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TECHNICAL NOTE

THE RESPONSE OF A DEEP RIGID ANCHOR DUE TO UNDRAINED ELASTIC DEFORMATION OF THE SURROUNDING SOIL MEDIUM

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SUMMARY

The equations governing the undrained linear elastic behaviour of a saturated soil are formally similar to the equations governing slow flows of an incompressible Newtonian viscous fluid. This principle of equivalence can then be effectively employed to obtain the load–deflection relationship for a deep rigid anchor with the shape of a solid of revolution which is embedded in bonded contact with an unbounded incompressible elastic medium. It is found that the load–deflection relationship for the deep rigid anchor can be directly recovered from the expression for the drag induced on an impermeable object with the same size and shape as the anchor, which is appropriately placed in a slow viscous flow region of uniform velocity.

INTRODUCTION

The behaviour of deep anchors is of importance in connection with the geotechnical study of foundations subject to uplift loads induced by wind, earthquake and other effects. (See for example, Hanna and Carr,[1] Adams and Klym,[2] Johnston and Ladanyi[3] and Selvadurai.[4]) A deep anchor is generally regarded as one in which the depth of embedment is considerably greater than the largest dimension of the anchor region. It is further assumed that the presence of external boundaries does not in any way influence the mechanical behaviour of the deep anchor. The extent to which this assumption is realized in theory, or in practice, will depend upon a number of factors, including the geometrical shape of the anchor, the relative deformability characteristics of the soil and the anchor, and the boundary conditions at the anchor–soil interface.

In this note we are primarily concerned with the analysis of the load–deflection behaviour of a rigid deep anchor due to undrained elastic behaviour of the surrounding soil medium. The mechanical response of both the anchor and the soil medium is generally quite complex; to a first approximation, however, the undrained behaviour of most saturated cohesive soils can be represented by incompressible elastic behaviour and the anchor region can be regarded as being rigid. The geometrical shape of the anchor is assumed to be that of a solid of revolution. the anchor, which is in bonded contact with the surrounding soil medium, is subjected to a resultant force directed along its axis of symmetry. This causes a rigid body displacement of the anchor. The relationship between the applied load and the resulting rigid body translation constitutes the load–displacement relationship for the rigid anchor. The assumption of bonded contact at the anchor–soil interface may not always be fully realized in practice. It is, however,

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reasonable to assume that as the depth of embedment increases the confining stresses due to the overburden may prevent any separation at the interface, especially at working loads. The analysis of such effects is beyond the scope of this note. In addition, we neglect any frictional restraint that may be offered by the anchor tie rod.

The solution of the deep anchor problem for the undrained elastic case can be approached by making use of the mathematical equivalence which exists between the equations governing slow viscous flows of a Newtonian viscous fluid expressed in terms of a 'stream function' and the equations governing the linear elastic behaviour of an incompressible solid in terms of a 'displacement function'. Using this principle it can be shown that the expression relating the load–deflection behaviour of a deep rigid anchor with the shape of a solid of revolution, which is in bonded contact with an unbounded incompressible elastic medium, can be directly recovered from the expression for the viscous drag on an impermeable object of the same size and shape situated in an unbounded viscous fluid flowing with uniform velocity in the direction of the axis of symmetry. This universal connection enables us to establish the load–deflection characteristics of deep anchors of various axisymmetric geometrical configurations by simply considering the solution to the appropriate slow viscous flow problem.

INCOMPRESSIBLE ELASTIC PROBLEM

It can be shown (see e.g. Selvadurai[15]) that the fundamental equations governing the axisymmetric deformations of an incompressible isotropic linearly elastic material can be reduced to the solution of two equations of the form

$$E^4\psi(R,\theta)=0; \qquad \nabla^2 p(R,\theta)=0 \tag{1}$$

where $\psi(R,\theta)$ and $p(R,\theta)$ are respectively a displacement function and a scalar isotropic stress, referred to a system of axisymmetric spherical polar coordinates R, θ. Also for axial symmetry the Stokes' operator E^2 and Laplace's operator ∇^2 take the forms

$$E^2=\frac{\partial^2}{\partial R^2}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{R^2}\frac{\partial}{\partial\theta}$$

$$\nabla^2=\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot\theta}{R^2}\frac{\partial}{\partial\theta} \tag{2}$$

The solution of (1), determined by satisfying the particular boundary conditions of the problem can then be utilized to generate the stress and displacement fields in the incompressible elastic medium. The general formulation of the incompressible elastic problem in terms of a displacement function can be effectively employed to analyse problems in which displacement boundary conditions are prescribed. In the rigid anchor problems discussed here, displacement boundary conditions are prescribed at the anchor–elastic medium interface. For future reference, we cite here the load–deflection relationships, derived by this technique, for rigid anchors with spherical and disc shapes (Figures 1(a) and 1(b), respectively).

(a) *Spherical anchor.* A rigid spherical anchor embedded in bonded contact with an infinite elastic medium is subjected to an axially symmetric load P_s in the direction of the z-axis (Figure 1(a)). The load–deflection relationship for this anchor is given by Joselin de Jong[6] and Selvadurai[4,5]

$$P_s=6\pi Ga\delta \tag{3a}$$

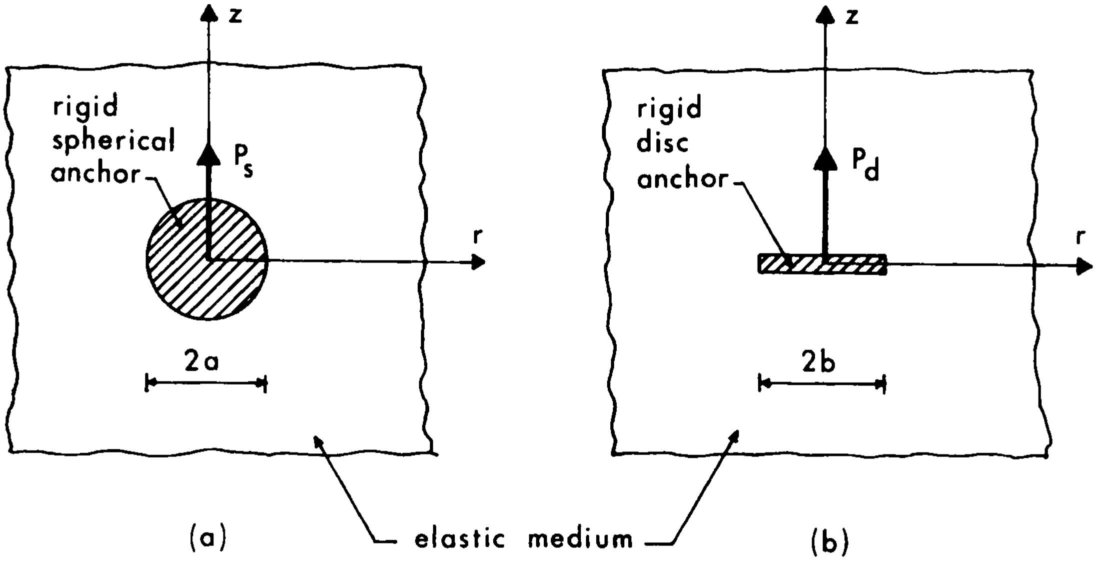

Figure 1. Spherical and disc rigid anchors

where δ is the rigid body translation of the anchor, a is the anchor radius and G is the linear elastic shear modulus of the incompressible medium. The result (3a) has also been obtained by Hill and Power[7] who have analysed the problem by an approximate method based on extremum principles.

(b) *Disc anchor.* A rigid circular disc of radius b and of infinite thickness is embedded in bonded contact with an incompressible elastic medium. It is subjected to a concentrated central load P_d. The solution to this problem occurs as a special case of the various generalized results given by Collins,[8] Hunter and Gamblen,[9] Kanwal and Sharma[10] and Selvadurai.[4] The appropriate load–deflection relationship is given by

$$P_d = 16Gb\delta \tag{3b}$$

THE VISCOUS FLOW PROBLEM

The determination of flow of an incompressible viscous fluid about a rigid impermeable body immersed therein requires the solution of the Navier–Stokes equations and the equation of continuity. these solutions are in turn subject to the condition that the velocity of flow coincides with that of the external boundary of the body at each of its points. The non-linear character of the Navier–Stokes equations renders the solution of this problem extremely difficult; as such, several plausible simplifying assumptions are made to render the problem more mathematically tractable. The oldest problem of this type is the so-called Stokes' flow problem in which the inertial effects are assumed to be negligible in comparison with those of viscosity, or more precisely, the Reynolds number (Re) of the flow is quite small. This situation is encountered when either the characteristic flow velocity or a typical body dimension is small, or when the viscosity of the fluid is large. Stokes[11] appears to have been the first to omit the inertial terms in the course of treating the steady motion of a rigid sphere in a viscous liquid.

The equations governing Stokes' flow or slow viscous flow of an incompressible fluid are then identical to (1) in which the displacement function $\psi(R, \theta)$ now corresponds to a stream function $\psi^*(R, \theta)$ and p corresponds to the hydrostatic stress p^*. Stokes' formulation can be used to analyse the viscous drag induced on impermeable rigid objects which are located in a stream of slow viscous flow of uniform velocity. Of particular interest here is the drag induced on rigid spherical and disc-shaped objects.

(a) *Rigid spherical object.* The viscous drag F_s exerted on a rigid sphere of radius a located in a stream of uniform velocity U and viscosity η is given by (Payne and Pell[12] and Langlois[13])

$$F_s = 6\pi\eta a U \tag{4a}$$

(b) *Rigid circular object.* The viscous drag F_d exerted on a circular disc of radius b situated in a slow viscous stream of uniform velocity is given by (Ray[14])

$$F_d = 16\eta b U \tag{4b}$$

In the derivation of results (4) it is explicitly assumed that no separation of flow takes place at the fluid–object interface.

THE ELASTIC-VISCOUS ANALOGY

The general similarity between the equations governing the two physically independent phenomena was first noticed by Rayleigh[15] and later adopted by Goodier,[16] Hill,[17] Prager,[18] Adkins,[19] Collins,[20] Richards[21] and others to examine various problems involving axially symmetric, plane stress and plate bending problems of infinitesimal elasticity theory. Selvadurai and Spencer[22] and Selvadurai[5,23] have shown that the displacement function techniques employed in the analysis of the linear problem in incompressible elasticity theory can be further extended to the analysis of incompressible deformations in materials exhibiting moderately large deformations.

Furthermore, we note that, for the illustrative examples considered here, the load–displacement relationship for the rigid anchor embedded in bonded contact with an incompressible elastic medium is identical in form to the viscous drag–velocity relationship for the same rigid geometric shape located in a uniform stream of flow. Similar correlations can be established between results for spheroidal rigid anchors embedded in imcompressible elastic media (Kanwal and Sharma[10] and Selvadurai[4]) and the results for spheroidal objects located in a uniform stream of fluid flow (Oberbeck,[24] Sampson[25] and Happel and Brenner[26]). A generalized proof of the analogy between the load–deflection relationship and the viscous drag–velocity relationship applicable for anchor regions of an arbitrary axisymmetric shape is beyond the scope of the present note. However, the slow viscous flow analogy for the determination of the undrained load–deflection characteristics of rigid anchors embedded in incompressible elastic soil media can be generalized to the following statement:

Proposition

The undrained load–deflection relationship for a deep rigid anchor with the shape of a solid of revolution, embedded in bonded contact with an unbounded isotropic incompressible elastic medium, can be directly recovered from the analogous problem of the drag induced on the same geometrical rigid shape with an impermeable surface, when placed in a stream of slow viscous flow of uniform velocity; the magnitudes of the uniform velocity, viscosity and drag force of the *latter problem correspond to the anchor displacement, the linear elastic shear modulus and the force applied (to the anchor), respectively.*

In Table I the load–displacement relationships for anchors of certain useful geometric shapes are listed. The load–displacement relationships for the oblate and prolate spheroidal

Table I. Load–deflection relationships for rigid anchors embedded in an incompressible elastic solid

Anchor shape	Load–deflection relationship
Sphere (Figure 1(a))	$P_s = 6\pi G a \delta$
Circular disc (Figure 1(b))	$P_d = 16 G b \delta$
Oblate spheroid (Figure 2(a))	$P = 8\pi G \delta c_0[\lambda_0 - (\lambda_0^2 - 1)\cot^{-1}\lambda_0]^{-1}$ $\left[c_0 = \{a_0^2 - b_0^2\}^{1/2};\ \lambda_0 = \left\{\left(\frac{a_0}{b_0}\right)^2 - 1\right\}^{-1/2}\right]$
Prolate spheroid (Figure 2(b))	$P = 8\pi G \delta c_p[(\tau_p^2 + 1)\coth^{-1}\tau_p - \tau_p]^{-1}$ $\left[c_p = \{a_p^2 - b_p^2\}^{1/2};\ \tau_p = \left\{1 - \left(\frac{b_p}{a_p}\right)^2\right\}^{1/2}\right]$

(P = load acting on the anchor; δ = anchor displacement)

anchors are directly obtained from the solutions for the analogous viscous flow problem. It should be noted that owing to the spatial symmetry of the deep rigid anchor problem, the load–displacement relationships obtained for the linear elastic problem are also valid for situations in which the surrounding soil medium experiences moderately large elastic deformations (see Selvadurai[5]).

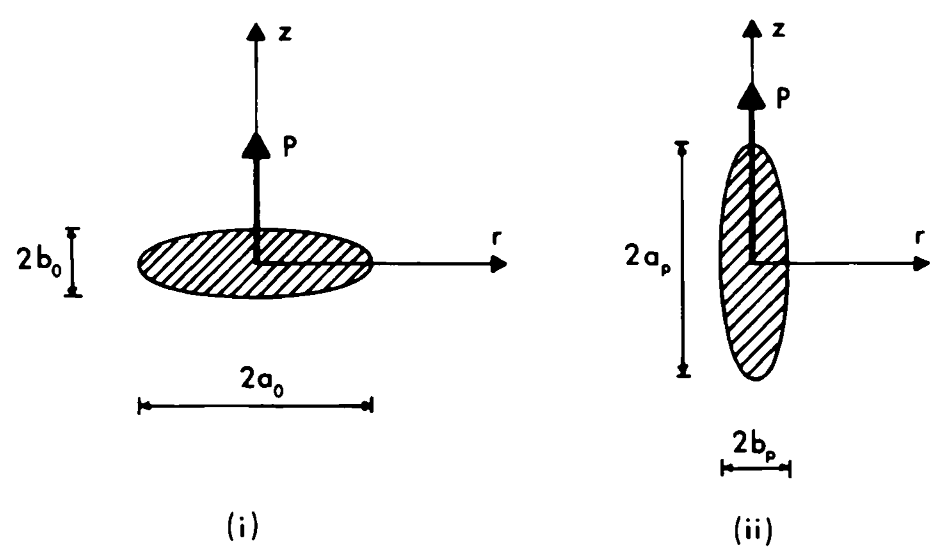

Figure 2. Rigid spheroidal anchors; (i) the oblate spheroidal anchor (ii) the prolate spheroidal anchor

CONTACT STRESS DISTRIBUTION AT THE ANCHOR–SOIL INTERFACE

The assumption of perfect continuity or bonding at the anchor–soil interface is central to the development of the *elastic–viscous analogy*. It is therefore of interest to investigate the extent to which this assumption may be realized in practice. Such an investigation is naturally concerned with the assessment of contact stresses which are developed at the anchor–soil interface due to the simultaneous action of stresses resulting from the self weight of the soil and the anchor load. For the purposes of illustration and for the sake of brevity we shall restrict our attention to the case of a deep rigid spherical anchor. It is assumed that the rigid spherical anchor is located at a large depth H ($\gg a$) from the free surface of a soil medium of unit weight γ. When the load, P, in the anchor is zero, the contact stresses which act at the interface are solely due to the self weight of the soil medium. A rigorous analysis of these contact stresses which takes into account the presence of the free surface is somewhat involved (see e.g. Datta[27]); however, when the anchor is located at a large depth (e.g. $H/a > 6$) the contact stresses at the interface can be estimated by making use of Goodier's solution[28] relating to a spherical inclusion contained in an isotropic infinite elastic medium which is subjected to arbitrary homogeneous states of stress at infinity. For the anchor problem posed here, these homogeneous stress states are assumed to approximately correspond to $\sigma_{zz} = \gamma H$ and $\sigma_{rr} = K_0 \gamma H$, where K_0 is the coefficient of earth pressure at rest. The contact stresses induced on the bonded spherical anchor interface due to these homogeneous stresses (and referred to a spherical polar coordinate system) take the relatively simple forms

$$\begin{aligned} \sigma_{RR}(a,\theta) &= \frac{\gamma H}{2}[(1-K_0)\{\tfrac{3}{2}+\tfrac{5}{2}\cos 2\theta\}+K_0] \\ \sigma_{R\theta}(a,\theta) &= \frac{\gamma H}{2}(1-K_0)\{-\tfrac{5}{2}\sin 2\theta\} \end{aligned} \tag{5}$$

where σ_{RR} and $\sigma_{R\theta}$ are, respectively, the normal and shear tractions on a surface R = constant. Similarly, the contact stress distribution due to the anchor load P is given by Selvadurai;[5] a superposition of these two states of contact stress gives the following expressions for the radial normal and tangential shear traction at the anchor interface $R = a$:

$$\frac{\sigma_{RR}(a,\theta)}{\gamma H} = K_0 + \frac{(1-K_0)}{2}\{\tfrac{3}{2}\cos 2\theta\} + \phi\frac{\cos\theta}{4} \tag{6a}$$

$$\frac{\sigma_{R\theta}(a,\theta)}{\gamma H} = \frac{(1-K_0)}{2}\{-\tfrac{5}{2}\sin 2\theta\} - \phi\frac{\sin\theta}{4} \tag{6b}$$

where

$$\phi = \frac{P}{\pi a^2 \gamma H} \tag{7}$$

The stress component of particular interest with regard to delamination separation at the anchor–soil interface is $\sigma_{RR}(a,\theta)$; the expression (6a) has therefore been evaluated for $K_0 = 1$ (this represents the correct value for the undranined case); $K_0 = 0$ (this value represents an extreme case for which the state of stress in the soil medium due to its self weight is purely one-dimensional, a result which is perhaps more consistent with a Poisson's ratio of zero rather than $\frac{1}{2}$) and for various values of ϕ. The computed results (Figure 3) indicate that the confining effects of the overburden stress with $K_0 = 1$ prevent the development of tensile normal traction at the anchor–soil interface for limited values of the anchor load P. The critical load at which

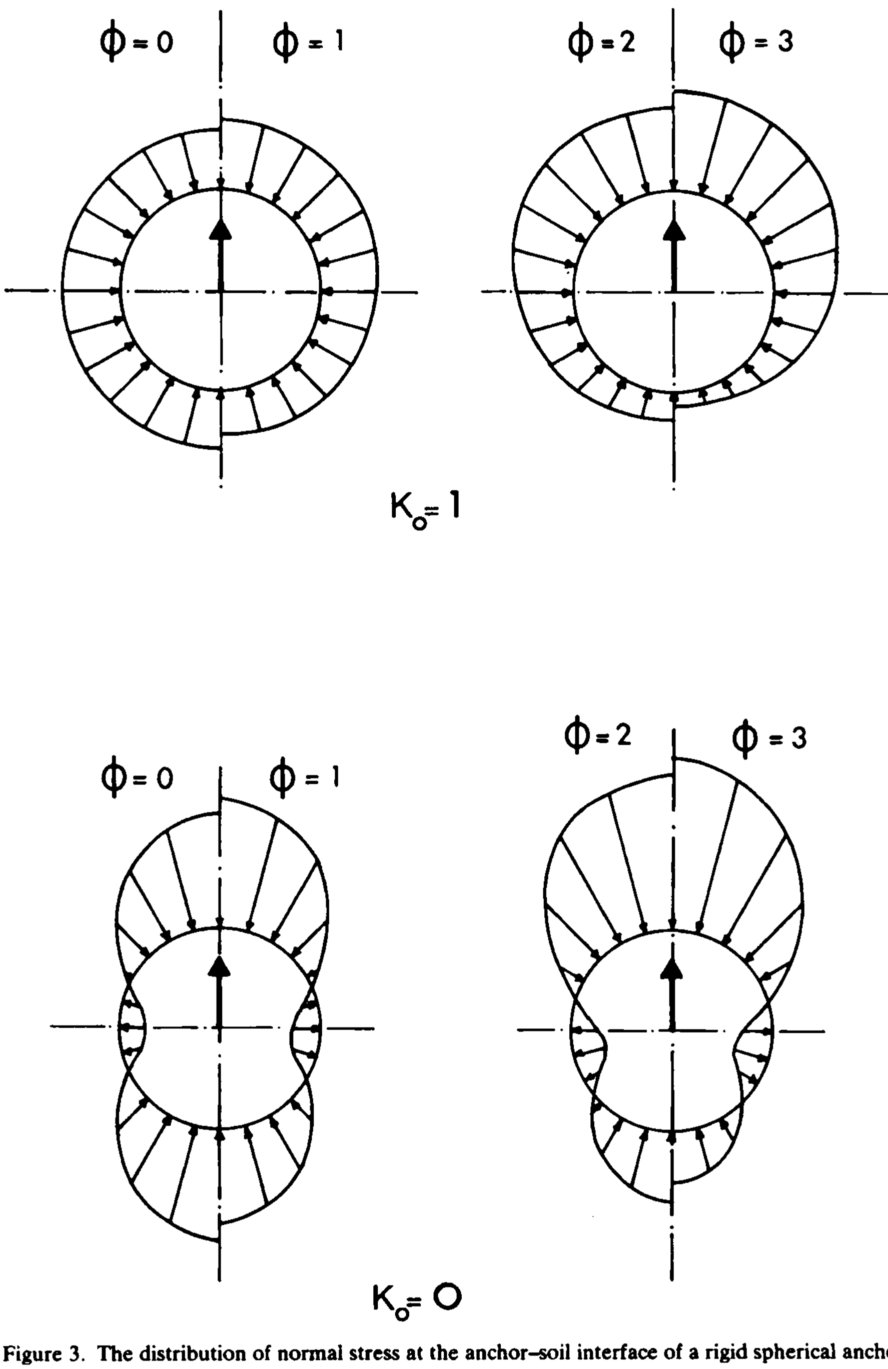

Figure 3. The distribution of normal stress at the anchor–soil interface of a rigid spherical anchor

tensile stresses occur at the interface in the undrained case given by $P = 4\pi a^2 \gamma H$. This load, which is approximately four times the weight of the soil above the spherical anchor, sets a realistic limit to the maximum safe load which can be supported by a spherical anchor buried at a large depth. Of related interest are the results provided by Hunter and Gamblen[9] for the case of a deep disc anchor located in an incompressible elastic soil medium; here, breakaway of the deep disc anchor occurs when $P \simeq 3\pi a^2 \gamma H$. We note that when $K_0 = 0$, tensile tractions develop at the anchor–soil interface even in the absence of the anchor load P. The presence of a three-dimensional state of confining stress is necessary, at least in the case of the spherical bonded anchor, to prevent breakaway at the anchor soil interface.

CONCLUSIONS

The technique outlined in this paper can be effectively employed to obtain the undrained load–deflection characteristics of deep rigid anchors embedded in bonded contact with an unbounded isotropic incompressible elastic medium.

In conclusion, it should be mentioned that solutions to other problems associated with deep anchors embedded in incompressible elastic media, namely, (i) the behaviour of groups of anchors, (ii) the frictional boundary conditions at the anchor–soil interface, (iii) the influence of neighbouring boundaries, etc., may have their viscous flow analogue already to be found in the literature on fluid mechanics. Such correlations, however, need further investigation. In this connection it should also be noted that experimental techniques associated with slow viscous fluid flow analysis can be used to great advantage to evaluate the load–deflection relationships for rigid anchors of complicated geometric shapes. The work described here forms part of a detailed theoretical study on the performance of a prolate spheroidal anchor embedded in an infinite medium. The load–deflection characteristics for such an anchor embedded in compressible elastic (see e.g. Selvadurai[4]), visco-elastic and consolidating media will be reported in subsequent papers.

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