Journal of Engineering Mathematics **34:** 403–416, 1998.

Some generalized anti-plane strain problems for an inhomogeneous elastic halfspace

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Received 20 January 1998; accepted in revised form 6 July 1998

Abstract. This paper develops a generalized formulation of the theory of anti-plane deformations of a linear elastic solid, the Lamé constants of which depend on a single spatial variable. The generalized theory is applied to the study of a generalized screw dislocation, a semi-infinite crack and a finite crack located in a halfspace region where the elastic inhomogeneity is depth dependent. The problems examined in this paper are relevant to the modelling of cracks located in inhomogeneous geological materials and the study of surface defects associated with functionally graded materials.

Keywords: inhomogeneous elastic media, anti-plane problems, semi-infinite crack, finite crack, screw dislocation.

1. Introduction

Materials which exhibit spatial variability in their mechanical properties are classified as inhomogeneous materials. Such material inhomogeneities can occur either naturally or could be introduced artificially. For example, with geomaterials, depositional effects and gravity stresses can introduce depth dependent inhomogeneities in the constitutive responses. Even a material such as wood which is often regarded as an orthotropic material is composed of an inhomogeneous material in the scale of a growth ring. With composites, bonded solids and functionally graded materials, inhomogeneities can be deliberately introduced to achieve certain functional requirements [1, 2]. The study of inhomogeneous elastic materials has been an important aspect of mechanics of solids over the past half-century and the renewal of interest largely stems from potential applications of the theories to geomechanics and advanced materials engineering. Accounts of developments in the applications of the theory of elasticity of inhomogeneous materials are given in [3–6].

Although the formal development of the theory of elasticity for inhomogeneous materials can be approached by assuming that the elasticity parameters are functions of the three spatial variables, little progress can be made in the application of the resulting equations to problems of scientific and engineering interest. Therefore attention is usually restricted to the consideration of problems where the elastic inhomogeneity is a function of only a single spatial variable. In the context of geomechanics, the elastic inhomogeneity which is dependent on a single spatial variable has been extensively applied to examine geomechanics problems where the elastic properties vary with depth. Several types of depth-dependent variations have been examined in the literature; these include half-space problems where the linear elastic shear

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modulus varies either linearly or exponentially with depth. References to these applications are given in [6]. With functionally graded materials and bonded solids, elastic inhomogeneities which depend on a single spatial variable can arise due to specialized treatments of the material. For example, both continuous and discontinuous variations in the elastic properties of materials can be introduced by surface treatments (*e.g.* ion plating, plasma spray coating etc.) and in the case of bonded solids, elastic inhomogeneities can be introduced as a result of diffusion of the adherent in the vicinity of the bonded interface [7].

Even with the simplifications provided by restricting the elastic inhomogeneity to a single spatial variable, attention is rarely focussed on the consideration of problems where both Lamé constants vary with the spatial variable. In the majority of cases, attention is usually restricted to situations where the linear elastic shear modulus exhibits a spatial variation [8–10]. This paper focuses on the application of the theory of elasticity for an inhomogeneous medium where both the Lamé elastic constants λ and μ are arbitrary functions of a single spatial variable. In particular, we examine the application of the resulting theory to the study of certain anti-plane strain problems occupying all or part of an inhomogeneous elastic half-space. The methodology used in the formulation and the solution of the resulting equations is derived from the studies originally developed by Rogers and Spencer [11–13] for the study of three-dimensional problems dealing with laminated plates. The paper presents a generalized formulation of the anti-plane strain problem where the Lamé constants are arbitrary functions of a single spatial variable. The paper develops exact closed-form solutions to problems involving a generalized screw dislocation, a semi-infinite crack undergoing anti-plane shear and a crack of finite length in a uniform field of anti-plane shear.

2. Formulation

We consider a linearly elastic, isotropic, inhomogeneous material occupying all or part of the half-space $z > 0$, where (x, y, z) are rectangular, cartesian coordinates. The mechanical inhomogeneity is such that the Lamé elastic constants λ and μ, or equivalently Young's modulus E and Poisson's ratio ν, are specified functions of z. This dependence on z need not be continuous, and is subject only to the usual requirement for positive-definiteness of the strain energy. Referred to (x, y, z) coordinates the components of displacement are denoted by u, v and w, the components of infinitesimal strain are, typically

$$e_{xx} = u,_x\,, \qquad e_{xy} = \tfrac{1}{2}(u,_y + v,_x) \tag{2.1}$$

and σ_{xx}, σ_{xy} etc. denote the stress components. Then the linearly elastic stress-strain relations can be expressed as

$$\begin{aligned}
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} &= \lambda(z)(u,_x + v,_y + w,_z) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\mu(z) \begin{bmatrix} u,_x \\ v,_y \\ w,_z \end{bmatrix}, \\
\begin{bmatrix} \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} &= \mu(z) \begin{bmatrix} v,_z + w,_y \\ w,_x + u,_z \\ u,_y + v,_x \end{bmatrix},
\end{aligned} \tag{2.2}$$

(where commas denote differentiation with respect to the indicated variables) or, equivalently, as

$$\begin{bmatrix} u,_x \\ v,_y \\ w,_z \end{bmatrix} = \frac{1+\nu(z)}{E(z)} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} - \frac{\nu(z)}{E(z)}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$
$$\begin{bmatrix} v,_z + w,_y \\ w,_x + u,_z \\ u,_y + v,_x \end{bmatrix} = \frac{2(1+\nu(z))}{E(z)} \begin{bmatrix} \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}. \tag{2.3}$$

The equations of equilibrium, with negligible body forces, are

$$\begin{aligned} &\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} = 0, \\ &\sigma_{xy,x} + \sigma_{yy,y} + \sigma_{yz,z} = 0, \\ &\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} = 0. \end{aligned} \tag{2.4}$$

Further, the strain components must satisfy the strain compatibility equations

$$\begin{aligned} &K_x \equiv 2e_{yz,yz} - e_{yy,zz} - e_{zz,yy} = 0, \\ &L_x \equiv e_{xx,yz} + e_{yz,xx} - e_{zx,xy} - e_{xy,zx} = 0 \end{aligned} \tag{2.5}$$

and the four equations derived from (2.5) by cyclic permutation of x, y and z. In terms of the stress components, (2.5) are

$$\begin{aligned} K_x &= \left\{\frac{2(1+\nu)\sigma_{yz}}{E}\right\},_{yz} - \left\{\frac{\sigma_{yy} - \nu(\sigma_{xx}+\sigma_{zz})}{E}\right\},_{zz} \\ &\quad - \left\{\frac{\sigma_{zz} - \nu(\sigma_{xx}+\sigma_{yy})}{E}\right\},_{yy} = 0, \\ L_x &= \left\{\frac{\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})}{E}\right\},_{yz} + \left\{\frac{(1+\nu)\sigma_{yz}}{E}\right\},_{xx} \\ &\quad - \left\{\frac{(1+\nu)\sigma_{zx}}{E}\right\},_{xy} - \left\{\frac{(1+\nu)\sigma_{xy}}{E}\right\},_{zx} = 0. \end{aligned} \tag{2.6}$$

It was shown in [11–16] that these equations have three-dimensional elasticity solutions, for arbitrary dependence of λ and μ on z, of the form

$$\begin{aligned} u(x,y,z) &= \overline{u}(x,y) + F(z)\Delta(x,y),_x + A(z)\overline{w}(x,y),_x + B(z)\nabla^2\overline{w}(x,y),_x, \\ v(x,y,z) &= \overline{v}(x,y) + F(z)\Delta(x,y),_y + A(z)\overline{w}(x,y),_y + B(z)\nabla^2\overline{w}(x,y),_y, \\ w(x,y,z) &= \overline{w}(x,y) + G(z)\Delta(x,y) + C(z)\nabla^2\overline{w}(x,y), \end{aligned} \tag{2.7}$$

where ∇^2 is the two-dimensional Laplacian operator, and $\overline{u}(x,y)$, $\overline{v}(x,y)$, $\overline{w}(x,y)$ satisfy the classical thin-plate (two-dimensional) equations

$$\begin{aligned} &\nabla^4\overline{w}(x,y) = 0; \qquad \kappa_1\Delta(x,y),_x - \Omega(x,y),_y + \kappa_2\nabla^2\overline{w}(x,y),_x = 0, \\ &\kappa_1\Delta(x,y),_y + \Omega(x,y),_x + \kappa_2\nabla^2\overline{w}(x,y),_y = 0, \end{aligned} \tag{2.8}$$

where

$$\Delta(x, y) = \overline{u}(x, y),_x + \overline{v}(x, y),_y , \qquad \Omega(x, y) = \overline{v}(x, y),_x - \overline{u}(x, y),_y .$$

It follows from (2.8) that

$$\nabla^2 \Delta(x, y) = 0, \qquad \nabla^2 \Omega(x, y) = 0.$$

Equations (2.8) are identical in form with the equations of classical thin-plate laminate theory (see for example Jones [17], Whitney [18]). However, here the field equations are satisfied for any values of the constants κ_1 and κ_2, whereas in classical laminate theory these constants take specific values in terms of weighted thorough-plate averages of $\lambda(z)$ and $\mu(z)$.

The constant κ_2 characterizes the coupling between bending and stretching deformations that arises in an inhomogeneous body. The coefficients $A(z)$, $B(z)$, $C(z)$, $F(z)$ and $G(z)$ can be obtained from the solution of the following

$$\begin{aligned}
&\{\mu(1 + A')\}' = 0,\\
&2\mu + \lambda(1 + G') + \left\{\mu(F' + G)\right\}' = \mu\kappa_1,\\
&\lambda(A + C') + 2\mu A + \left\{\mu(B' + C)\right\}' = \mu\kappa_2,\\
&\{\lambda(1 + G') + 2\mu G'\}' = 0,\\
&\mu(1 + A') + \left\{\lambda A + (\lambda + 2\mu)C'\right\}' = 0,
\end{aligned} \tag{2.9}$$

where primes denote differentiation with respect to z. When $\lambda(z)$ and $\mu(z)$ are specified, all five coefficients can be determined by consecutive quadratures. The integration constants that arise, and the constants κ_1 and κ_2, can be chosen to satisfy certain boundary conditions. For example, if the surface $z = 0$ of the half-space $z \geqslant 0$ is traction-free, so that

$$\sigma_{xz} = 0, \qquad \sigma_{yz} = 0, \qquad \sigma_{zz} = 0 \qquad (z = 0),$$

then it follows from (2.2) and (2.7) that

$$\begin{aligned}
&\lambda + (\lambda + 2\mu)G' = 0, \qquad \lambda A + (\lambda + 2\mu)C' = 0,\\
&F' + G = 0, \qquad A' + 1 = 0, \qquad B' + C = 0,
\end{aligned} \tag{2.10}$$

all at $z = 0$. It follows from $(2.9)_{1,4,5}$ that

$$\begin{aligned}
&A = -z + A_0, \qquad \lambda(z) + \{\lambda(z) + 2\mu(z)\}\, G'(z) = 0,\\
&\lambda(z)A(z) + \{\lambda(z) + 2\mu(z)\}C'(z) = 0, \quad (z \geqslant 0),
\end{aligned} \tag{2.11}$$

where A_0 is constant. Then from $(2.9)_{2,3}$ and (2.10), we have

$$\mu(F' + G) = \int_0^z \{\mu\kappa_1 - (\lambda + 2\mu) - \lambda G'\}\, \mathrm{d}z,$$

$$\mu(B' + C) = \int_0^z \{\mu\kappa_2 - (\lambda + 2\mu)A - \lambda C'\}\, \mathrm{d}z,$$

and hence, from (2.10)

$$\begin{aligned} \mu(F'+G) &= \kappa_1\int_0^z \mu\,\mathrm{d}z - 4\int_0^z \frac{\mu(\lambda+\mu)}{\lambda+2\mu}\,\mathrm{d}z, \\ \mu(B'+C) &= \kappa_2\int_0^z \mu\,\mathrm{d}z - 4\int_0^z \frac{\mu(\lambda+\mu)A(z)}{\lambda+2\mu}\,\mathrm{d}z. \end{aligned} \tag{2.12}$$

In the case of a plate of uniform thickness with traction-free surfaces, it was shown in [15] that it follows from (2.12) that the constants κ_1 and κ_2 assume the values that they take, in Equations (2.8), in classical laminate theory.

In the case of a half-space $z \geqslant 0$, the implications of (2.11) depend on the behaviour of the stress as $z \to \infty$. If, for example, σ_{xz} and σ_{yz} are bounded as $z \to \infty$, then $F' + G$ and $B' + C$ are bounded as $z \to \infty$. If also μ and $(\lambda + 2\mu)$ remain finite and positive as $z \to \infty$ then the constants κ_1 and κ_2 are given by

$$\kappa_1 = \lim_{z\to\infty}\left\{4\int_0^z \frac{\mu(s)\,\{\lambda(s)+\mu(s)\}}{\lambda(s)+2\mu(s)}\,\mathrm{d}s \Big/ \int_0^z \mu(s)\,\mathrm{d}s\right\},$$

$$\kappa_2 = \lim_{z\to\infty}\left\{-4\int_0^z \frac{s\mu(s)\,\{\lambda(s)+\mu(s)\}}{\lambda(s)+2\mu(s)}\,\mathrm{d}s \Big/ \int_0^z \mu(s)\,\mathrm{d}s\right\},$$

provided the limits exist. In particular, if λ and μ tend to finite limits as $z \to \infty$, then κ_1 is finite but κ_2 is unbounded. It then follows from (2.8) that $\nabla^2\overline{w}(x, y) = 0$, which is the case considered in Sections 3–6.

An alternative, and essentially equivalent, formulation given in [16] expresses the solution in terms of a 'stress function' $\chi(x, y)$ as follows

$$\begin{aligned} \sigma_{xx} &= P(z)\chi_{,yy} + Q(z)\nabla^2\chi + R(z)\nabla^2\chi_{,yy}, \\ \sigma_{yy} &= P(z)\chi_{,xx} + Q(z)\nabla^2\chi + R(z)\nabla^2\chi_{,xx}, \\ \sigma_{xy} &= -P(z)\chi_{,xy} - R(z)\nabla^2\chi_{,xy}, \\ \sigma_{zz} &= M(z)\nabla^2\chi, \\ \sigma_{xz} &= K(z)\chi_{,x} + J(z)\nabla^2\chi_{,x}, \\ \sigma_{yz} &= K(z)\chi_{,y} + J(z)\nabla^2\chi_{,y}. \end{aligned} \tag{2.13}$$

These stress components satisfy the equilibrium, stress-strain and strain-compatibility equations provided that χ is a biharmonic function

$$\nabla^4\chi = 0, \tag{2.14}$$

and that the coefficients $P(z), Q(z), R(z), M(z), J(z), K(z)$ satisfy

$$\begin{aligned} & Q(z) + J'(z) = 0, \qquad K(z) + M'(z) = 0, \qquad K'(z) = 0, \\ & \left\{\frac{P + Q(1-\nu) - \nu M}{E}\right\}' = 0, \\ & \left\{\frac{R(1+\nu)}{E}\right\}'' + \left\{\frac{2J(1+\nu)}{E}\right\}' + \left\{\frac{\nu(P+2Q) - M}{E}\right\} = 0, \\ & \left\{\frac{P(1+\nu)}{E}\right\}'' + \left\{\frac{2K(1+\nu)}{E}\right\}' = 0. \end{aligned} \tag{2.15}$$

For this system also the six coefficients can be evaluated consecutively by quadratures, and the integration constants chosen to satisfy appropriate boundary conditions.

Equilibrium, and continuity of displacement, require that $A(z)$, $B(z)$, $C(z)$, $F(z)$, $G(z)$, $M(z)$, $K(z)$ and $J(z)$ are continuous functions of z, even at points where λ, μ, E and ν are discontinuous (as in a laminated or layered medium) but $P(z)$, $Q(z)$ and $R(z)$ may be discontinuous at such points. The appropriate jump conditions are readily established.

In many problems it is convenient to employ cylindrical polar coordinates (r, θ, z) rather than cartesian coordinates. In terms of (r, θ, z) (2.7) becomes

$$\begin{aligned} u_r(r,\theta,z) &= \overline{u}_r(r,\theta) + F(z)\Delta(r,\theta)_{,r} + A(z)\overline{w}(r,\theta)_{,r} + B(z)(\nabla^2\overline{w}(r,\theta))_{,r}\,, \\ u_\theta(r,\theta,z) &= \overline{u}_\theta(r,\theta) + r^{-1}F(z)\Delta(r,\theta)_{,\theta} \\ &\qquad + r^{-1}A(z)\overline{w}(r,\theta)_{,\theta} + r^{-1}B(z)(\nabla^2\overline{w}(r,\theta))_{,\theta}\,, \\ w(r,\theta,z) &= \overline{w}(r,\theta) + G(z)\Delta(r,\theta) + C(z)\nabla^2\Delta(r,\theta), \end{aligned} \tag{2.16}$$

where u_r, u_θ are r and θ components of displacement respectively, and now

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{2.17}$$

$$\Delta(r,\theta) = \overline{u}_{r,r} + r^{-1}\overline{u}_r + r^{-1}\overline{u}_{\theta,\theta}, \qquad \Omega(r,\theta) = \overline{u}_{\theta,r} - r^{-1}\overline{u}_{r,\theta} - r^{-1}\overline{u}_\theta, \tag{2.18}$$

whilst $\overline{u}_r(r,\theta)$, $\overline{u}_\theta(r,\theta)$, $\overline{w}(r,\theta)$ are solutions of the thin-plate equations

$$\begin{aligned} &\kappa_1\Delta(r,\theta)_{,r} - r^{-1}\Omega(r,\theta)_{,\theta} + \kappa_2(\nabla^2\overline{w}(r,\theta))_{,r} = 0, \\ &r^{-1}\kappa_1\Delta(r,\theta)_{,\theta} + \Omega(r,\theta)_{,r} + r^{-1}\kappa_2(\nabla^2\overline{w}(r,\theta))_{,\theta} = 0, \\ &\nabla^4\overline{w}(r,\theta) = 0, \end{aligned} \tag{2.19}$$

and κ_1, κ_2, $A(z)$, $B(z)$, $C(z)$, $F(z)$ and $G(z)$ are as before.

Alternatively, the stress function formulation in cylindrical polar coordinates is

$$\begin{aligned} \sigma_{rr} &= P(z)\{r^{-2}\chi_{,\theta\theta} + r^{-1}\chi_{,r}\} + Q(z)\nabla^2\chi + R(z)\{r^{-2}(\nabla^2\chi)_{,\theta\theta} + r^{-1}(\nabla^2\chi)_{,r}\}, \\ \sigma_{\theta\theta} &= P(z)\chi_{,rr} + Q(z)\nabla^2\chi + R(z)(\nabla^2\chi)_{,rr}\,, \\ \sigma_{r\theta} &= -P(z)(r^{-1}\chi_{,\theta})_{,r} - R(z)\{r^{-1}(\nabla^2\chi)_{,\theta}\}_{,r}\,, \\ \sigma_{zz} &= M(z)\nabla^2\chi, \qquad \sigma_{rz} = K(z)\chi_{,r} + J(z)(\nabla^2\chi)_{,r}\,, \\ \sigma_{\theta z} &= r^{-1}K(z)\chi_{,\theta} + r^{-1}J(z)(\nabla^2\chi)_{,\theta}\,, \end{aligned} \tag{2.20}$$

where $\chi(r,\theta)$ is any biharmonic function

$$\nabla^4\chi(r,\theta) = 0. \tag{2.21}$$

3. Generalized anti-plane strain

In this section we consider a sub-class of the solutions described in Section 2, namely deformations in which $\overline{w}$ and χ are harmonic functions (and consequently, of course, biharmonic), and $\overline{u}$ and $\overline{v}$ are zero, thus

$$\nabla^2\overline{w}(x,y)=0, \qquad \nabla^2\chi(x,y)=0, \qquad \overline{u}(x,y)=0, \qquad \overline{v}(x,y)=0. \tag{3.1}$$

It then follows from (2.7) that

$$\begin{aligned} u(x,y,z) &= A(z)\overline{w}(x,y)_{,x}\,, \\ v(x,y,z) &= A(z)\overline{w}(x,y)_{,y}\,, \\ w(x,y,z) &= \overline{w}(x,y), \end{aligned} \tag{3.2}$$

and from (2.13) that

$$\begin{aligned} \sigma_{xx} &= P(z)\chi_{,yy}\,, \qquad \sigma_{yy}=P(z)\chi_{,xx}\,, \qquad \sigma_{xy}=-P(z)\chi_{,xy}\,, \\ \sigma_{zz} &= 0, \qquad \sigma_{xz}=K(z)\chi_{,x}\,, \qquad \sigma_{yz}=K(z)\chi_{,y}\,, \end{aligned} \tag{3.3}$$

with

$$K'(z)=0, \qquad \left(\frac{P}{\mu}\right)''+\left(\frac{2K}{\mu}\right)'=0, \qquad \left\{\mu(1+A')\right\}'=0, \tag{3.4}$$

where we have used the relation

$$\mu=\frac{E}{2(1+\nu)}.$$

By integrating the first and third of (3.4), we have

$$K(z)=K_0, \qquad A(z)=A_0+\int_0^z\left(\frac{\mu_0}{\mu}-1\right)\mathrm{d}z, \tag{3.5}$$

where K_0, μ_0 and A_0 are constants of integration. For consistency between (3.2) and (3.3), it is necessary that

$$K_0\chi=\mu_0\overline{w}, \qquad P(z)\chi=-2\mu A(z)\overline{w}. \tag{3.6}$$

The choice $K_0=1$ can be made without loss of generality. With this choice

$$\chi=\mu_0\overline{w}, \qquad P(z)=-\frac{2\mu}{\mu_0}A(z)=-\frac{2\mu}{\mu_0}\left\{A_0+\int_0^z\left(\frac{\mu_0}{\mu}-1\right)\mathrm{d}z\right\}. \tag{3.7}$$

It is easily confirmed that this expression for $P(z)$ satisfies the second of (3.4).

To summarize, for this class of deformations, the displacement is given by (3.2) and the stress by

$$\begin{aligned} \sigma_{xx} &= 2\mu A(z)\overline{w}_{,xx}\,, \qquad \sigma_{yy}=2\mu A(z)\overline{w}_{,yy}\,, \\ \sigma_{xy} &= 2\mu A(z)\overline{w}_{,xy}\,, \qquad \sigma_{zz}=0, \\ \sigma_{xz} &= \mu_0\overline{w}_{,x}\,, \qquad \sigma_{yz}=\mu_0\overline{w}_{,y}\,, \end{aligned} \tag{3.8}$$

where $\overline{w}(x, y)$ is any harmonic function and $A(z)$ is given by (3.5).

The standard anti-plane strain theory for a homogeneous material with constant shear modulus, is recovered by setting $A_0 = 0$ and $\mu = \mu_0$, which results in $A(z) = 0$ and leaves σ_{xz} and σ_{yz} as the only nonzero stress components.

In many problems of geotechnical interest it is reasonable to assume that μ tends to a constant value as $z \to \infty$. Then it is natural to take μ_0 to be this limiting value. Further, the choice

$$A_0 = -\int_0^\infty \left(\frac{\mu_0}{\mu} - 1\right) \mathrm{d}z$$

(provided that the integral is finite) gives $A(z) \to 0$ as $z \to \infty$, and the generalized anti-plane strain solutions tend asymptotically to the corresponding homogeneous anti-plane strain solutions as $z \to \infty$. For example, if it is supposed that μ can be described with sufficient accuracy by an expression of the form

$$\frac{1}{\mu} = \frac{1}{\mu_0} + \left(\frac{1}{\mu_1} - \frac{1}{\mu_0}\right) \mathrm{e}^{-\gamma z}, \tag{3.9}$$

so that $\mu = \mu_1$ at $z = 0$ and $\mu \to \mu_0$ as $z \to \infty$, and γ is a constant, then

$$A_0 = -\frac{1}{\gamma}\left(\frac{\mu_0}{\mu_1} - 1\right)$$

and

$$A(z) = -\frac{1}{\gamma}\left(\frac{\mu_0}{\mu_1} - 1\right) \mathrm{e}^{-\gamma z}. \tag{3.10}$$

In cylindrical polar coordinates (3.8) are

$$\begin{aligned}
&u_r(r, \theta, z) = A(z)\overline{w}(r, \theta)_{,r}, \qquad u_\theta(r, \theta, z) = A(z) r^{-1}\overline{w}(r, \theta)_{,\theta} \\
&w(r, \theta, z) = \overline{w}(r, \theta), \\
&\sigma_{rr} = 2\mu A(z)\overline{w}(r, \theta)_{,rr}, \qquad \sigma_{\theta\theta} = 2\mu A(z)\left\{r^{-1}\overline{w}(r, \theta)_{,r} + r^{-2}\overline{w}(r, \theta)_{,\theta\theta}\right\}, \\
&\sigma_{r\theta} = 2\mu A(z)(r^{-1}\overline{w}(r, \theta)_{,\theta})_{,r}, \qquad \sigma_{zz} = 0, \\
&\sigma_{rz} = \mu_0\overline{w}(r, \theta)_{,r}, \qquad \sigma_{\theta z} = \mu_0 r^{-1}\overline{w}(r, \theta)_{,\theta}.
\end{aligned} \tag{3.11}$$

4. Generalized screw-dislocation

The mathematical theory of dislocations has been successfully applied to examine a variety of problems in materials science and geophysics. References to applications in these areas are given by Nabarro [19], Eshelby [20], Hirth and Lothe [21] and Mura [22]. The article by Eshelby [20], in particular, gives further references to geophysical applications of the screw dislocation. The displacement

$$\overline{w} = \theta(\alpha + \beta(\log r)), \qquad \overline{u}_r = 0, \qquad \overline{u}_\theta = 0, \tag{4.1}$$

(where α and β are constants) describes a screw dislocation in anti-plane strain theory for homogeneous materials. If θ is specified to lie in the range $0 \leqslant \theta < 2\pi$, then (4.1) implies a discontinuity in w across the initial line $\theta = 0$.

For an inhomogeneous material with $\mu = \mu(z)$, the expressions (4.1) generate the displacement and stress fields

$$\begin{aligned}
&u_r = A(z)\frac{\beta\theta}{r}, \qquad u_\theta = A(z)\frac{\alpha + \beta\log r}{r}, \qquad w = \theta(\alpha + \beta\log r),\\
&\sigma_{rr} = -2\mu A(z)\frac{\beta\theta}{r^2}, \qquad \sigma_{\theta\theta} = 2\mu A(z)\frac{\beta\theta}{r^2}, \qquad \sigma_{zz} = 0,\\
&\sigma_{r\theta} = 2\mu A(z)\frac{(\beta - \alpha) - \beta\log r}{r^2}, \qquad \sigma_{rz} = \mu_0\frac{\beta\theta}{r}, \qquad \sigma_{\theta z} = \mu_0\frac{\alpha + \beta\log r}{r}.
\end{aligned} \tag{4.2}$$

Thus, unlike the case of a homogeneous material, the discontinuity in $\overline{w}$ generates a discontinuity in u_r (*i.e.* an edge dislocation) at the initial line, and there is an essential coupling between the edge and screw dislocations. Furthermore, there are jumps in σ_{rr} and $\sigma_{\theta\theta}$ at $\theta = 0$ that are not present when the material is homogeneous. Also the singularities at the axis $r = 0$ are of higher order than they are in a homogeneous material. As is usual in the theory of crystal dislocations, it is necessary to exclude a region in the neighbourhood of $r = 0$ to ensure that the stress and displacement remain finite (Figure 1).

5. A semi-infinite crack

For a homogeneous material, the displacement

$$\overline{w} = Kr^{1/2}\sin\tfrac{1}{2}\theta, \qquad \overline{u}_r = 0, \qquad \overline{u}_\theta = 0 \tag{5.1}$$

describes the displacement due to a crack lying in the vertical plane $y = 0$, extending from $x = -\infty$ to its tip on the line $x = y = 0$, and subject to shear in the z-direction on the plane $y = 0$. The only nonzero stress components associated with this deformation are

$$\sigma_{rz} = \tfrac{1}{2}\mu K r^{-1/2}\sin\tfrac{1}{2}\theta, \qquad \sigma_{\theta z} = \tfrac{1}{2}\mu K r^{-1/2}\cos\tfrac{1}{2}\theta. \tag{5.2}$$

Thus the crack surfaces $\theta = \pm\pi$ are traction-free, $\sigma_{\theta z}$ has the characteristic inverse square root tip singularity ahead of the crack on $\theta = 0$, and the crack faces $\theta = \pm\pi$ are displaced in the z-direction relatively to one another. In fracture mechanics terminology, this represents a 'Mode III' crack.

For an inhomogeneous elastic material, (5.1) generates the displacement and stress fields

$$\begin{aligned}
&u_r = \tfrac{1}{2}KA(z)r^{-1/2}\sin\tfrac{1}{2}\theta, \qquad u_\theta = \tfrac{1}{2}KA(z)r^{-1/2}\cos\tfrac{1}{2}\theta,\\
&w = Kr^{1/2}\sin\tfrac{1}{2}\theta,\\
&\sigma_{rr} = -\tfrac{1}{2}\mu KA(z)r^{-3/2}\sin\tfrac{1}{2}\theta, \qquad \sigma_{\theta\theta} = \tfrac{1}{2}\mu KA(z)r^{-3/2}\sin\tfrac{1}{2}\theta,\\
&\sigma_{r\theta} = -\tfrac{1}{2}\mu KA(z)r^{-3/2}\cos\tfrac{1}{2}\theta, \qquad \sigma_{zz} = 0,\\
&\sigma_{rz} = \tfrac{1}{2}\mu_0 K r^{-1/2}\sin\tfrac{1}{2}\theta, \qquad \sigma_{\theta z} = \tfrac{1}{2}\mu_0 K r^{-1/2}\cos\tfrac{1}{2}\theta.
\end{aligned} \tag{5.3}$$

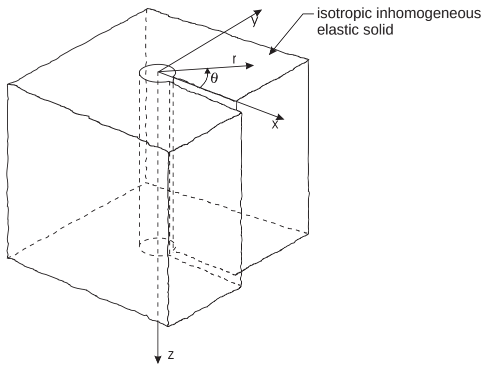

Figure 1. Generalized screw dislocation in an inhomogeneous elastic medium.

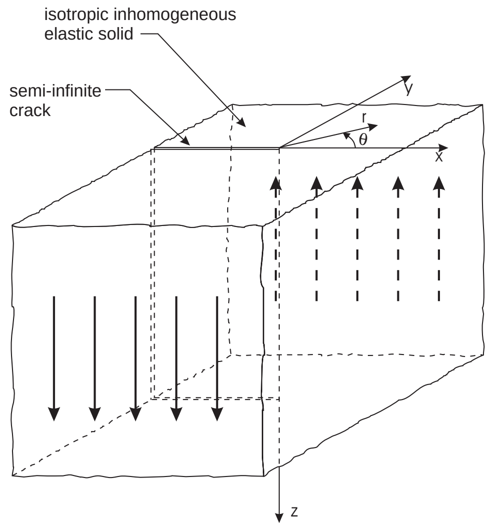

Figure 2. Semi-infinite crack in an inhomogeneous elastic solid subjected to shear.

In this problem also there is a coupling between in-plane and anti-plane deformation, because the expression for u_r shows that, for $A(z) \neq 0$, there is a horizontal shear on the crack face associated with the imposed vertical shear. Also σ_{rr} and $\sigma_{\theta\theta}$ are discontinuous at $\theta = \pm\pi$. Furthermore the singularities as $r \to 0$ are intensified compared to those that arise in homogeneous material, so for physically meaningful solutions it is necessary to exclude a neighbourhood of $r = 0$. In the geomechanical context this may not be too serious a restriction because the concept of a sharp crack is not a very realistic one in a geological material. The theory does seem to suggest a concentration of strain energy in the vicinity of $r = 0$, and depending on $A(z)$ (and hence on the degree of inhomogeneity), which may have geophysical implications.

6. Finite-length crack in uniform anti-plane shear field

Several investigators including Kassir [23], Erdogan *et al*. [24], Craster and Atkinson [25], Clements *et al*. [26], Ozturk and Erdogan [27,28] and Selvadurai and Lan [29] have examined problems related to both mode III and mode I behaviour of cracks in inhomogeneous elastic media. In these investigations the elastic inhomogeneities are restricted to very simple forms for the variations of μ as a function of a single spatial variable.

We now consider a vertical crack of length $2a$, lying in the plane $y = 0$ from $x = a$ to $x = -a$, in a field of uniform anti-plane shear. For this and many other problems the solution is most conveniently expressed in terms of a complex variable ζ as

$$\overline{w}(x, y) = \operatorname{Re} f(\zeta), \qquad \zeta = x + iy. \tag{6.1}$$

Then, for a homogeneous material

$$w(x, y) = \operatorname{Re} f(\zeta), \qquad u = 0, \qquad v = 0, \tag{6.2}$$

and the only nonzero stress components are given by

$$\sigma_{xz} - i\sigma_{yz} = \mu f'(\zeta). \tag{6.3}$$

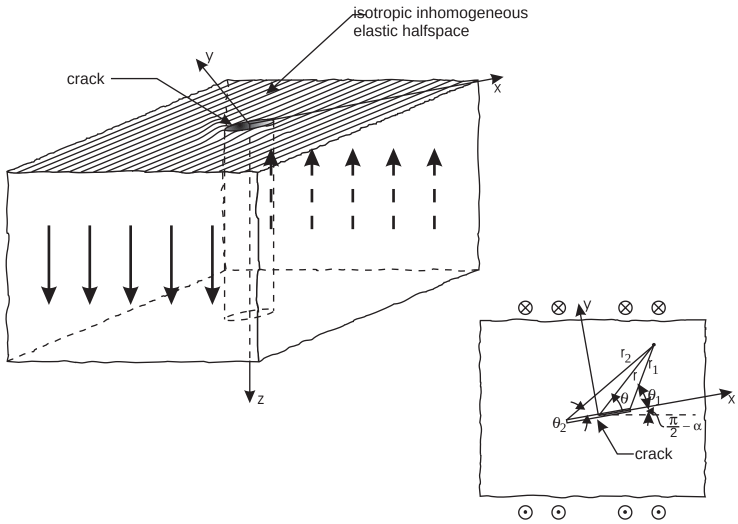

Figure 3. A finite length crack in a uniform anti-plane shear field.

For an inhomogeneous material, from (3.8)

$$\begin{aligned} &u(x,y,z) - iv(x,y,z) = A(z)f'(\zeta), \qquad w(x,y) = \operatorname{Re} f(\zeta),\\ &\sigma_{xx}+\sigma_{yy}=0, \qquad \sigma_{xx}-\sigma_{yy}-2i\sigma_{xy} = 4\mu A(z) f''(\zeta),\\ &\sigma_{xz} - i\sigma_{yz} = \mu_0 f'(\zeta). \end{aligned} \tag{6.4}$$

The potential

$$f(\zeta) = K\zeta\, \mathrm{e}^{i\alpha} \tag{6.5}$$

gives the displacement and stress fields, for a homogeneous material, as

$$w = K(x\cos\alpha - y\sin\alpha), \qquad \sigma_{xz} = \mu K\cos\alpha, \qquad \sigma_{yz} = -\mu K\sin\alpha, \tag{6.6}$$

and so represents a uniform shear in the z-direction on planes with normal unit vector $(\cos\alpha, -\sin\alpha, 0)$. For an inhomogeneous material, from (3.8)

$$\begin{aligned} &w = K(x\cos\alpha - y\sin\alpha), \qquad u = KA(z)\cos\alpha, \qquad v = -KA(z)\sin\alpha,\\ &\sigma_{xx}=0, \quad \sigma_{yy}=0, \quad \sigma_{xy}=0, \quad \sigma_{xz}=\mu_0 K\cos\alpha, \quad \sigma_{yz} = -\mu_0 K\sin\alpha. \end{aligned} \tag{6.7}$$

Now disturb this uniform field by introducing a crack in the vertical plane $y = 0$ from $x = -a$ to $x = a$. The appropriate potential is

$$f(\zeta) = K(\zeta^2 - a^2)^{1/2}\,\mathrm{e}^{i\alpha}. \tag{6.8}$$

Note that $f(\zeta) \sim K\zeta\,\mathrm{e}^{i\alpha}$ as $|\zeta| \to \infty$, so that the uniform field is obtained at large distances from the crack. It is convenient to introduce polar coordinates (r_1, θ_1) and (r_2, θ_2) with origins at the crack tips, so that

$$\zeta - a = r_1\,\mathrm{e}^{i\theta_1}, \qquad \zeta + a = r_2\,\mathrm{e}^{i\theta_2}, \tag{6.9}$$

(see Figure 3). Then for a homogeneous material (6.3) and (6.8) give

$$
\begin{aligned}
w &= K(r_1r_2)^{1/2}\cos(\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_2 + \alpha),\\
\sigma_{xz} &= \frac{K\mu r}{(r_1r_2)^{1/2}}\cos(\theta + \alpha - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2),\\
\sigma_{yz} &= -\frac{K\mu r}{(r_1r_2)^{1/2}}\sin(\theta + \alpha - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2),
\end{aligned}
\tag{6.10}
$$

with the remaining stress and displacement components zero. Hence, unless $\alpha = 0$ or $\alpha = \pi$, w is discontinuous across the crack faces $y = \pm 0$, $|x| < a$, and the solution represents an anti-plane shear crack, or a mode III crack in fracture mechanics terminology. In general, σ_{xz} and σ_{yz} are also discontinuous across the crack faces.

In the case of an inhomogeneous material, (6.4) and (6.8) give

$$
\begin{aligned}
w &= K(r_1r_2)^{1/2}\cos(\tfrac{1}{2}\theta_1 + \tfrac{1}{2}\theta_2 + \alpha),\\
u &= \frac{KA(z)r}{(r_1r_2)^{1/2}}\cos(\theta + \alpha - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2),\\
v &= -\frac{KA(z)r}{(r_1r_2)^{1/2}}\sin(\theta + \alpha - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2),\\
\sigma_{xx} &= -\sigma_{yy} = -2\frac{\mu A(z)a^2K}{(r_1r_2)^{3/2}}\cos(\alpha - \tfrac{3}{2}\theta_1 - \tfrac{3}{2}\theta_2),\\
\sigma_{xy} &= 2\frac{\mu A(z)a^2K}{(r_1r_2)^{3/2}}\sin(\alpha - \tfrac{3}{2}\theta_1 - \tfrac{3}{2}\theta_2),\\
\sigma_{xz} &= \frac{K\mu_0 r}{(r_1r_2)^{1/2}}\cos(\theta + \alpha - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2),\\
\sigma_{yz} &= -\frac{K\mu_0 r}{(r_1r_2)^{1/2}}\sin(\theta + \alpha - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2).
\end{aligned}
\tag{6.11}
$$

We observe that for $A(z) \neq 0$, u and v are, in general, also discontinuous across the crack, and so the anti-plane shear crack (mode III) is, for our inhomogeneous material, coupled to in-plane opening and shear cracks (modes I and II).

The case of most interest is that in which the crack lies in the shear planes of the underlying shear field, which corresponds to $\alpha = \frac{1}{2}\pi$. In this case (6.11) becomes

$$
\begin{aligned}
w &= -K(r_1r_2)^{1/2}\sin\tfrac{1}{2}(\theta_1 + \theta_2),\\
u &= -\frac{KA(z)r}{(r_1r_2)^{1/2}}\sin(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2), \qquad v = -\frac{KA(z)r}{(r_1r_2)^{1/2}}\cos(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2),\\
\sigma_{xx} &= -\sigma_{yy} = -2\frac{\mu A(z)a^2K}{(r_1r_2)^{3/2}}\sin\tfrac{3}{2}(\theta_1 + \theta_2),
\end{aligned}
\tag{6.12}
$$

$$\sigma_{xy} = 2\frac{\mu A(z)a^2 K}{(r_1 r_2)^{3/2}} \cos \tfrac{3}{2}\,(\theta_1 + \theta_2)\,,$$

$$\sigma_{xz} = -\frac{K\mu_0 r}{(r_1 r_2)^{1/2}} \sin(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2), \qquad \sigma_{yz} = -\frac{K\mu_0 r}{(r_1 r_2)^{1/2}} \cos(\theta - \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_2).$$

As in the case of the semi-infinite crack, there are $r_1^{-3/2}$ and $r_2^{-3/2}$ order singularities in σ_{xx}, σ_{xy} and σ_{xz} at the crack tips, which suggests that material inhomogeneities result in the high concentrations of strain energy in the vicinities of the crack tips when $A(z) \neq 0$.

7. Conclusions

The majority of investigations which deal with the mechanics of inhomogeneous elastic solids focus primarily on rather simplified representations of the elastic inhomogeneity where the Lamé parameters are either linear or exponential variations of a single spatial variable. These representations of the elastic inhomogeneity have been extensively applied in the literature in solid mechanics to the study of crack and contact problems in elasticity. In this paper certain antiplane problems associated with an inhomogeneous elastic medium are examined. It is shown that the formulation of this class of problems can be approached in a general fashion by using the procedures developed by Rogers, Spencer and Mian [11–16] for the study of problems involving inhomogeneous elastic layered media. The generalized formulation is used to examine antiplane problems related to generalized screw dislocations, semi-infinite cracks and cracks of finite length located in inhomogeneous elastic media. The solutions to these problems are obtained in exact closed form. For example, in the case of anti-plane crack problems, it is shown that the presence of the axial elastic inhomogeneity introduces additional coupled modes of crack opening behaviour which are absent in the equivalent anti-plane problems associated with the crack situated in a homogeneous elastic medium. Furthermore the singular behaviour at the crack tip is amplified as a result of the elastic inhomogeneity which suggests the possibility of localization of the strain energy consistent with the nature of the axial elastic inhomogeneity.

Acknowledgements

A.J.M.S. thanks the Leverhulme Trust for an Emeritus Fellowship, and the Department of Civil Engineering and Applied Mechanics, McGill University, for their hospitality during the preparation of this paper.

References

1. M. Yamanouchi, M. Koizumi, T. Hirai and I. Shiota (eds.) FGM ’90, *Proc. 1st Int. Symp. on Functionally Gradient Materials*, Japan (1990).
2. F. Y. Shiau, Y. Zuo, X. Y. Zeng, J. C. Line and Y. A. Chang, Interfacial reactions between Co and $GaAs$. In: D. M. Mattox, J. E. E. Baglin, R. J. Gottshall and C. D. Batich, (eds), *Adhesion in Solids*, *Mat. Res. Soc. Symp. Proc.* 119 (1988) 171–176.
3. W. Olszak (ed.), *Nonhomogeneity in Elasticity and Plasticity.* Proc. IUTAM Symp., Warsaw, Poland, Pergamon Press, Oxford (1959) 528 pp.
4. J. J. Golecki and R. J. Knops, Introduction to a linear elastostatics with variable Poisson’s ratio. *Akad. gorn.-Hutn. Krakowie* 30 (1969) 81–92.
5. G. M. L. Gladwell, *Contact Problems in the Classical Theory of Elasticity*. Alphen aan den Rijn: Sijthoff and Noordhoff (1980) 716 pp.

6. A. P. S. Selvadurai, The settlement of a rigid circular foundation resting on a half-space exhibiting a near surface elastic nonhomogeneity. *Int. J. Num. Analyt. Meth. Geomech.* 20 (1996) 351–364.
7. D. L. Houck (ed.), *Thermal Spray: Advances in Coating Technology*, Proc. National Thermal Spray Conference, Orlando, (FL).: ASM International (1988) 426 pp.
8. G. Ia. Popov, Bending of an unbounded plate supported by an elastic halfspace with a modulus of elasticity varying with depth. *J. Appl. Math. Mech.* 23 (1959) 1566–1573.
9. M. K. Kassir and M. F. Chuaprasert, A rigid punch in contact with an inhomogeneous elastic solid. *J. Appl. Mech.* **41**, (1974) 1019–1024.
10. R. E. Gibson, Some results concerning displacements and stresses in an inhomogeneous elastic layer. *Geotechnique* 17 (1989) 58–67.
11. T. G. Rogers and A. J. M. Spencer, Thermoelastic stress analysis of moderately thick homogeneous and laminated plates. *Int. J. Solid Structures* 25 (1989) 1467–1482.
12. T. G. Rogers, Exact three-dimensional bending solutions for inhomogeneous and laminated elastic plates, In: G. Eason and R. W. Ogden (eds.), *Elasticity, Mathematical Methods and Applications*. Chichester: Ellis Horwood (1990) pp. 301–313.
13. A. J. M. Spencer, Three-dimensional elasticity solutions for stretching of inhomogeneous and laminated elastic plates, In: G. Eason and R. W. Ogden, (eds.), *Elasticity, Mathematical Methods and Applications.* Chichester: Ellis Horwood (1990) 347–356.
14. A. J. M. Spencer and M. Abid Mian, Exact solutions for functionally graded and laminated elastic materials (Abstract). *XIX International Congress of Theoretical and Applied Mechanics*, Kyoto (1996) p. 445.
15. M. Abid Mian and A. J. M. Spencer, Exact solutions for functionally graded and laminated elastic materials. *J. Mech. Phys. Solids* (in press).
16. A. J. M. Spencer, A stress function formulation for a class of exact solutions for functionally graded elastic plates. In: Y. H. Bahei-el-Din and G. J. Dvorak (eds.), *Proceedings of the IUTAM Symposium on Transformation Problems in Composite and Active Materials*. Dordrecht: Kluwer Academic Publishers (1998) pp. 161–172.
17. R. M. Jones, *Mechanics of Composite Materials*. Washington: Scripta Book Co. (1975) 355 pp.
18. J. M. Whitney, *Structural Analysis of Laminated Anisotropic Plates*. Lancaster (PA): Technomic Publ.Co. (1987) 342 pp.
19. F. R. N. Nabarro, *Theory of Crystal Dislocations*. Oxford: Clarendon Press (1967) 821 pp.
20. J. D. Eshelby, Dislocation theory for geophysical applications. *Phil. Trans. Roy. Soc. London* A274 (1973) 331–338.
21. J. P. Hirth and J. Lothe, *Theory of Dislocations*. New York: John Wiley (1982) 857 pp.
22. T. Mura, *Micromechanics of Defects in Solids*. 2nd ed. Dordrecht: Martinus Nijhoff Publ. (1987) 587 pp.
23. M. K. Kassir, A note on the twisting deformation of a inhomogeneous shaft containing a circular crack. *Int. J. Fracture Mech.* 8 (1972) 325–334.
24. F. Erdogan, A. C. Kaya and P. F. Joseph, The mode III crack problem in bonded materials with a inhomogeneous interfacial zone. *J. Appl. Mech.* 58 (1991) 419–427.
25. R. V. Craster and C. Atkinson, Mixed boundary-value problems in inhomogeneous elastic materials. *Q. J. Mech. Appl. Mech.* 47 (1994) 183–206.
26. D. L. Clements, W. T. Ang and J. Kusuma, A note on antiplane deformations of inhomogeneous elastic materials. *Int. J. Engng. Sci.* 35 (1997) 593–601.
27. M. Ozturk and F. Erdogan, An axisymmetric crack in bonded materials with a nonhomogeneous interfacial zone under torsion. *J. Appl. Mech.* 62 (1995) 116–125.
28. M. Ozturk and F. Erdogan, Mode I crack problem in an inhomogeneous orthotropic medium. *Int. J. Engng. Sci.* 35 (1997) 869–883.
29. A. P. S. Selvadurai and Q. Lan, Axisymmetric mixed boundary value problems for an elastic halfspace with a periodic nonhomogeneity, *Int. J. Solids Structures* 35 (1998) 1813–1826.