\eta)+\frac{1}{\pi^2}\int_0^1\frac{\chi(\xi)}{(\eta^2-\xi^2)}\left\{\eta\ln\left|\frac{\delta-\eta}{\delta+\eta}\right|-\xi\ln\left|\frac{\delta-\xi}{\delta+\xi}\right|\right\}\mathrm{d}\xi-\frac{(1-2\nu)^2}{2(1-\nu)^2}\frac{1}{\gamma}\int_0^1\chi(\xi)\,\mathrm{d}\xi$$

$$=-\frac{2G\Delta}{\sigma_0 a}-\frac{(1-2\nu)^2\pi}{16(1-\nu)\gamma}+(1-\nu)\left(1-\eta^2\right)^{1/2}\left\{1+\frac{2}{\pi}\tan^{-1}\left(\frac{\delta^2-1}{1-\eta^2}\right)^{1/2}\right\}$$

$$-\frac{2(1-\nu)}{\pi}\left\{\delta+\left(\delta^2-1\right)^{1/2}+\frac{\eta}{2}\ln\left|\frac{\delta-\eta}{\delta+\eta}\right|\right\} \tag{82}$$

and condition (71) can be written as

$$\int_0^1\frac{\chi(\xi)\,\mathrm{d}\xi}{(\delta^2-\xi^2)}+\frac{(1-\nu)\pi\left\{\delta+(\delta^2-1)^{1/2}\right\}}{\delta}=0, \tag{83}$$

where

$$\xi=\frac{x}{a},\quad \eta=\frac{r}{a},\quad \delta=\frac{b}{a},\quad \gamma=\frac{c}{a},\quad \chi(\eta)=\frac{S_1(r)}{a}. \tag{84}$$

The interval [0,1] associated with the integrations in (82) and (83) is divided into N equal segments. The integral equation (82) can be replaced by its discretized version in the form

$$[A_{ij}]\ \{\chi_j\}=\{f_i\},\quad i,j=1,2,\ldots,N, \tag{85}$$

where the coefficients A_{ij} are given by

$$A_{ij}=\delta_{ij}+\frac{1}{\pi^2 N}\frac{1}{\left(\xi_i^2-\xi_j^2\right)}\left\{\xi_i\ln\left|\frac{\delta-\xi_i}{\delta+\xi_i}\right|-\xi_j\ln\left|\frac{\delta-\xi_j}{\delta+\xi_j}\right|\right\}-\frac{(1-2\nu)^2}{8(1-\nu)^2\gamma N}, \tag{86}$$

$\chi_j=\chi(\xi_j)$; and a limiting value of (86) has to be considered when $\xi_j\to\xi_i$. Also the right-hand side of (85) is given by

$$f_i=\frac{-2G\Delta}{\sigma_0 a}-\frac{(1-2\nu)^2}{16(1-\nu)}\frac{\pi}{\gamma}+(1-\nu)\left(1-\xi_i^2\right)^{1/2}\left\{1+\frac{2}{\pi}\tan^{-1}\left(\frac{\delta^2-1}{1-\xi_i^2}\right)^{1/2}\right\}$$

$$-\frac{2(1-\nu)}{\pi}\left\{\delta+\left(\delta^2-1\right)^{1/2}+\frac{\xi_i}{2}\ln\left|\frac{\delta-\xi_i}{\delta+\xi_i}\right|\right\}. \tag{87}$$

In (85) and (86), ξ_j is given by

$$\xi_j = \frac{(2j-1)}{2N}, \quad j = 1, 2, \ldots, N. \tag{88}$$

The constraint (83) is replaced by the condition

$$\epsilon = \frac{1}{N}\sum_{j=1}^{N} \frac{\chi_j}{\left(\delta^2 - \xi_j^2\right)} + \pi(1-v)\left\{\frac{\delta + (\delta^2-1)^{1/2}}{\delta}\right\}. \tag{89}$$

An error term ϵ is added to (89) to account for the non-linearity of the problem, associated with the determination of the radius of the separation zone $r = b$. A standard iterative procedure can be applied to obtain an error ϵ associated with (89), which is assigned as $|\epsilon| < 10^{-6}$. Convergence is established within 25 iterations. In the case where $b \to c$ the condition (83) is not satisfied and the error bound can be directly used (see e.g. (80)) to calculate the value of $K_{\rm I}^c$. The discretized equivalents for (73), (80) and (81) are given by

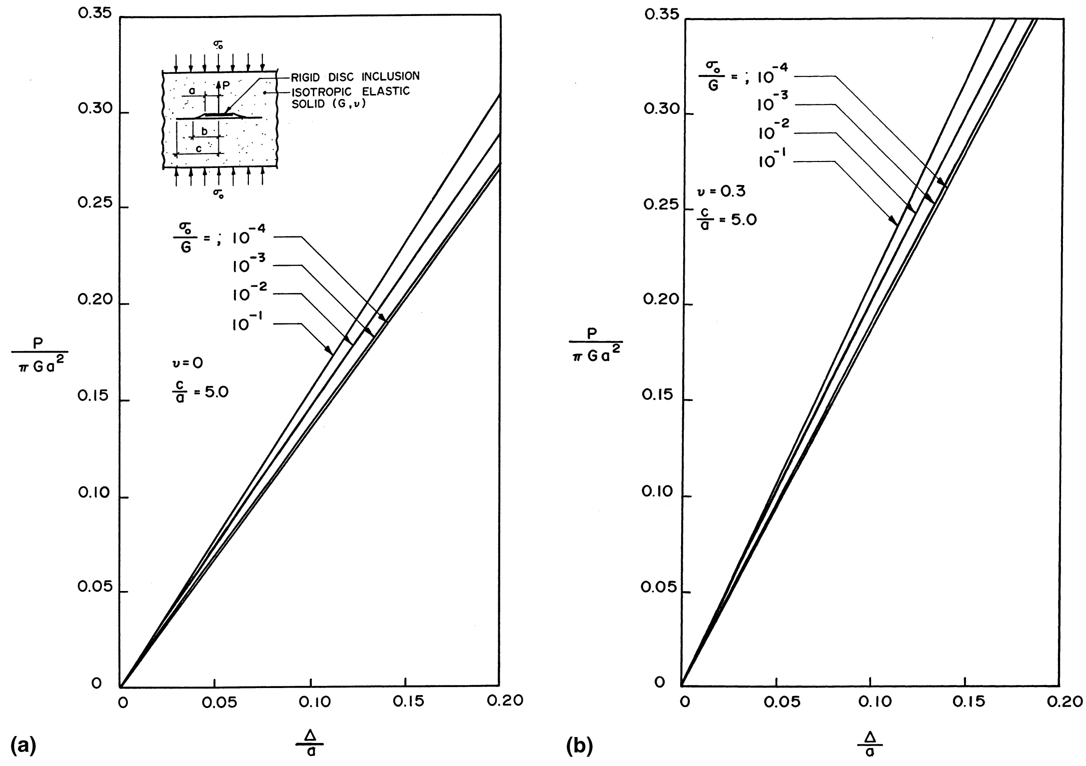

Fig. 2. The load–displacement relationship for the inclusion.

$$\frac{P}{G\pi a^2} = \frac{2\sigma_0}{(1-\nu)G\pi}\frac{1}{N}\sum_{j=1}^{N}\chi_j, \tag{90}$$

$$\overline{K}_{\mathrm{I}}^c = -\frac{2\sigma_0 c^2 \epsilon}{(1-\nu)P}, \tag{91}$$

$$\overline{K}_{\mathrm{II}}^c = 1 + \frac{\pi a^2 \sigma_0}{P}. \tag{92}$$

The preceding numerical procedures are used to evaluate the influence of the precompression on the axial stiffness of the inclusion and the stress intensity factors at the boundary of the crack tip. The non-dimensional parameters influencing the problem include σ_0/G, ν and c/a. The numerical results presented in the paper will be restricted to specific values of these parameters.

Owing to the receding contact associated with the indentation of the precompressed crack, the load–displacement relationship for the indenting inclusion is non-linear. Figs. 2 and 3 illustrate the load–displacement relationships for the disc inclusion for variable values of σ_0/G and c/a and

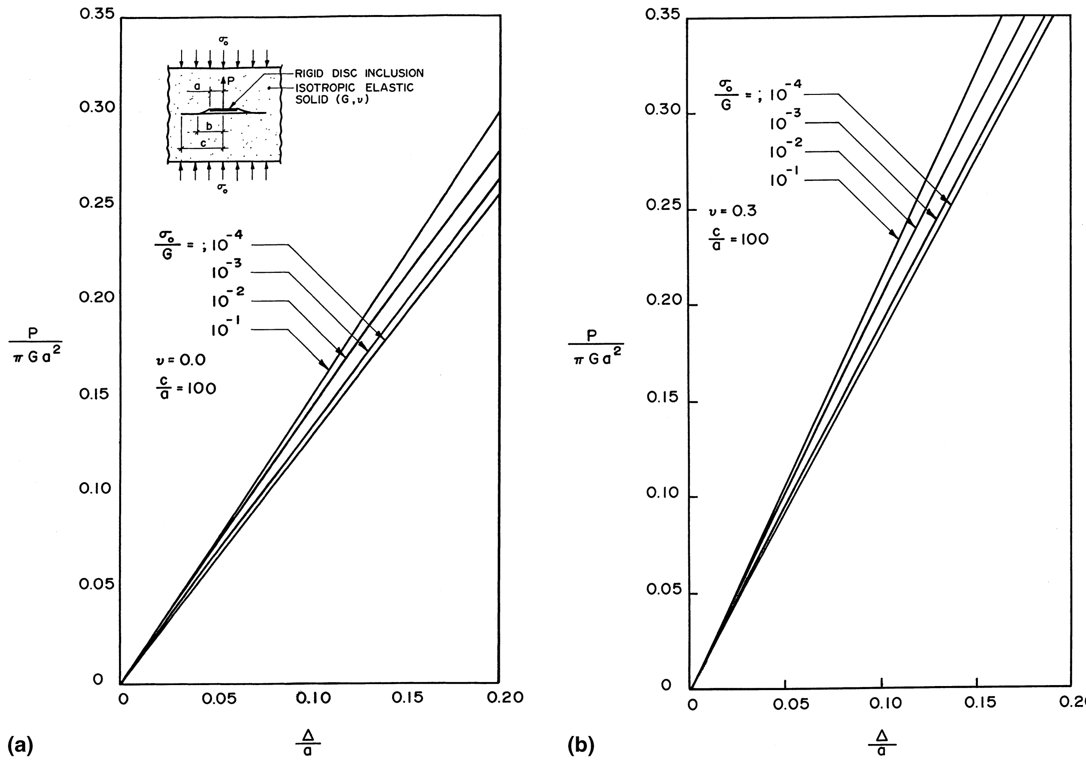

Fig. 3. The load–displacement relationship for the inclusion.

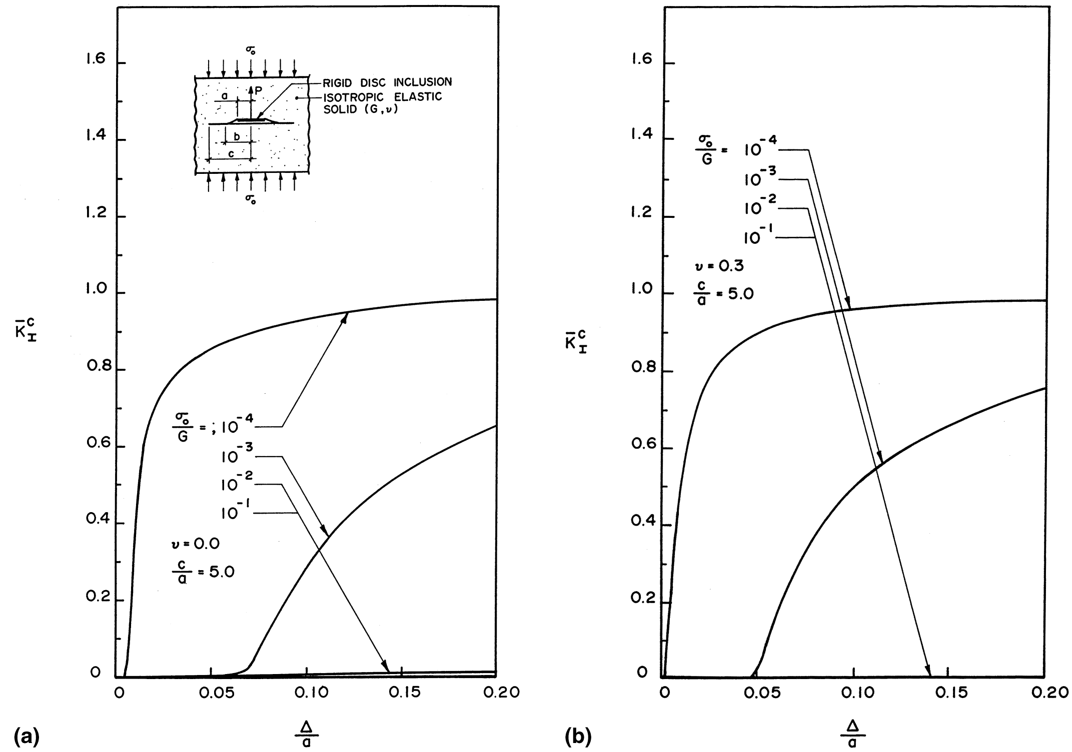

Fig. 4. The mode I stress intensity factor at the boundary of the internally indented precompressed penny-shaped crack.

a specific value of v. It is clear that although there is receding contact at the precompressed crack, this does not result in a load–displacement behaviour with a pronounced non-linearity. As c/a becomes large and σ_0/G becomes small, the problem reduces to that of the indentation of a surface of a halfspace region by a smooth rigid punch. The results of the computations shown in Fig. 3 ($c/a = 100$; $\sigma_0/G = 10^{-4}$) agree closely with Boussinesq's result for the indentation of the surface or halfspace region. Fig. 4 illustrates the variation in the normalized crack opening mode stress intensity factor $\overline{K}_{\mathrm{I}}^{c}$ at the crack tip $r = c$ with that of the displacement of the rigid inclusion. It is evident that crack closure is maintained for values of the order of $\sigma_0/G > 10^{-1}$. The normalized crack shearing mode stress intensity factor $\overline{K}_{\mathrm{II}}^{c}$ is dependent on only the value of σ_0/P and independent of the crack-inclusion radius ratio c/a. Consequently the results for $\overline{K}_{\mathrm{II}}^{c}$ can be represented by a single set of curves applicable to variations in σ_0/G. The relevant results are shown in Fig. 5.

7. Conclusions

In conventional treatment of cracks, it is invariably assumed that the crack tip remains open even for situations involving compressive loads which act normal to the plane of the crack. An

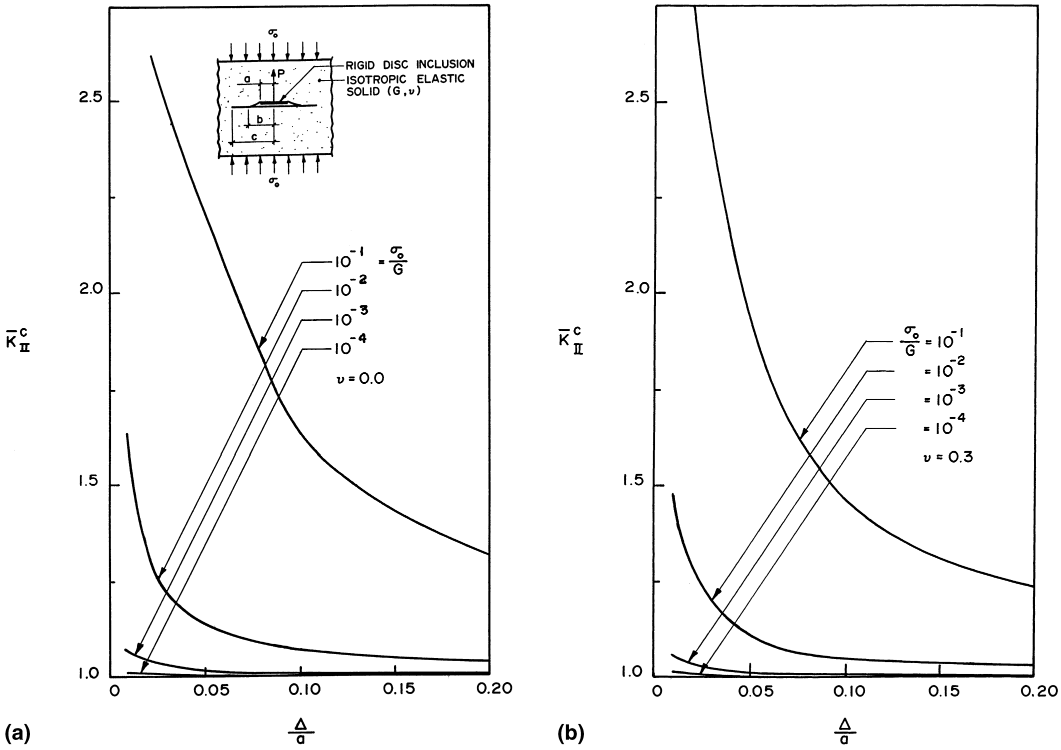

Fig. 5. The mode II stress intensity factor at the boundary of the internally indented precompressed penny-shaped crack.

alternative to this would be to consider cracks which experience closure at the tip particularly with compressive stress fields. This paper examines the situation where the precompressed smooth crack experiences partial closure due to the indentation of the single surface of the crack by a rigid disc inclusion. For an axisymmetric situation, the resulting unilateral contact problem can be formulated as a mixed boundary of value problem in elasticity. It is shown that this procedure yields a single integral equation of the Fredholm type and a constraint for the separation which can be solved to evaluate results of engineering interest. In particular, the results of the numerical computations indicate that the axial compliance of the inclusion is relatively insensitive to the precompression. The stress intensity factors at the crack tip experience a precompression stress dependency in that the activation of the flaw opening or mode I stress intensity factor is dependent upon the ratio σ_0/G and that the flaw shearing or mode II stress intensity factor is non-zero for all choices of σ_0/G.

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