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An inclusion at a bi-material elastic interface

A. P. S. SELVADURAI
Department of Civil Engineering and Applied Mechanics, McGill University, 817 Sherbrooke Street West Montreal, Quebec, Canada H3A 2K6

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Abstract. This paper examines the problem of the axial displacement of a rigid circular disk-shaped inclusion which is embedded at the interface between two bonded dissimilar isotropic elastic solids. The analysis focusses on the determination of the axial stiffness of the embedded inclusion, which is evaluated by a numerical solution of two coupled Fredholm integral equations of the second-kind derived from the reduced mixed boundary-value problem for the interface. The results for the axial stiffness are also compared with certain 'bounds' which are developed by imposing constraints on the variation of either the traction or the displacements at the interface.

Key words: embedded inclusion, bi-material interface, stiffness bounds, integral equations.

1. Introduction

Problems which deal with inclusions embedded in elastic media have occupied a prominent place in the development of the theory of elasticity. Such solutions also have a wide range of engineering applications including the study of stress amplification at inhomogeneities in solids and in the evaluation of the effective properties of multiphase solids. Classical solutions for spherical, spheroidal and ellipsoidal inhomogeneity problems developed by Goodier [1], Dewey [2], Edwards [3] Eshelby [4], Lur'e [5] and others have been applied quite extensively to the modelling of composite solids. The solutions to inclusion problems have also been adopted for the modelling of anchoring mechanisms employed in geomechanical applications. Studies by de Josselin de Jong [6], Kanwal and Sharma [7] Selvadurai [8] and Zureick [9] dealing with three-dimensional inclusions which are subjected to axial forces have been used to estimate the axial stiffness properties of anchors embedded in geomaterials. Comprehensive accounts of developments pertaining to three-dimensional inclusion problems are given by Willis [10], Walpole [11] and Mura [12, pp. 63–203], [13].

The disk inclusion or the penny-shaped inclusion is a particular limiting case of the general problem of a three-dimensional inhomogeneity. Owing to the simplified geometry of the disk inclusion problem it is possible to develop a variety of solutions which have applications to mechanics of composite materials and geomechanics. The studies by Collins [14], Keer [15] and Kassir and Sih [16] have developed solutions for rigid-disk inclusions with circular and elliptical planforms which are embedded in elastic media and subjected to loads which act directly on the inclusion. In an extensive series of studies (see *e.g.* Selvadurai [17]) the work on the disk-inclusion problem has been extended to include the influence of material anisotropy, inclusion flexibility, interaction of inclusions with cracks, influence of external boundaries and the interaction of inclusions with nuclei of strain. Both the exact closed form solutions and numerical results developed in connection with these problems have found extensive application in the estimation of the stiffness characteristics of disk-shaped anchoring mechanisms.

In this paper we consider the problem of a rigid-disk inclusion which is embedded in bonded contact at the bonded interface between two isotropic elastic solids. From the perspective of applications in geomechanics, the inclusion can be regarded as a region of higher stiffness which can be introduced at the interface due to migration of a bonding material such as epoxy which is injected under pressure to create the anchoring mechanism. The analysis focuses on the evaluation of the axial stiffness characteristics of the rigid disk inclusion and in particular the evaluation of the influence of the mis-match in the elasticity properties of the bonded elastic halfspace regions on the axial stiffness of the rigid disk inclusion. The mathematical modelling of the problem is essentially reduced to the consideration of a mixed boundary problem derived from the consideration of the continuity conditions both within and exterior to the embedded inclusion region. When dealing with elastic bi-material regions and bonded regions which are located at the surface of an elastic halfspace region it is well known [18–20], [21, pp. 117–177] that the stress singularity at discontinuities can exhibit oscillatory phenomena. In the modelling of the disk inclusion embedded in bonded contact at the boundary between two bonded dissimilar halfspace regions, the stress singularity can exhibit an oscillatory form, depending upon the manner in which the boundary configuration of the inclusion is idealized. Consequently, in situations where exact stress distributions at the inclusion boundary are of interest, it is necessary to perform the analysis by appeal to a formulation based on the Hilbert problem where the stress singularity is accurately modelled [22–24], [25, pp. 481–496]. This is particularly the case when the elastic modulus of one of the bonded regions reduces to zero, in which case, the inclusion problem is identical to the problem of a rigid punch which is bonded to the surface of a halfspace region. The stress singularity at the boundary of the rigid punch will have an oscillatory form which depends on Poisson's ratio of the halfspace region. Since the primary focus of the analysis is on the evaluation of 'global results' pertaining to the stiffness of the embedded rigid inclusion, the contribution from the oscillatory form of the stress singularity is excluded from the analysis. We will illustrate the rationale for this approach by examining the problem of a rigid punch in adhesive contact with a halfspace region. The formulation of the problem is achieved by considering Love's strain-function approach [26, pp. 274–276] for the solution of axisymmetric problems in the classical theory of elasticity. The mixed boundary-value problem derived by considering the continuity of tractions and displacements at the bonded interface is reduced to the solution of a pair of Fredhom-type integrals of the second kind which are solved via a numerical technique. The numerical scheme is used to develop results for the axial stiffness of the disk inclusion. The numerical results are compared with the bounds for the axial stiffness derived from a method proposed by Selvadurai [27] where the axial stiffness is calculated by imposing either traction or displacement constraints at the interface between the bounded halfspace regions.

2. The inclusion problem

We consider the axisymmetric elasticity problem related to an infinite space which is composed of a bonded contact between two homogeneous isotropic elastic halfspace regions. A rigid circular-disk inclusion embedded in bonded contact with the two halfspace regions is subjected to an axial displacement Δ in the z-direction (Figure 1). Since the disk inclusion

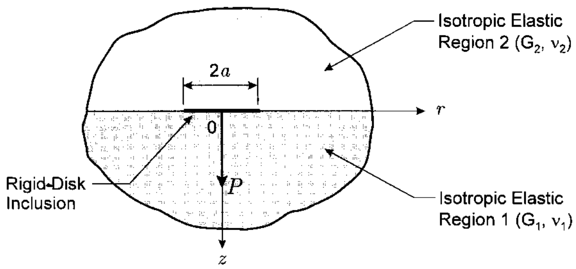

Figure 1. Rigid-disk inclusion at a bi-material elastic interface.

problem exhibits axial symmetry, it is convenient to use Love's strain function approach for the formulation of the problem. It can be shown [26, pp. 274–276] that the solution to the displacement equations of equilibrium can be expressed in terms of a single function $\Phi^{(i)}(r, z)$ which satisfies

$$\nabla^2\nabla^2\Phi^{(i)}(r, z) = 0, \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is the axisymmetric form of Laplace's operator and the superscripts i refer to the halfspace regions 1 ($z \geqslant 0$) and 2 ($z \leqslant 0$) which are bonded at the plane $z = 0$.

The displacement and stress components in the elastic media can be evaluated in terms of the strain potentials $\Phi^{(i)}$ (r, z); we have

$$2G_i u_r^{(i)}(r, z) = -\frac{\partial^2\Phi^{(i)}}{\partial r\partial z}, \tag{3}$$

$$2G_i u_z^{(i)}(r, z) = 2(1 - \nu_i)\nabla^2\Phi^{(i)} - \frac{\partial^2\Phi^{(i)}}{\partial z^2} \tag{4}$$

and

$$\sigma_{rr}^{(i)} = \frac{\partial}{\partial z}\left\{\nu_i\nabla^2\Phi^{(i)} - \frac{\partial^2\Phi^{(i)}}{\partial r^2}\right\}, \tag{5}$$

$$\sigma_{\theta\theta}^{(i)} = \frac{\partial}{\partial z}\left\{\nu_i\nabla^2\Phi^{(i)} - \frac{1}{r}\frac{\partial\Phi^{(i)}}{\partial r}\right\}, \tag{6}$$

$$\sigma_{zz}^{(i)} = \frac{\partial}{\partial z}\left\{(2 - \nu_i)\nabla^2\Phi^{(i)} - \frac{\partial^2\Phi^{(i)}}{\partial z^2}\right\}, \tag{7}$$

$$\sigma_{rz}^{(i)} = \frac{\partial}{\partial r}\left\{(1 - \nu_i)\nabla^2\Phi^{(i)} - \frac{\partial^2\Phi^{(i)}}{\partial z^2}\right\}, \tag{8}$$

where G_i and ν_i $(i = 1, 2)$ are the shear moduli and Poisson's ratios of the respective halfspace regions. The boundary conditions governing the mixed boundary-value problem related to the bonded interface $z = 0$ are as follows;

$$u_z^{(1)}(r, 0) = \Delta; \quad 0 \leqslant r \leqslant a, \tag{9}$$

$$u_z^{(2)}(r, 0) = \Delta; \quad 0 \leqslant r \leqslant a, \tag{10}$$

$$u_r^{(1)}(r, 0) = 0; \quad 0 \leqslant r \leqslant a, \tag{11}$$

$$u_r^{(2)}(r, 0) = 0; \quad 0 \leqslant r \leqslant a, \tag{12}$$

$$u_z^{(1)}(r, 0) = u_z^{(2)}(r, 0); \quad a \leqslant r < \infty, \tag{13}$$

$$u_r^{(1)}(r, 0) = u_r^{(2)}(r, 0); \quad a \leqslant r < \infty, \tag{14}$$

$$\sigma_{zz}^{(1)}(r, 0) = \sigma_{zz}^{(2)}(r, 0); \quad a < r < \infty, \tag{15}$$

$$\sigma_{rz}^{(1)}(r, 0) = \sigma_{rz}^{(2)}(r, 0); \quad a < r < \infty. \tag{16}$$

Owing to the axisymmetric nature of the inclusion problem, it is convenient to adopt a solution based on the Hankel transforms development of the governing partial differential equation (see *e.g.* Sneddon [28, pp. 327–328], [29, Chapter 4]). The relevant solutions applicable to the halfspace regions $z \geqslant 0$ and $z \leqslant 0$ can be written as

$$\Phi^{(1)}(r, z) = \int_0^\infty \xi[A(\xi) + B(\xi)z]\, \mathrm{e}^{-\xi z} J_0(\xi r)\, \mathrm{d}\xi \tag{17}$$

and

$$\Phi^{(2)}(r, z) = \int_0^\infty \xi[C(\xi) + D(\xi)z]\, \mathrm{e}^{\xi z} J_0(\xi r)\, \mathrm{d}\xi, \tag{18}$$

where $A(\xi)$, $B(\xi)$... etc., are arbitrary functions. The corresponding displacement and stress components applicable to the regions 1 and 2 can be obtained from (17) and (18) and the expressions (3–8). We have

$$2G_1 u_r^{(1)}(r, 0) = \int_0^\infty \xi^2[-\xi A(\xi) + B(\xi)] J_1(\xi r)\, \mathrm{d}\xi, \tag{19}$$

$$2G_1 u_z^{(1)}(r, 0) = -\int_0^\infty \xi^2[\xi A(\xi) + 2(1 - 2\nu_1)B(\xi)] J_0(\xi r)\, \mathrm{d}\xi, \tag{20}$$

$$\sigma_{zz}^{(1)}(r, z) = \int_0^\infty \xi^3[\xi A(\xi) + 2(1 - 2\nu_1)B(\xi)] J_0(\xi r)\, \mathrm{d}\xi, \tag{21}$$

$$\sigma_{rz}^{(1)}(r, 0) = \int_0^\infty \xi^3[\xi A(\xi) - 2\nu_1 B(\xi)] J_1(\xi r)\, \mathrm{d}\xi. \tag{22}$$

and

$$2G_2 u_r^{(2)}(r,0) = \int_0^\infty \xi^2 [\xi C(\xi) + D(\xi)] J_1(\xi r)\, \mathrm{d}\xi, \tag{23}$$

$$2G_2 u_z^{(2)}(r,0) = -\int_0^\infty \xi^2 [\xi C(\xi) - 2(1-\nu_2) D(\xi)] J_0(\xi r)\, \mathrm{d}\xi, \tag{24}$$

$$\sigma_{zz}^{(2)}(r,0) = \int_0^\infty \xi^3 [-\xi C(\xi) + (1-2\nu_2) D(\xi)] J_0(\xi r)\, \mathrm{d}\xi, \tag{25}$$

$$\sigma_{rz}^{(2)}(r,0) = \int_0^\infty \xi^3 [\xi C(\xi) + 2\nu_2 D(\xi)] J_1(\xi r)\, \mathrm{d}\xi. \tag{26}$$

Using (19) to (26), we can reduce the boundary conditions (9–16) to the following system of integral equations

$$\begin{aligned}\int_0^\infty & [L(\xi)\{R_1 + 2(1-2\nu_1)P_1\} + M(\xi)\{R_2 + 2(1-2\nu_1)P_2\} \\ & + N(\xi)\{R_3 + 2(1-2\nu_1)P_3\} + Q(\xi)\{R_4 + 2(1-2\nu_1)P_4\}] J_0(\xi r)\, \mathrm{d}\xi \\ & = -2G_1\Delta; \quad 0 \leqslant r \leqslant a,\end{aligned} \tag{27}$$

$$\begin{aligned}\int_0^\infty & [L(\xi)\{\alpha + \Omega\} + M(\xi)\{\beta - \Omega\} + N(\xi)\{\gamma + \Omega\} \\ & + Q\{\delta - \Omega\}] J_0(\xi r)\, \mathrm{d}\xi = -2G_2\Delta; \quad 0 \leqslant r \leqslant a,\end{aligned} \tag{28}$$

$$\begin{aligned}\int_0^\infty & [L(\xi)\{P_1 - R_1\} + M(\xi)\{P_2 - R_2\} + N(\xi)\{P_3 - R_3\} \\ & + Q(\xi)\{P_4 - R_4\}] J_1(\xi r)\, \mathrm{d}\xi = 0; \quad 0 \leqslant r \leqslant a,\end{aligned} \tag{29}$$

$$\begin{aligned}\int_0^\infty & [L(\xi)\{\alpha - \psi\} + M(\xi)\{\beta + \psi\} + N(\xi)\{\gamma - \psi\} \\ & + Q(\xi)\{\delta + \psi\}] J_1(\xi r)\, \mathrm{d}\xi = 0; \quad 0 \leqslant r \leqslant a,\end{aligned} \tag{30}$$

$$\int_0^\infty M(\xi) J_1(\xi r)\, \mathrm{d}\xi = 0; \quad a < r < \infty, \tag{31}$$

$$\int_0^\infty \xi N(\xi) J_0(\xi r)\, \mathrm{d}\xi = 0; \quad a < r < \infty, \tag{32}$$

$$\int_0^\infty \xi Q(\xi) J_0(\xi r)\, \mathrm{d}\xi = 0; \quad a < r < \infty, \tag{33}$$

where the substitution functions $L(\xi)$, $M(\xi)$, $N(\xi)$ and $P(\xi)$ can be related to $A(\xi)$, $B(\xi)$ through the relationships

$$\xi^3 A(\xi) = R_1 L(\xi) + R_2 M(\xi) + R_3 N(\xi) + R_4 Q(\xi), \tag{34}$$

$$\xi^2 B(\xi) = P_1 L(\xi) + P_2 M(\xi) + P_3 N(\xi) + P_4 Q(\xi), \tag{35}$$

$$\xi^3 C(\xi) = \alpha L(\xi) + \beta M(\xi) + \gamma N(\xi) + \delta Q(\xi), \tag{36}$$

$$\xi^2 D(\xi) = \psi[M(\xi) + Q(\xi) - L(\xi) + N(\xi)] \tag{37}$$

and the constants $\alpha, \beta, \dots, P_1, P_2, \dots, R_1, R_2, \dots$ etc. depend only on the elastic constants of the bi-material region and explicit expressions are given in Appendix A. The use of a Love strain-function formulation of the problem, unfortunately results in a rather awkward set of constants characterizing the influence of the bimaterial parameters. It is quite likely that other representations such as the Neuber–Papkovich formulation of the solution of elasticity problems in terms of harmonic functions could result in more managable forms of the material parameter combinations. The results given in Appendix A are, however, mathematically correct. They could, of course, be further reduced by the use of symbolic manipulation schemes such as *Mathematica*® *Maple*®. Since the solution scheme ultimately involves numerical procedures, the specific combinations of the material parameters can be chosen in the presentation of numerical results associated with the problem. The functions $L(\xi)$, $M(\xi), \dots$ etc., can in turn be expressed in the forms

$$L(\xi) = \frac{1}{G_2}[\xi^2 G_1\{\xi C(\xi) - 2(1-2\nu_2)D(\xi)\} - \xi^2 G_2\{\xi A(\xi) + 2(1-2\nu_1)B(\xi)\}], \tag{38}$$

$$M(\xi) = \frac{1}{G_2}[\xi^2 G_1\{\xi C(\xi) + D(\xi)\} - \xi^2 G_2\{-\xi A(\xi) + B(\xi)\}], \tag{39}$$

$$N(\xi) = \xi^2\{\xi A(\xi) + (1-2\nu_1)B(\xi)\} + \xi^2\{\xi C(\xi) - (1-2\nu_2)D(\xi)\}, \tag{40}$$

$$Q(\xi) = \xi^2\{-\xi A(\xi) + 2\nu_1 B(\xi)\} + \xi^2\{\xi C(\xi) + 2\nu_2 D(\xi)\} \tag{41}$$

The Equations (31), (32) and (33) can be satisfied if we assume representations of the form

$$M(\xi) = -2\Delta G_2 \int_0^a u\varphi_2(u) J_1(\xi u)\,\mathrm{d}u, \tag{42}$$

$$N(\xi) = -2\Delta G_1 \int_0^a \varphi_3(u)\,\cos(\xi u)\,\mathrm{d}u, \tag{43}$$

$$Q(\xi) = -2\Delta G_1 \int_0^a \varphi_4(u)\,\sin(\xi u)\,\mathrm{d}u, \tag{44}$$

where $\varphi_2(u)$, $\varphi_3(u)$ and $\varphi_4(u)$ are unknown functions. Substituting the values of $A(\xi)$, $B(\xi)$, $C(\xi)$ and $D(\xi)$ defined by (34) to (37) into (38) we note that

$$L(\xi) \equiv 0. \tag{45}$$

The Equations (27), (28) and (29) now yield the following system of coupled integral equations for the functions φ_2, φ_3 and φ_4; *i.e.*

$$\frac{G_2\{R_2 + 2(1-2\nu_1)P_2\}}{G_1}\varphi_2(r) + \{R_3 + 2(1-2\nu_1)P_3\}\int_0^a \frac{\varphi_3(u)H(r-t)\,\mathrm{d}u}{\sqrt{r^2-u^2}}$$
$$+\{R_4 + 2(1-2\nu_1)P_4\}\int_0^a \frac{\varphi_4(u)H(u-r)\,\mathrm{d}u}{\sqrt{u^2-r^2}} = 1; \quad 0 \leqslant r \leqslant a, \tag{46}$$

$$\{P_4 - R_4\} \int_0^a \frac{t\varphi_4(t)H(r-t)\,\mathrm{d}t}{\sqrt{r^2-t^2}} = -\frac{G_2\{P_2 - R_2\}r\varphi_2(r)}{G_1}$$
$$-\{P_3 - R_3\} \int_0^a \varphi_3(t)\left[1 - \frac{tH(t-r)}{\sqrt{t^2-r^2}}\right]\mathrm{d}t; \quad 0 < r < a, \tag{47}$$

$$\{\delta + \psi\} \int_0^a \frac{t\varphi_4(t)H(r-t)\,\mathrm{d}t}{\sqrt{r^2-t^2}} = -\frac{G_2}{G_1}\{\beta + \psi\}r\varphi_{2(}(r)$$
$$-\{\gamma - \psi\} \int_0^a \left[1 - \frac{tH(t-r)}{\sqrt{t^2-r^2}}\right]\varphi_3(t)\,\mathrm{d}t; \quad 0 < r < a \tag{48}$$

where $H(x)$ is the Heaviside unit function. The analysis of the disk-inclusion embedded at the bi-material elastic interface and subjected to the rigid displacement Δ along the z-direction is now reduced to the solution of the three integral Equations (46–48). These integral equations are not amenable to exact solution and their numerical evaluation will be discussed in a subsequent section.

A result of importance to engineering applications involves the evaluation of the force-displacement relationship for the rigid-disk inclusion. The force P required to induce the rigid displacement Δ can be evaluated by considering the tractions acting on the faces of the inclusion in contact with the two halfspace regions. We have

$$P = 2\pi \int_0^a [\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0)]r\,\mathrm{d}r \tag{49}$$

From the result (32) we have

$$\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0) = \int_0^\infty \xi N(\xi) J_0(\xi r)\,\mathrm{d}\xi. \tag{50}$$

From (50) and (51) we obtain

$$P = 2\pi \int_0^a \left[\frac{\partial}{\partial r}\int_0^\infty \xi N(\xi) J_1(\xi r)\,\mathrm{d}\xi\right]\mathrm{d}r. \tag{51}$$

Using the result (43) in (51) we have

$$P = -4\pi G_1 \Delta \int_0^a \varphi_3(t)\,\mathrm{d}t. \tag{52}$$

The numerical solution of the integral Equations (46) to (48) can be directly utilized to evaluate the load-displacement relationship for the embedded rigid-disk inclusion.

3. The numerical evaluation of the governing integral equations

The structure of the governing integral equations is such that they do not appear to be amenable to exact solution. As a result it is necessary to adopt a numerical scheme to generate the relevant results. A variety of techniques have been proposed for the numerical solution of coupled systems of integral equations of the general type described by (46) to (48). These are fully documented by Atkinson [30, p. 230], Baker [31, Chapters 4 and 5] and Delves and Mohamed [32, Chapters 4, 7, 10]. We write the coupled integral equations (46) to (48) in the generalized Fredholm forms

$$C_{12}\varphi_2(r) + C_{13}\int_0^a \varphi_3(t)K_{13}(t,r)\,\mathrm{d}t + C_{14}\int_0^a \varphi_4(t)K_{14}(t,r)\,\mathrm{d}t = 1; \quad 0 \leqslant r \leqslant a \tag{53}$$

$$C_{32}r\varphi_2(r) + C_{33}\int_0^a \varphi_3(t)K_{33}(t,r)\,\mathrm{d}t + C_{34}\int_0^a \varphi_4(t)K_{34}(t,r)\,\mathrm{d}t = 0; \quad 0 \leqslant r \leqslant a, \tag{54}$$

$$C_{42}r\varphi_2(r) + C_{43}\int_0^a \varphi_3(t)K_{43}(t,r)\,\mathrm{d}t + C_{44}\int_0^a \varphi_4(t)K_{44}(t,r)\,\mathrm{d}t = 0; \quad 0 \leqslant r \leqslant a, \tag{55}$$

where the constants C_{12}, C_{13} etc. and the kernel functions K_{13}, $K_{14}, \ldots$ etc., are given by

$$\begin{aligned} C_{12} &= \frac{G_2}{G_1}[R_2 + 2(1-2\nu_1)P_2], \qquad C_{13} = [R_3 + 2(1-2\nu_1)P_3],\\ C_{14} &= [R_4 + 2(1-2\nu_1)P_4], \end{aligned} \tag{56}$$

$$C_{32} = \frac{G_2}{G_1}(P_2 - R_2); \qquad C_{33} = (P_3 - R_3), \qquad C_{34} = (P_4 - R_4),$$

$$C_{42} = \frac{G_2}{G_1}[\beta + \Omega), \qquad C_{43} = [\gamma - \Omega], \qquad C_{44} = [\delta + \Omega]$$

and

$$\begin{aligned} K_{13}(t,r) &= \frac{H(r-t)}{\sqrt{r^2 - t^2}}, \qquad K_{14}(r,t) = \frac{H(t-r)}{\sqrt{t^2 - r^2}},\\ K_{33}(t,r) &= \left[1 - \frac{tH(r-t)}{\sqrt{t^2 - r^2}}\right], \qquad K_{34}(t,r) = \frac{tH(t-r)}{\sqrt{r^2 - t^2}},\\ K_{43}(t,r) &= K_{33}(r,t); \qquad K_{44}(t,r) = K_{34}(t,r). \end{aligned} \tag{57}$$

We can eliminate $\varphi_2(r)$ between the Equations (53), (54) and (55) and obtain a system of coupled Fredholm-type equations in terms of $\varphi_3(r)$ and $\varphi_4(r)$; *i.e.*

$$\begin{aligned} &\int_0^a \varphi_3(t)\{C_{32}C_{13}rK_{13}(t,r) - C_{33}C_{12}K_{33}(t,r)\}\,\mathrm{d}t\\ &\quad + \int_0^a \varphi_4(t)\{C_{32}C_{14}rK_{14}(t,r) - C_{34}C_{12}K_{34}(t,r)\}\,\mathrm{d}t = C_{32}r \end{aligned} \tag{58}$$

and

$$\begin{aligned} &\int_0^a \varphi_3(t)\{C_{42}C_{13}rK_{13}(t,r) - C_{43}C_{12}K_{43}(t,r)\}\,\mathrm{d}t\\ &\quad + \int_0^a \varphi_4(t)\{C_{42}C_{14}rK_{13}(t,r) - C_{44}C_{12}K_{44}(t,r)\}\,\mathrm{d}t = C_{42}r. \end{aligned} \tag{59}$$

In order to solve this coupled system of integral equations numerically, the interval $[0, a]$ is divided into N segments with end points $[t_{n,} t_{n+1}]$ where $t_n = (n-1)a/N$. For the N locations of r_n, where

$$r_n = \left(\frac{t_n + t_{n+1}}{2}\right), \tag{60}$$

the discretized system corresponding to (57) and (58) can be written as

$$[A_{\ell m}]\{X_m\} = \{B_\ell\}, \tag{61}$$

where $\ell, m = 1, 2, \ldots, N$. The matrix $[A_{\ell m}]$ is a (2×2) sub-matrix which can be written as

$$[A_{\ell m}] = \begin{bmatrix} C_{32}C_{13}r_\ell I_1 - C_{33}C_{12}(I_3 - I_5) & C_{32}C_{14}r_\ell I_2 - C_{34}C_{12}I_4 \\ C_{42}C_{13}r_\ell I_1 - C_{43}C_{12}(I_3 - I_5) & C_{42}C_{14}r_\ell I_2 - C_{44}C_{12}I_4 \end{bmatrix}, \tag{62}$$

the sub-vector $\{B_\ell\}$ is given by

$$\{B_\ell\} = \begin{Bmatrix} C_{32}r_\ell \\ C_{42}r_\ell \end{Bmatrix} \tag{63}$$

and the unknown matrix is

$$\{X_m\} = \begin{Bmatrix} \varphi_3(r_{m)} \\ \varphi_4(r_{m)} \end{Bmatrix}. \tag{64}$$

The integral functions I_k $(k = 1, 2, \ldots, 5)$ defined in (62) are given by

$$I_1(r_\ell, t_m, t_{m+1}) = \begin{cases} 0; \quad 0 < r_\ell < t_m, \\ \left[\frac{\pi}{2} - \sin^{-1}\left(\frac{t_m}{r_\ell}\right)\right]; \quad t_m < r_\ell < t_{m+1}, \\ \left[\sin^{-1}\left(\frac{t_{m+1}}{r_\ell}\right) - \sin^{-1}\left(\frac{t_m}{r_\ell}\right)\right]; \quad t_m < r_\ell < a, \end{cases} \tag{65}$$

$$\begin{aligned} &I_2(r_\ell, t_m, t_{m+1}) \\ &= \begin{cases} \left\{\log|t_{m+1} + \sqrt{t_{m+1}^2 - r_\ell^2}| - \log|t_m + \sqrt{t_m^2 - r_\ell^2}|\right\}; \quad 0 < r_\ell < t_m, \\ \left\{\log|t_{m+1} + \sqrt{t_{m+1}^2 - r_\ell^2}| - \log(r_\ell)\right\}; \quad t_m < r_\ell < t_{m+1}, \\ 0; \quad t_{m+1} < r_\ell < a, \end{cases} \end{aligned} \tag{66}$$

$$I_3(t_m, t_{m+1}) = (t_{m+1} - t_m), \tag{67}$$

$$I_4 = \begin{cases} 0; \quad 0 < r_\ell < t_m, \\ \sqrt{r_\ell^2 - t_m^2}; \quad t_m < r_\ell < t_{m+1}, \\ \left\{\sqrt{r_\ell^2 - t_m^2} - \sqrt{r_\ell^2 - t_{m+1}^2}\right\}; \quad t_{m+1} < r_\ell < a, \end{cases} \tag{68}$$

and

$$I_5 = \begin{cases} \left\{\sqrt{t_{m+1}^2 - r_\ell^2} - \sqrt{t_m^2 - r_\ell^2}\right\}; \quad 0 < r_\ell < t_m, \\ \sqrt{t_{m+1}^2 - r_\ell^2}; \quad t_m < r_\ell < t_{m+1}, \\ 0; \quad t_{m+1} < r_\ell < a. \end{cases} \tag{69}$$

The numerical solution of the coupled integral Equations (58) and (59) is essentially reduced to the solution of the matrix equation (61) where the value of N is changed to achieve a desired accuracy. The forms of the kernel functions are such that a Gaussian quadrature scheme can be used quite effectively and for $N = 20$, the numerical results for the associated halfspace solutions are obtained to within an accuracy of 0·06%.

4. Numerical results

Prior to the presentation of results for the stiffness of the rigid-disk inclusion derived via the numerical scheme described previously, it is instructive to assess the degree of error introduced by the omission of the oscillatory form of the singularity which will occur at the boundary of the rigid-disk inclusion especially when the elastic modulus of one of the regions approaches zero. To illustrate this, we consider the problem of a rigid circular punch of radius a which is bonded to the surface of a halfspace region. It is subjected to an axial force P_B which induces a displacement Δ_B in the z-direction. The mixed boundary conditions associated with this indentation problem are

$$u_z(r, 0) = \Delta_B; \quad 0 \leqslant r \leqslant a, \tag{70}$$

$$u_r(r, 0) = 0; \quad 0 \leqslant r \leqslant a, \tag{71}$$

$$\sigma_{zz}(r, 0) = 0; \quad a < r < \infty, \tag{72}$$

$$\sigma_{rz}(r, 0) = 0; \quad a < r < \infty. \tag{73}$$

The solution of this mixed boundary-value problem was presented by Ufliand [22] and further expositions of the method of solution are given by Mossakovskii [23] and Gladwell [24]. The result of particular interest concerns the load-displacement behaviour of the rigid circular punch, which can be evaluated in exact closed form. Using the formulation based on the Hilbert problem, we can evaluate this result in exact closed form as follows:

$$\frac{P_B}{4G\Delta_B a} = \frac{\ell n(3 - 4\nu)}{1 - 2\nu}, \tag{74}$$

Table 1. Axial stiffness of a rigid circular indentor bonded to a halfspace region.

	$P_B/(8G\Delta_B a)$		
Poisson's ratio ν	Results based on the Hilbert-problem approach	Results based on the Fredholm integral--equation approach	% Error
0	0·5490	0·5493	0·0546
0·1	0·5970	0·5972	0·0335
0·2	0·6569	0·6570	0·0152
0·3	0·7346	0·7347	0·0136
0·4	0·8411	0·8412	0·0119
0·5	1·0000	1·0000	0

where G and ν are respectively the linear elastic shear modulus and Poisson's ratio of the halfspace region. An alternative approach is to use the basic methodologies outlined previously and to reduce the problem to the solution of a single Fredholm integral equation of the second kind. This integral equation can be solved numerically to generate the equivalent results for the load-displacement relationship for the bonded circular punch, albeit without the implementation of the oscillatory form of the stress singularity at the boundary of the punch. The comparison between the result (74) and the results obtained from the solution of the Fredholm integral equation is presented in Table 1. It is evident that the discrepancy between the two sets of results is within limits acceptable for engineering applications of the results.

To further aid the presentation of the results we cite here a 'bounding technique' for the calculation of the stiffness of inclusions embedded at bi-material interface regions first proposed by Selvadurai [27]. The bounding technique involves the application of either displacement constraints or traction constraints at the interface between the two bi-material elastic halfspace regions. In the bound involving the kinematic constraint, it is explicitly assumed that the interface is inextensible in the plane of the inclusion and continuity of displacements is maintained normal to the interface between the bonded bi-material elastic halfspace regions. In developing the bound involving traction constraints, it is assumed that the inclusion is embedded in smooth contact at the interface between two pre-compressed elastic halfspace regions. The precompression normal to the smooth interface is assumed to be sufficient to maintain continuity of displacement normal to the bi-material boundary. Selvadurai [27] applied this bounding to technique to develop solutions to the problem of an elliptical disk inclusion which was embedded in bonded contact between two dissimilar transversely isotropic elastic solids. These results give the following bounds for the axial stiffness of a rigid circular inclusion which is embedded in bonded contact with two isotropic elastic halfspace regions, *i.e.*

$$\left\{\frac{\Gamma(1-\nu_2)+(1-\nu_{1)}}{2(1-\nu_1)(1-\nu_2)(1+\Gamma)}\right\} \leqslant \frac{P}{8(G_1+G_2)\Delta a}$$
$$\leqslant \frac{2[\Gamma(1-\nu_1)(3-4\nu_2)+(1-\nu_2)(3-4\nu_1)]}{(3-4\nu_1)(3-4\nu_2)(1+\Gamma)}, \tag{75}$$

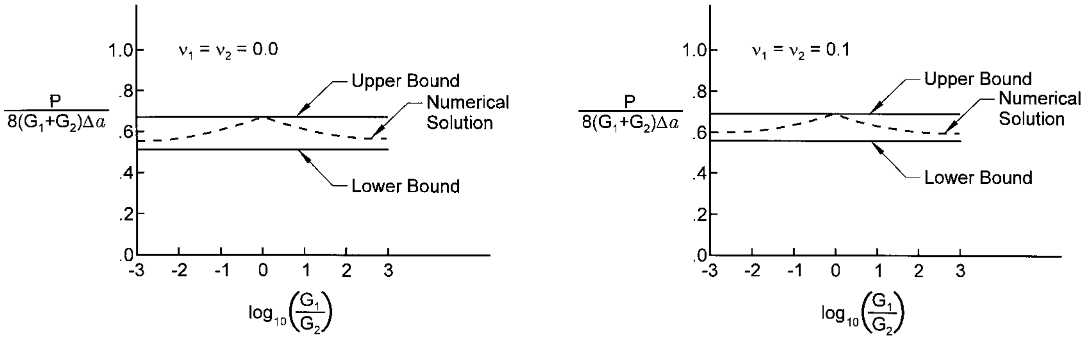

Figure 2. Normalized axial stiffness of the disk inclusion.

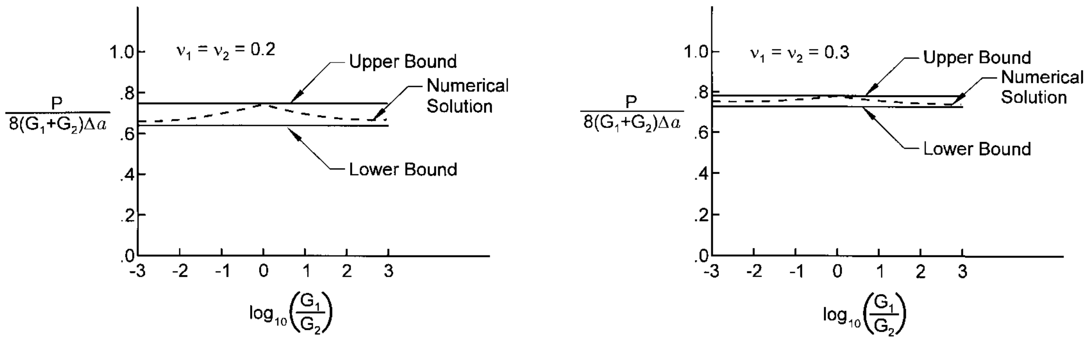

Figure 3. Normalized axial stiffness of the disk inclusion.

where

$$\Gamma = \frac{G_1}{G_2}. \tag{76}$$

It may be noted that in the limit of material incompressibility for both halfspace regions (*i.e.* $\nu_1 = \nu_2 = \frac{1}{2}$), both bounds converge to the exact result

$$P = 8(G_1 + G_2)\Delta a. \tag{77}$$

The numerical procedure outlined in the previous section was used to develop estimates for the axial stiffness of the rigid disk inclusion. The numerical results can be presented for a variety of combinations of ν_1, ν_2 and the modular ratio Γ. The Figures 2 to 6 illustrate the variation in $\overline{P}$, where

$$\overline{P} = \frac{P}{8(G_1 + G_2)\Delta a} \tag{78}$$

for various choices of ν_1, ν_2 and Γ. It is evident that the numerical estimates are consistent with the results derived via the bounding technique. In connection with applications to problems in geomechanics, the limit of material incompressibility describes the behaviour of the

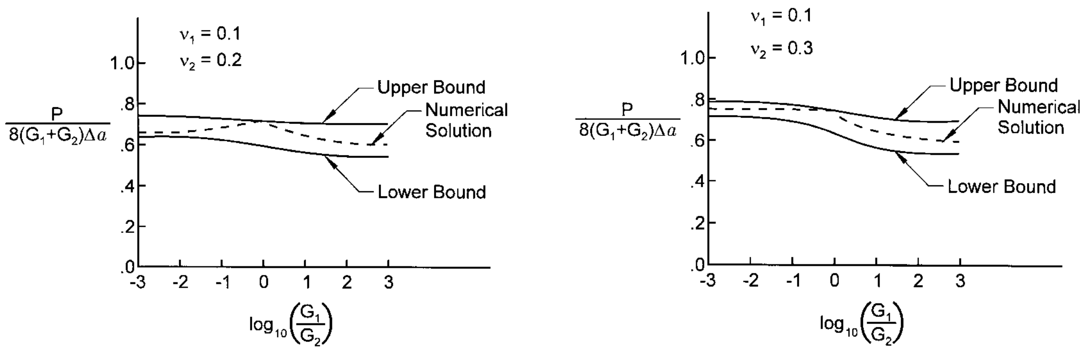

Figure 4. Normalized axial stiffness of the disk inclusion.

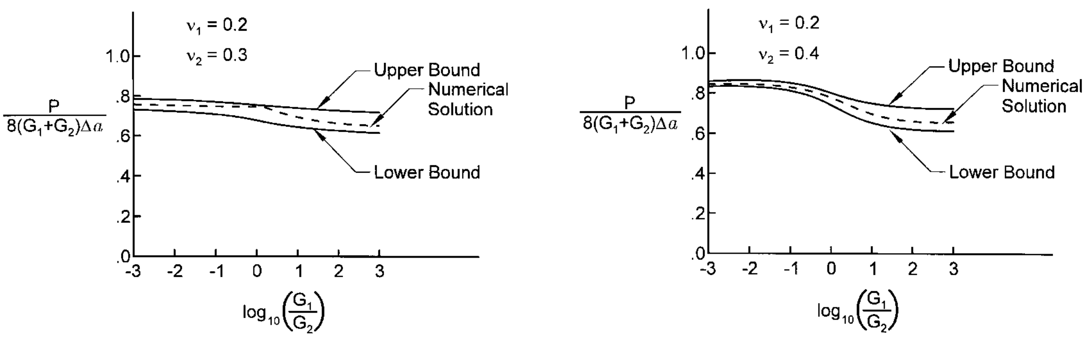

Figure 5. Normalized axial stiffness of the disk inclusion.

mechanical response of fluid saturated poroelastic geomaterials at the commencement of the pore fluid diffusion process, and usually referred to as an 'undrained response' [33-34], [35, pp. 33–45]. In these circumstances the mechanical response of the axial load displacement behaviour of the inclusion can be conveniently derived from result (77).

5. Concluding remarks

The axial load-displacement response of a rigid-disk inclusion embedded in bonded contact at the bonded interface between two dissimilar elastic solids can be examined as a mixed boundary-value problem derived by considering the displacement conditions within the inclusion region and continuity conditions exterior to the inclusion region. At the boundary of the inclusion–bi-material region interface the stress singularity has to be defined by considering the local geometry. In particular, the stress singularity should exhibit the transition from a regular $1/\sqrt{r}$ type for identical materials to the oscillatory form when the elastic modulus of one of the materials reduces to zero. The developments presented in this paper are primarily focussed on procedures which can be used to evaluate the load-displacement behaviour of the embedded rigid-disk inclusion. It is shown that when a regular $1/\sqrt{r}$ type stress singularity is incorporated at the inclusion boundary, the problem is reduced to the solution of a pair of Fredholm integral equations of the second kind which can be solved in a numerical fashion to develop the load-displacement relationship of the inclusion. Furthermore, it is shown that the

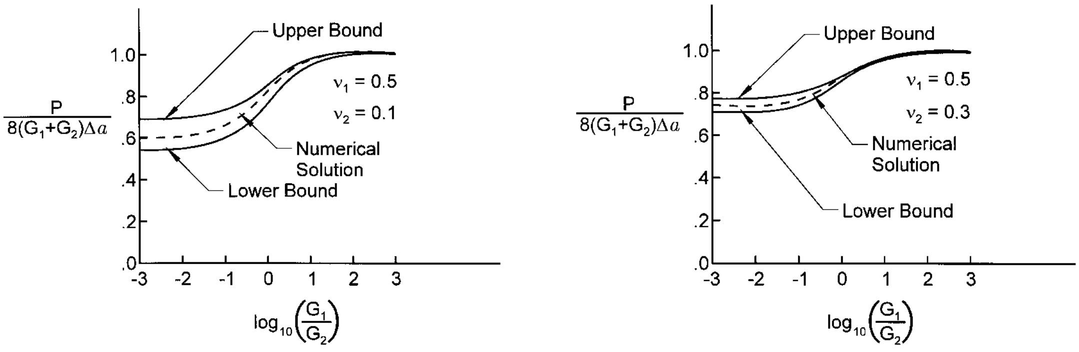

Figure 6. Normalized axial stiffness of the disk inclusion.

oscillatory form of the stress singularity at the boundary of the bonded rigid inclusion has no appreciable contribution to a global result such as a load-displacement relationship. Comparisons between a Hilbert-problem approach and regular Fredholm integral-equation approach for a bonded punch problem for a halfspace indicates that the maximum error between the exact solution which incorporates an oscillatory stress singularity and the approximate result is less than 0·0546%. The approaches converge to the same result as $\nu_i \to \frac{1}{2}$. The accuracy of the numerical scheme is also verified by comparison with certain bounds which have been developed by incorporating either kinematic or mechanical constraints at the bonded bi-material interface.

Appendix A

The substitution constants $\alpha, \beta, \ldots, P_1, P_2, \ldots$ etc. are defined as follows:

$$\alpha = \frac{G_2}{[2G_1 + 2G_2(3-4\nu_1)]}\left[1 - \frac{(1-4\nu_2)G_1 - G_2(3-4\nu_1)(4\nu_2-1)}{(3G_1+G_2-4G_1\nu_2)}\right],$$

$$\beta = \frac{G_2}{[2G_1 + 2G_2(3-4\nu_1)]}\left[1 + \frac{(1-4\nu_2)G_1 - G_2(3-4\nu_1)(4\nu_2-1)}{(3G_1+G_2-4G_1\nu_2)}\right],$$

$$\gamma = \frac{1}{[2G_1 + 2(3-4\nu_1)G_2]}\left[(3-4\nu_1)G_2 - \frac{G_2\{(1-4\nu_2)G_1 - G_2(3-4\nu_1)(4\nu_2-1)\}}{(3G_1+G_2-4G_1\nu_2)}\right],$$

$$\delta = \frac{1}{[2G_1 + 2(3-4\nu_1)G_2]}\left[(3-4\nu_1)G_2 + \frac{G_2\{(1-4\nu_2)G_1 - G_2(3-4\nu_1)(4\nu_2-1)\}}{(3G_1+G_2-4G_1\nu_2)}\right],$$

$$P_1 = \frac{1}{G_2(1-2\nu_1)}\left[\alpha(G_1+G_2) - G_2 + \frac{G_2(1-2\nu_2)(2G_1+G_2)}{(3G_1+G_2-4G_1\nu_2)}\right],$$

$$P_2 = \frac{1}{G_2(1-2\nu_1)}\left[\beta(G_1+G_2) - \frac{G_2(1-2\nu_2)(2G_1+G_2)}{(3G_1+G_2-4G_1\nu_2)}\right],$$

$$P_3 = \frac{1}{G_2(1-2\nu_1)}\left[\gamma(G_1+G_2) - G_2\frac{G_2(1-2\nu_2)(2G_1+G_2)}{(3G_1+G_2-4G_1\nu_2)}\right]$$

$$P_4 = \frac{1}{G_2(1-2\nu_1)}\left[\delta(G_1+G_2) - \frac{G_2(1-2\nu_2)(2G_1+G_2)}{(3G_1+G_2-4G_1\nu_2)}\right],$$

$$R_1 = \frac{1}{G_2}\left[\alpha G_1 + \frac{2G_1G_2(1-2\nu_2)}{(3G_1+G_2-4\nu_2G_1)} - 2(1-2\nu_1)P_1G_2\right],$$

$$R_2 = \frac{1}{G_2}\left[\beta G_1 - \frac{2G_1G_2(1-2\nu_2)}{(3G_1+G_2-4\nu_2G_1)} - 2(1-2\nu_1)P_2G_2\right],$$

$$R_3 = \frac{1}{G_2}\left[\gamma G_1 + \frac{2G_1G_2(1-2\nu_2)}{(3G_1+G_2-4\nu_2G_1)} - 2(1-2\nu_1)P_3G_2\right],$$

$$R_4 = \frac{1}{G_2}\left[\delta G_1 - \frac{2G_1G_2(1-2\nu_2)}{(3G_1+G_2-4\nu_2G_1)} - 2(1-2\nu_1)P_4G_2\right],$$

$$\Omega = \frac{2(1-2\nu_2)G_2}{(3G_1+G_2-4\nu_2G_1)}, \qquad \psi = \frac{G_2}{(3G_1+G_2-4\nu_2G_1)},$$

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