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A DISC INCLUSION AT A NONHOMOGENEOUS ELASTIC INTERFACE

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INTRODUCTION

The behaviour of inclusions embedded in elastic media is of interest to the modelling of multiphase composite materials where the inclusion serves as a strengthening element. The advances in this area with special emphasis on multiphase materials are documented in the studies by Eshelby [1], Willis [2], and Mura [3]. The disc inclusion is a particular simplification of the general class of three-dimensional inclusion problem where the inclusion essentially has a two-dimensional configuration. The approximation in terms of a disc inclusion is a useful model for multiphase composites which are reinforced with plate-like inclusions which could be either flexible or rigid. The theory of the disc inclusion problem has also been extensively studied owing to potential applications in geomechanics and in the study of anchorages and other load transfer devices. Examples of these applications are given by Selvadurai [4]. The majority of problems dealing with disc inclusions embedded in elastic media have largely been restricted to the examination of homogeneous elastic media. Rajapakse and Selvadurai [5] have examined a problem related to a disc inclusion which is embedded in a non-homogeneous elastic halfspace where the elastic shear modulus varies linearly with depth. This model was used to examine the axial stiffness of both rigid and flexible circular anchor plates embedded in the non-homogeneous elastic halfspace region.

In this paper we examine the axisymmetric problem involving a rigid circular disc inclusion which is embedded at the interface of identical nonhomogeneous elastic halfspace regions. The nonhomogeneity pertains to the axial variations in the elastic shear modulus in an exponential fashion. the particular form of the exponential nonhomogeneity gives rise to finite values of the shear modulus as the axial coordinate $z \in (-\infty, \infty)$. The problem is of interest to the study of anchoring devices which are embedded between bonded elastic regions where diffusion of the adherent into the pore structure of the jointed regions can result in an alteration of the elastic properties. Alternatively, a change in the elastic properties of the bonded regions can result from chemical interactions between the adhesive and elastic material. In general, such alterations are usually restricted to the vicinity of the bonded interface. For this reason it is desirable to consider variations in the elastic nonhomogeneity which have an exponential variation along an axis of symmetry but reduces to the original values at distances remote from the bonded interface. This particular model of the inhomogeneity is also of some interest to the analysis of functionally graded materials where the elastic properties are deliberately altered to achieve desirable performance characteristics. In general, both elastic constants can experience a variation due to the induced nonhomogeneity. In this study, however, we examine the special case where only the shear modulus varies with the axial coordinate normal to the bonded interface between the two nonhomogeneous elastic halfspace regions. The bonded rigid circular disc inclusion located at the interface is subjected to an axial load P (Figure 1). The paper develops the integral equation governing the inclusion problem by formulating the inclusion problem as a mixed boundary value problem related to a single halfspace region. The integral equation is solved in a numerical fashion to evaluate the load-displacement relationship for the embedded rigid disc inclusion.

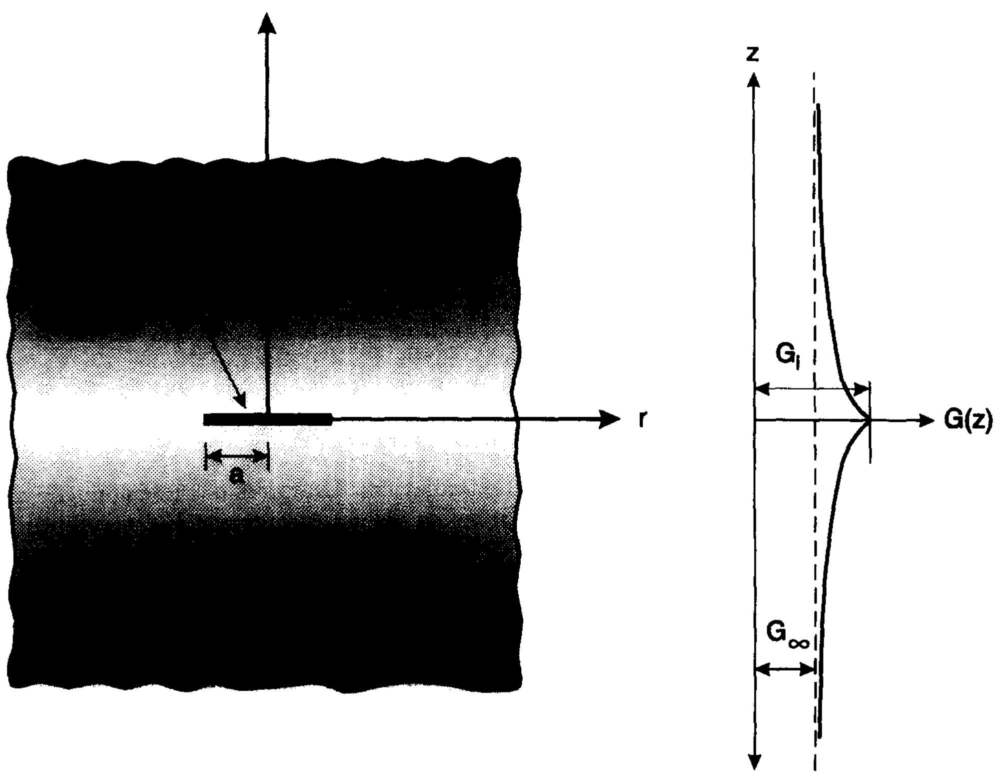

Figure 1: Rigid circular disc inclusion embedded at a non-homogeneous elastic interface.

GOVERNING EQUATIONS

We consider the axisymmetric deformation of a non-homogeneous elastic medium in which the linear elastic shear modulus is a function of the axial variable z and Poisson's ratio is assumed to be constant. For axisymmetric deformations

$$u_i = (u_r, 0, u_z), \tag{1}$$

and the non-zero components of the strain and stress tensors are are given by

$$\epsilon_{ij} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & 0 & \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ 0 & \frac{u_r}{r} & 0 \\ \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) & 0 & \frac{\partial u_z}{\partial z} \end{bmatrix}, \tag{2}$$

and

$$\sigma_{ij} = \begin{bmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{rz} & 0 & \sigma_{zz} \end{bmatrix}. \tag{3}$$

The stress-strain relationship for the non-homogeneous elastic medium takes the form

$$\sigma_{ij} = 2G(z)\epsilon_{ij} + \frac{2\nu G(z)}{1-2\nu}\epsilon_{kk}\delta_{ij}, \tag{4}$$

where $\epsilon_{kk} = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz} = e$; δ_{ij} is Kronecker's delta function and $G(z)$ and ν are, respectively, a spatially variable linear elastic shear modulus and a constant Poisson's ratio. We specifically consider exponential variations in the shear modulus such that

$$G(z) = G_\infty + (G_I - G_\infty)e^{-\zeta|z|} \tag{5}$$

where G_∞ is the far field value of the shear modulus ($|z| \to \infty$) and G_I is the shear modulus at the interface between the two halfspace regions which occupies $z = 0$. The thermodynamic constraints on the elastic constants which ensure positive definiteness of the elastic strain energy are

$$G(z) > 0; \quad -1 < \nu < \frac{1}{2}. \tag{6}$$

Using the stress-strain relation (4), the equations of equilibrium can be reduced to the following

$$\nabla^2 u_r + \frac{1}{1-2\nu}\frac{\partial e}{\partial r} - \frac{u_r}{r^2} + \frac{1}{G}\frac{dG}{dz}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0, \tag{7}$$

$$\nabla^2 u_z + \frac{1}{1-2\nu}\frac{\partial e}{\partial z} + \frac{2}{G}\frac{dG}{dz}\left(\frac{\partial u_z}{\partial z} + \frac{\nu e}{1-2\nu}\right) = 0, \tag{8}$$

where ∇^2 is the axisymmetric form of Laplace's operator referred to the cylindrical coordinate system; *i.e.*

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{9}$$

We seek solutions of (7) and (8) where the displacement components can be represented in terms of Hankel transforms (Sneddon [6])

$$u_r(r,z) = \int_0^\infty U(s,z)A(s)J_1(rs)ds, \tag{10}$$

$$u_z(r,z) = \int_0^\infty W(s,z)A(s)J_0(rs)ds, \tag{11}$$

where $A(s)$ is an arbitrary function and $J_n(rs)$ are Bessel functions of the order n ($n = 0, 1$). Using the above displacement representations, the equilibrium equations (7) and (8) can be reduced to the following forms

$$\frac{d^2U}{dz^2} + q(z)\frac{dU}{dz} - \frac{2(1-\nu)}{1-2\nu}s^2U - \frac{s}{1-2\nu}\frac{dW}{dz} - q(z)sW = 0, \tag{12}$$

$$\frac{d^2W}{dz^2} + q(z)\frac{dW}{dz} - \frac{1-2\nu}{2(1-\nu)}s^2W + \frac{s}{2(1-\nu)}\frac{dU}{dz} + q(z)\frac{s\nu U}{1-\nu} = 0, \tag{13}$$

where

$$q(z) = \frac{1}{G(z)}\frac{dG(z)}{dz}. \tag{14}$$

The expression for σ_{zz} required to formulate the embedded rigid disc inclusion problem can be expressed in the form

$$\sigma_{zz}(r,z) = \frac{2G(z)[1-\nu]}{1-2\nu}\int_0^\infty \left[\frac{dW}{dz} + \frac{\nu sU}{1-\nu}\right] A(s)J_0(rs)ds. \tag{15}$$

THE INCLUSION PROBLEM

In this section we examine the problem of the axisymmetric loading of a rigid circular disc inclusion which is embedded at the interface between two bonded non-homogeneous elastic halfspace regions where the shear modulus varies according to the relationship (5). Owing to the asymmetry of the deformation induced by the rigid displacement of the embedded disc inclusion, we can formulate the inclusion problem as a mixed boundary value problem related to the halfspace region $0 \le z < \infty$. The mixed boundary conditions governing the inclusion problem are

$$u_z(r,0) = \Delta, \quad r \le a, \tag{16}$$

$$u_r(r,0) = 0, \quad r \ge 0. \tag{17}$$

$$\sigma_{zz}(r,0) = 0, \quad r > a, \tag{18}$$

where Δ is the rigid displacement of the disc inclusion. Considering the boundary condition (17) and the regularity conditions applicable to u_z as $z \to \infty$, we obtain the following boundary conditions which are applicable to the differential equations (12) and (13)

$$U(s,0) = 0, \quad W(s,0) = 1, \quad U(s,\infty) = W(s,\infty) = 0 \tag{19}$$

Considering the expressions for u_z and σ_{zz}, (11) and (15), the constraints (19), and the substitutions

$$sR(s) = \left[\frac{dW}{dz}\right]_{z=0}, \quad A(s)R(s) = B(s) \tag{20}$$

the mixed boundary conditions (16) and (18) can be reduced to the forms

$$\int_0^\infty \left[\frac{W}{R}\right]_{z=0} B(s)J_0(rs)ds = \Delta, \quad r \le a, \tag{21}$$

$$\int_0^\infty sB(s)J_0(rs)ds = 0, \quad r > a. \tag{22}$$

Introducing the finite Fourier transform for $B(s)$ in the form

$$B(s) = \frac{2\Delta}{\pi} \int_0^a \phi(t) \cos(st) dt, \tag{23}$$

the system of dual integral equations (21) and (22) can be reduced to a Fredholm integral equation of the second kind

$$\frac{3-4\nu}{2(2\nu-1)} \phi(x) + \int_0^a K(x,t) \phi(t) dt = 1, \tag{24}$$

where the kernel function $K(x,t)$ is given by

$$K(x,t) = \frac{2}{\pi} \int_0^\infty \left[\left(\frac{W}{R} \right)_{z=0} - \frac{3-4\nu}{2(2\nu-1)} \right] \cos(st) \cos(sx) ds. \tag{25}$$

The ordinary differential equations (12) and (13) associated with the embedded disc inclusion problem can be solved numerically by using an ordinary differential equation solver COLSYS given by Ascher et al [7]. With the help of these numerical solutions, it is possible to evaluate the kernel function of the Fredholm integral equation. The resulting Fredholm integral equation can then be solved numerically by employing a standard technique (see Delves and Mohamed [8]).

NUMERICAL RESULTS

The load-displacement relationship for the rigid disc inclusion can be obtained by considering the overall equilibrium of the embedded disc inclusion, *i.e.*

$$P = 2\pi \int_0^a [\sigma_{zz}(r, 0^+) - \sigma_{zz}(r, 0^-)] r dr, \tag{26}$$

where the designations 0^+ and 0^- refer to planes of the disc inclusion in contact with the halfspace regions $z > 0$ and $z < 0$ respectively. It can be shown that

$$P = \frac{16G(0)\Delta(1-\nu)}{(1-2\nu)} \int_0^a \phi(t) dt. \tag{27}$$

Figure 2 illustrates the manner in which the normalized axial stiffness of the embedded disc inclusion is influenced by the elastic nonhomogeneity in the shear modulus. The non-dimensional elastic stiffness $\bar{P}$ is defined by

$$\bar{P} = \frac{P}{P_0} = -\frac{3-4\nu}{2(1-2\nu)} \frac{G_I}{G_\infty} \int_0^1 \phi(ax) dx, \tag{28}$$

where $P_0 = [32 G_\infty \Delta a(1-\nu)/(3-4\nu)]$ is the total force required to initiate a rigid displacement Δ in a the homogeneous medium with shear modulus G_∞. All solutions are given for $\nu = 0.3$. Fig.2 indicates that all the results are bounded between the results which correspond, respectively, to the two limiting cases $\bar{\zeta} = a\zeta = 0$ and $\bar{\zeta} \to \infty$.

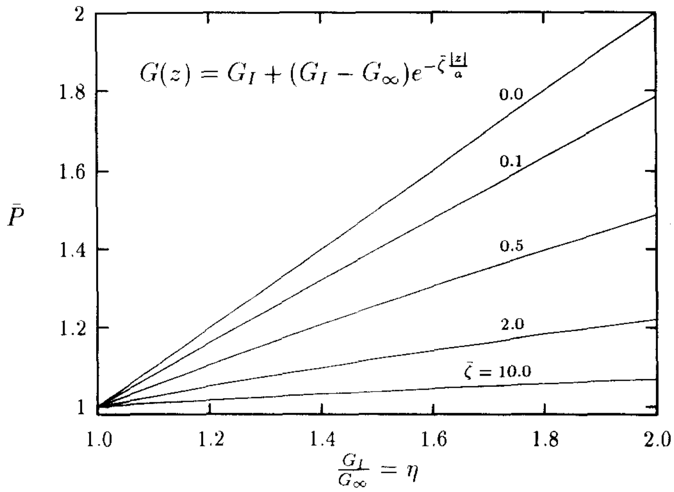

Figure 2: Variation of the normalized axial stiffness $\bar{P}$ of the rigid disc inclusion with respect to $\eta = \frac{G_I}{G_\infty}$ for $\nu = 0.3$ and various values of $\bar{\zeta} = a\zeta$.

REFERENCES

[1] J.D. Eshelby, *Progress in Solid Mechanics* **2**, (eds. I.N. Sneddon and R. Hill), p.89. North-Holland, Armsterdam(1961).

[2] J. R. Willis, *J. Mech. Phys. Solids* **13**, 377(1965).

[3] T. Mura, *Micromechanics of Defects in Solids*, 2nd ed., Martinus Nijhoff Publ., Netherlands(1987).

[4] A.P.S. Selvadurai, *Computer Methods and Advances in Geomechanics*, (eds. H. J. Siriwardane and M.M. Zaman), Vol. 1, p.305, Balkema, Rotterdam (1994).

[5] R.K.N.D. Rajapakse and A.P.S. Selvadurai, *Int. J. Numer. Analy. Mech. Geomech.* **15**, 457 (1991).

[6] I.N. Sneddon, *Fourier Transforms*, McGraw-Hill, New York (1951).

[7] U. Ascher, J. Christiansen and R.D. Russell, *Acm Trans. Math Software* **7**, 209(1981).

[8] L.M. Delves and J. L. Mohamed, *Computational Methods for Integral Equations*, Cambridge Univ. Press, Cambridge(1985).