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Computational modeling of the indentation of a cracked poroelastic half-space

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Abstract. The paper examines the computational modelling of the surface identation of a poroelastic half-space region which is weakened either by a cylindrical crack or a penny-shaped crack. The axisymmetric problems associated with those situations are examined using a finite element procedure where special singularity elements are incorporated at the crack tip and appropriate interaction conditions are incorporated on the faces of the crack. The results presented in the paper illustrate the influence of the extent of fracture and the pore pressure boundary conditions on the various surfaces, on the time dependent evolution of the stress intensity factors and the time dependent consolidation settlement of the axisymmetric indentor. The analysis is extended to the consideration of crack extension in poroelastic materials where displacement, traction and pore water pressure boundary conditions are altered to take into account the evolving crack. The path of crack extension is established by mixed mode crack extension criteria applicable to porous fabric. The computational procedure associated with this approach is used to examine the problem of the surface indentation of a half-space by a rigid circular indentor.

1. Introduction

The mathematical model of the one-dimensional consolidation of a fluid saturated porous elastic medium was first developed by Terzaghi (1925) and later extended by Biot (1941; 1956) to provide a rational framework for the theory of poroelasticity. In its classical form, the theory of poroelasticity accounts for the pore pressure gradient-induced transport of a compressible fluid through the pore structure of an ideal linearly elastic material. The theory has formed the basis for a variety of applications in the areas of energy resources recovery, bio-mechanics, mechanics of biological materials and most notably in geomechanics (Selvadurai, 1996). The seminal studies in the area of geomechanics originated with the application of integral transform techniques to the problem of surface loading of half-space regions (McNamee and Gibson, 1960; Schiffman and Fungaroli, 1965) and has continued with applications to mixed boundary value problems involving the indentation of fluid saturated half-space regions and inclusion problems related to infinite domains (Agbezuge and Deresiewicz, 1974; 1975; Chiarella and Booker, 1975; Szefer and Gaszynski, 1975; Booker and Small, 1975; Selvadurai and Yue, 1994; Yue and Selvadurai, 1994; 1995).

The current paper re-examines the classical indentation problem associated with a fluid saturated half-space region. The indentation of the surface of an elastic half-space by a smooth flat indentor was first examined by Boussinesq (1885) by applying results of potential theory. The problem was re-examined by Harding and Sneddon (1945) who reduced the axisymmetric problem to the solution of a pair of dual integral equations. The literature in contact mechanics contains extensive accounts of the subject of indentation problems and such studies are given by Galin (1961), Lur'e (1964), Selvadurai (1979), Gladwell (1980) and Johnson (1985). In the classical treatment of the axisymmetric contact problem, it is invariably assumed that the elastic medium is free of defects, most notably cracks, which can invariably initiate in the vicinity of the indentation zone. The primary reason is that the presence of defects even in their axisymmetric forms entail a great deal of analytical effort in the solution of the final forms of the integral equation (Selvadurai, 1997). The applications of the classical theory of poroelasticity to problems of indentation in the vicinity of a defect makes the problem unwieldy particularly due to the convolution nature of the final integral equations governing the poroelasticity problem (Lan and Selvadurai, 1996). For this reason, recourse must be made to the use of numerical schemes when dealing with problems associated with interaction of indentors and defects such as cracks in the vicinity of the indentation zone. Both finite element techniques and boundary element techniques have been successfully applied in the solution of problems in poroelasticity. The application of finite element procedures to the solution of problems in poroelasticity are given by a number of authors including Sandhu and Wilson (1969) who were the first to apply finite element procedures to examine soil consolidation problems. Similarly, the application of boundary element procedures are documented by Cheng and Liggett (1984).

The present paper deals with the analysis of the axisymmetric identation of a saturated poroelastic half-space which is weakened by either a stationary crack or a crack which experiences extension in a quasi-static manner. The geometry of the stationary crack is considered to be axisymmetric and two configurations of the crack are considered; the first involves a cylindrical crack which is located at a specified radius from the rigid indentor (Figure 1) and the second involves a flat penny shaped co-axial crack which is located at a finite depth from the rigid indentor (Figure 2). The paper also develops a numerical procedure which can be used to examine quasi-static crack extension within a poroelastic medium, particularly from a region at the edge of the indentor on the poroelastic medium. In the context of analytical studies the treatment of cracks occurring in poroelastic geomaterials has received only limited attention. The problem of the fracture of a poroelastic material was investigated by Rudnicki (1985) in connection with fault initiation and propagation in porous rock. These problems were examined in view of their potential use in the study of earthquake mechanisms at the source location and in connection with enhanced energy resources recovery by hydraulic fracturing techniques (see also Ingraffea and Boone (1988)).

Recently Atkinson and Craster (1991) and Craster and Atkinson (1996) have examined

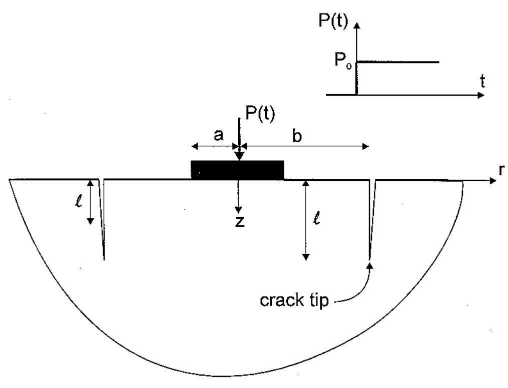

Figure 1. Indentation of a poroelastic half-space containing a cylindrical crack.

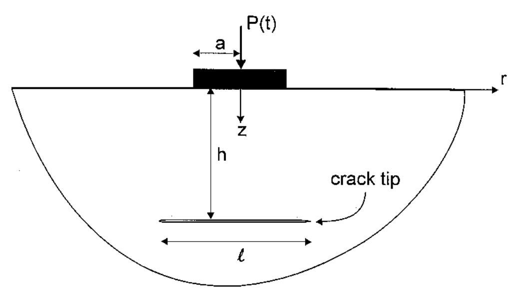

Figure 2. Indentation of a poroelastic half-space containing a flat penny-shaped co-axial crack.

the problems of stationary and steadily propagating semi-infinite cracks embedded in poroelastic media. These studies include the examination of crack behaviour in the presence of variable pore pressure boundary conditions at the faces of the crack and their potential influence on the crack tip pore pressure fields and stress intensity factors. In this paper the computational modelling of the problems primarily focuses on addressing two questions; the first relates to the examination of the influence of the defect and pore fluid pressure boundary conditions on the time-dependent displacements of the indentor and on the stress intensity factor at the crack tip. The radius of the indentor in relation to the radius and/or length of the crack is taken as variable. The second relates to the examination of non-planar quasi-static extension of a cylindrical crack emanating from the edge of the indentor into poroelastic medium. The extension of the crack follows a mixed mode crack extension criterion applicable to the porous skeleton and the pore pressure, displacement and traction boundary conditions are modified in the crack extension zone. The numerical procedure also prevents overlapping closure of the crack in a time-dependent fashion. The numerical results presented in the paper for a variety of problems indicate the influence of crack-indentor interaction and crack extension pattern on the stress intensity factors and consolidation rates.

2. Governing equations

The basic equations governing Biot's theory of poroelasticity are summarized for completeness. The constitutive equations governing the quasi-static response of a poroelastic medium, which consists of a porous isotropic elastic soil skeleton saturated with a *compressible pore fluid* take the forms

$$\sigma_{ij} = \frac{2\mu\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)}p\delta_{ij}, \tag{1a}$$

$$p = \frac{2\mu B^2(1-2\nu)(1+\nu_u)^2}{9(\nu_u-\nu)(1-2\nu_u)}\zeta_v - \frac{2\mu B(1+\nu_u)}{3(1-2\nu_u)}\varepsilon_{kk}, \tag{1b}$$

where σ_{ij} is the total stress tensor; p is the pore fluid pressure; ζ_v is the volumetric strain in the compressible pore fluid; ν is Poisson's ratio of the porous fabric; ν_u is the undrained Poisson's ratio; μ is the shear modulus; B is Skempton's pore pressure coefficient; $\kappa = k/\gamma_w$ where k is the coefficient of hydraulic conductivity and γ_w is the unit weight of pore fluid. Also ε_{ij} is the soil skeletal strain tensor defined by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{2}$$

where u_i are the displacement components, and a comma denotes a partial derivative with respect to a spatial variable. In the absence of body forces, the quasi-static equations of equilibrium take the forms

$$\sigma_{ij,j} = 0. \tag{3}$$

the fluid transport within the pores of the medium is governed by Darcy's law which can be written as

$$\upsilon_j = -\kappa p_{,i}, \tag{4}$$

where υ_i is the specific discharge vector in the pore fluid. The equation of continuity associated with quasi-static fluid flow is

$$\frac{\partial \zeta_v}{\partial t} + \upsilon_{i,i} = 0. \tag{5}$$

Considering requirements for a positive definite strain energy potential (Rice and Cleary, 1976), it can be shown that the material parameters should satisfy the following thermodynamical constraints: $\mu > 0; 0 \leqslant B \leqslant 1; -1 < \nu < \nu_u \leqslant 0.5; \kappa > 0$.

The resulting equations of equilibrium for a poroelastic medium as introduced by Biot (1941) and reformulated in more physically relevant variables by Rice and Cleary (1976), can be written in terms of the displacements and pore pressure as

$$\mu \nabla^2 u_i + \frac{\mu}{(1-2\nu)} \varepsilon_{kk,i} - \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)} p_{,i} = 0, \tag{6a}$$

$$\frac{\partial p}{\partial t} - \frac{2\kappa\mu B^2(1-2\nu)(1+\nu_u)^2}{9(\nu_u - \nu)(1-2\nu_u)} \nabla^2 p = -\frac{2\mu B(1+\nu_u)}{3(1-2\nu_u)} \frac{\partial \varepsilon_{kk}}{\partial t}. \tag{6b}$$

Equation (6a) corresponds to the usual Navier equation of linear, isotropic elasticity together with the coupling term of induced stresses by the pore fluid. Equation (6b) is a diffusion-type equation for the pore fluid.

The following limiting conditions can be recovered from the generalized results:

1. The undrained behaviour immediately after the loading corresponds to $\zeta_v = 0$, consequently (6a) reduces to the usual elasticity equation with undrained Poisson's ratio ν_u.
2. In achieving the fully drained state, the excess pore pressure dissipates completely (i.e. $p \to 0$) and again (6a) reduces to the usual elasticity equation with drained values of the elastic constants; μ and ν.
3. For most naturally occuring geomaterials, the saturated porous medium can be modelled as a medium consisting of incompressible solid particles and an incompressible pore fluid. In this case $\zeta_v = \varepsilon_{kk}$ and the governing equations reduce to:

$$\mu \nabla^2 u_i + \frac{\mu}{(1-2\nu)} \varepsilon_{kk,i} - p_{,i} = 0, \tag{7a}$$

$$\frac{\partial \varepsilon_{kk}}{\partial t} = \frac{2\kappa\mu B^2(1-\nu)(1+\nu_u)^2}{9(1-\nu_u)(\nu_u - \nu)} \nabla^2 \varepsilon_{kk}. \tag{7b}$$

For a well posed problem, boundary conditions and initial conditions on the variables u_i, σ_{ij} and p also need to be prescribed.

3. Numerical approach

Finite element methods offer efficient procedures for solution of problems associated with the consolidation of saturated poroelastic media. Sandhu and Wilson (1969) were the first to apply finite element methods to the study of problems associated with consolidating geomaterials. Ghaboussi and Wilson (1973), Booker and Small (1975) and Sandhu et al. (1985) have developed finite element procedures for the analysis of problems associated with surface loading of semi-infinite media. Selvadurai and Gopal (1986) and Schrefler and Simoni (1987) have used the finite element method to investigate the consolidation behaviour of media of infinite extent, which are modeled by appeal to infinite elements. A review of both analytical and numerical approaches to the study of soil consolidation related to poroelastic media are given by Schiffman (1984), Lewis and Schrefler (1987) and Selvadurai (1996).

In this section we present a brief review of the finite element formulation of the poroelasticity problem with special reference to the incorporation of singular elements to model the singular stress field at the crack tip.

Equations (6a) and (6b) can be approximated by a matrix equation by adopting a standard Galerkin finite element procedure. The details of these procedures are well documented by Sandhu and Wilson (1969), Aboustit et al. (1985), Schrefler and Simoni (1987), Lewis and Schrefler (1987), Selvadurai and Kapurapu (1989) and more recently by Selvadurai and Nguyen (1995) in connection with the finite element modelling of isothermal consolidation of sparsely jointed porous media.

3.1. Finite element procedure

Equation (6a) and (6b) are discretized on the spatial domain using a standard finite element procedure. The approximation used for the basic variables u_i and p are the following:

$$\boldsymbol{u} = \boldsymbol{N}^u \bar{\boldsymbol{u}}; \qquad \boldsymbol{p} = \boldsymbol{N}^p \bar{\boldsymbol{p}} \tag{8}$$

where $\boldsymbol{N}^u$ are the interpolation functions for the displacement field; $\boldsymbol{N}^p$ are the interpolation functions for the pore pressure field and $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{p}}$ are nodal value vectors of the respective variables. In general $\boldsymbol{N}^u$ and $\boldsymbol{N}^p$ can be different but must exhibit $\mathbf{C}_0$ continuity. The element chosen to represent the intact region of the poroelastic medium is the eight-noded axisymmetric isoparametric element where the displacements within the element are interpolated as functions of the 8 nodes, whereas the pore pressures are interpolated as a function of the four corner nodes i, k, m, and o only (Figure 3). The rationale for this procedure is now well documented (see e.g. Lewis and Schrefler, 1987 and Smith and Griffiths, 1988).

In addition to the axisymmetric solid element it is also necessary to introduce a singular element to model crack tip behaviour. The quarter point singularity element introduced by Henshell and Shaw (1975) and Barsoum (1976) has been adopted for this purpose. It has been

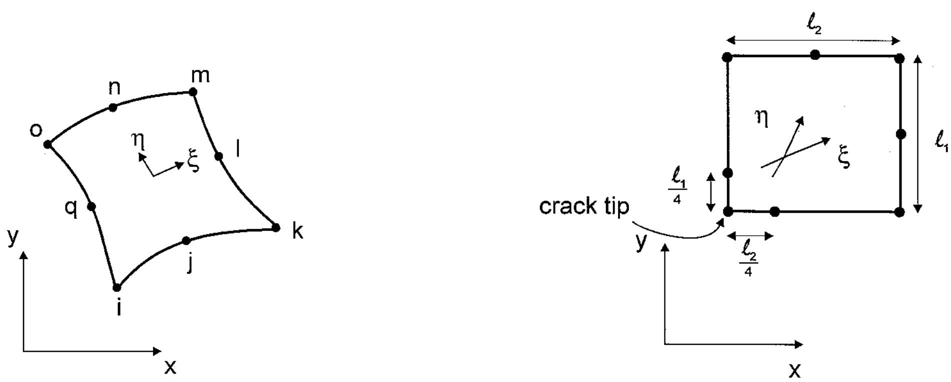

Figure 3 (a) Plane isoparametric element; and (b) Quarter point isoparametric element.

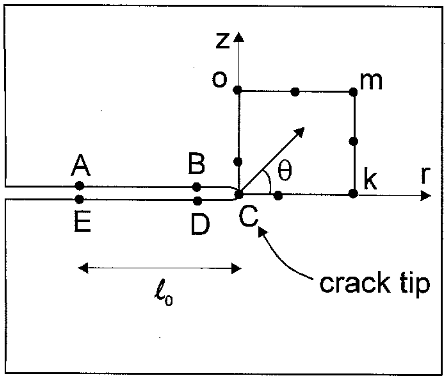

Figure 4. Details of the crack tip and node arrangement for computation of the stress intensity factors.

successfully used in modelling of axisymmetric crack problems in elasticity. The order of the effective stress singularity at the crack tip for the porous skeleton modelled by this approach is $r^{-1/2}$, while the pore pressure field around the crack tip is modeled by conventional isoparametric elements. However, the pore pressure singularity at the crack tip can also be modeled by a similar approach. The stress intensity factors at the crack tip are evaluated by applying the displacement correlation method which utilizes the nodal displacements at four locations A, B, E, D and the crack tip C (Figure 4).

The application of a Galerkin procedure to the governing equations (see e.g. Sandhu and Wilson, 1969) gives rise to the following discretized forms of the equations governing poroelastic media:

$$\begin{bmatrix} \mathbf{K} & \mathbf{C} \\ \mathbf{C}^T & -\gamma\Delta t\mathbf{H} + \mathbf{E} \end{bmatrix} \begin{Bmatrix} u_{t+\Delta t} \\ p_{t+\Delta t} \end{Bmatrix} = \begin{bmatrix} \mathbf{K} & \mathbf{C} \\ \mathbf{C}^T & (1-\gamma)\Delta t\mathbf{H} + \mathbf{E} \end{bmatrix} \begin{Bmatrix} u_t \\ p_t \end{Bmatrix} + \{\mathbf{F}\}, \tag{9}$$

where **K** is the stiffness matrix of the soil skeleton; **C** is the stiffness matrix due to interaction between soil and the pore fluid; **E** is the compressibility matrix of fluid; **H** is the permeability matrix; **F** is the force vector due to external tractions, body forces and flows; u_t, p_t are the nodal displacements and pore pressures; Δt is the time increment and γ is a time integration constant.

Expression for the matrices $\mathbf{K}, \mathbf{C}$ etc. are given in the Appendix. The time integration constant γ varies between 0 and 1. The criteria for the stability of the integration scheme given in Booker and Small (1975) require that $\gamma \geqslant 1/2$. According to Lewis and Schrefler (1987) and Selvadurai and Nguyen (1995), the stability of solution can generally be achieved by selecting values of γ close to 1.

4. Numerical procedures

In this paper we focus on the incorporation of the singular stress field crack tip element to model the $r^{-1/2}$ type singularity at the crack tip in poroelastic medium and also to examine the quasi-static problem associated with the mixed mode crack extension. The result of particular interest is the evaluation of the stress intensity factor at the crack tip. For axisymmetric problems the mode I and mode II stress intensity factors can be evaluated by the displacement correlation method incorporating the nodes A, B, E, D, and the crack tip C (Figure 4), i.e.,

$$K_{\mathrm{I}} = \frac{\mu}{(1+k_\alpha)}\sqrt{\frac{2\pi}{\ell_0}}\{4[u_z(B) - u_z(D)] + u_z(E) - u_z(A)\}, \tag{10}$$

$$K_{\mathrm{II}} = \frac{\mu}{(1+k_\alpha)}\sqrt{\frac{2\pi}{\ell_0}}\{4[u_r(B) - u_r(D)] + u_r(E) - u_r(A)\}, \tag{11}$$

where $k_\alpha = (3 - 4\nu)$ and ℓ_0 is the length of the crack tip element.

Box 1. Crack extension algorithm

(i) Compute K_{I} and K_{II} (Equations 10 and 11):

IF $K_{\mathrm{I}} < 0$; Add *rigid links* between crack faces. Return to (i).
ELSE; Proceed to (ii).

(ii) Check fracture criterion:

$$\cos\frac{1}{2}\theta\left[\frac{K_{\mathrm{I}}}{K_{\mathrm{I}C}}\cos^2\frac{1}{2}\theta - \frac{3K_{\mathrm{II}}}{2K_{\mathrm{I}C}}\sin\theta\right] = 1,$$
$$K_{\mathrm{I}}\sin\theta - K_{\mathrm{II}}(3\cos\theta - 1) = 0.$$

Yes: Crack extension. Proceed to (iii).
No: No further crack extension. Exit.

(iii) Locate new crack tip (CT) by following approximations:

IF $0 \leqslant \theta < \frac{1}{8}\pi$; SET $\theta = 0 \Rightarrow$ CT moves to node k (Figure 4).

IF $\frac{3}{8}\pi \leqslant \theta < \frac{3}{8}\pi$; SET $\theta = \frac{1}{4}\pi \Rightarrow$ CT moves to node m.

IF $\frac{3}{8}\pi \leqslant \theta < \frac{5}{8}\pi$; SET $\theta = \frac{1}{2}\pi \Rightarrow$ CT moves to node o.

Split double nodes and generate quarterpoint singular elements.

(iv) Update pore pressure boundary conditions at crack faces:

IF $K_{\mathrm{I}} > 0 \Rightarrow$ SET $p = 0$ (permeable).

IF $K_{\mathrm{I}} = 0 \Rightarrow$ SET $\begin{cases} p_A = p_E \\ p_B = p_D \end{cases}$ (Figure 4) (continuous pore pressure).

The quasi-static crack extension in poroelastic media is a moving boundary-type problem where the pore pressure boundary conditions change depending upon the extent of crack opening. For the opening mode $(K_{\mathrm{I}} > 0)$ of crack extension, the pore pressure boundary condition is adjusted in the opened region to reflect a zero pore pressure boundary condition

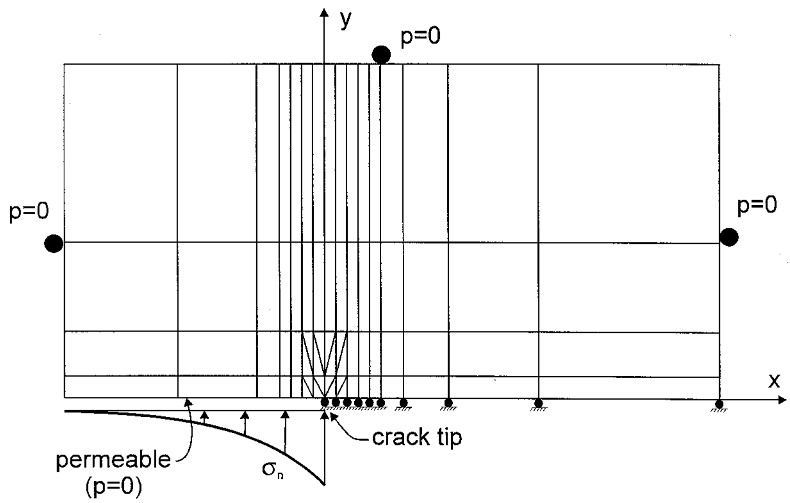

Figure 5. Finite element discretization of a semi infinite crack in a poroelastic medium.

(permeable fracture). Alternatively, it is assumed that the pore pressure field remains continuous for cracks which extend in a shear mode but remain closed (impermeable fracture). In this paper a computational method is developed for the mixed mode extension of a crack in a poroelastic medium. In the procedure employed, there is no remeshing implemented and the crack is allowed to extend through either the inter-element boundaries or through the element with the provision of splitting the double nodes. The detail of the crack extension algorithm is given in Box 1. The mixed mode fracture criterion employed is the one given by Erdogan and Sih (1963) which can be stated as follows: The crack in the poroelastic medium will extend when the stress intensity factors applicable to the singular effective stresses at the crack tip satisfies the condition:

$$\cos\frac{\theta}{2}\left[\frac{K_{\mathrm{I}}}{K_{\mathrm{I}C}}\cos^2\frac{\theta}{2}-\frac{3K_{\mathrm{II}}}{2K_{\mathrm{I}C}}\sin\theta\right]=1, \tag{12a}$$

where θ is the orientation of crack extension which can be obtained from the equation

$$K_{\mathrm{I}}\sin\theta - K_{\mathrm{II}}(3\cos\theta - 1) = 0 \tag{12b}$$

and $K_{\mathrm{I}C}$ is the critical intensity factor.

4.1. Calibration of the computational scheme

The computational scheme developed in connection with the crack-indentor interaction problem has been calibrated by appeal to an analytical result. We consider the problem of a semi-infinite crack in an isotropic poroelastic medium, the faces of which are subjected to an exponentially decaying internal normal total stress

$$\sigma_{yy} = \sigma_0\,\mathrm{e}^{x/d}H(t), \quad -\infty < x \leqslant 0, \tag{13}$$

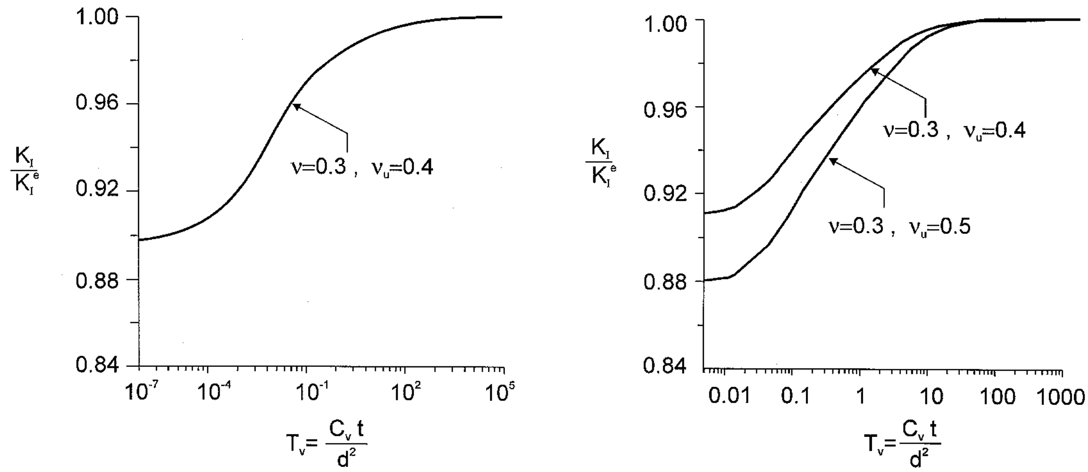

Figure 6. Time dependent variation of the stress intensity factor K_{I} for semi infinite crack with permeable faces; (a) Atkinson and Craster (1991) and Craster and Atkinson (1996); and (b) Finite element analysis.

where $H(t)$ is the Heaviside step function, and d is a constant. Admittedly, the problem does not have an axisymmetric configuration consistent with other problems examined in this paper; nonetheless, it helps to identify the accuracy of the local field in the axisymmetric problem which would correspond closely to the two-dimensional plane strain situation. Figure 5 illustrates the geometry of the problem and finite element discretization used in the computations. In Figure 6 the results derived from the finite element technique for the crack opening-mode stress intensity factor K_{I} are compared with the analytical results given in Atkinson and Craster (1991) and Craster and Atkinson (1996) for the case where the faces of the crack are permeable. The results indicate that there is good agreement between the analytical results and computational results particularly with respect to the time variation of the stress intensity factor. The maximum discrepancy (2%) between the two sets of results ($\nu = 0.3, \nu_u = 0.4$) could be attributed to the influence of the finite domain when dealing with the finite element modelling.

The axisymmetric indentation of a standard poroelastic half-space weakened by an axisymmetric crack of either a cylindrical configuration (Figure 1) or a co-axial penny-shaped configuration (Figure 2) is considered in the following sections.

4.2. The cylindrical crack problem

The problem of a deep region of a saturated poroelastic geomaterial is considered. The surface of the poroelastic medium is indented by a rigid indentor of diameter $2a$ which is subjected to a load $P(t)$ that is kept constant (i.e. a Heaviside step function of total load) as follows

$$P(t) = \int_0^a \sigma_{zz}(r, 0, t) = P_0 H(t), \quad 0 \leqslant r \leqslant a, \tag{14}$$

where σ_{zz} is the normal contact stress between the geomaterial and the indentor. The region contains a cylindrical crack of length ℓ located at a finite radius b from the center of the rigid indentor (Figure 1). The faces of fracture are considered to be smooth and permeable. The location of the outer boundary of the poroelastic medium is specified in relation to the radius of the rigid indentor.

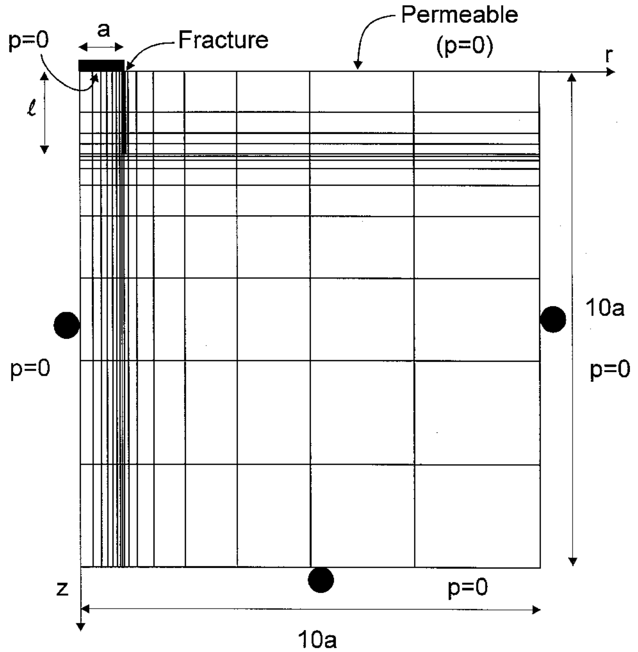

Figure 7. Finite element discretization of the poroelastic medium containing a cylindrical crack of $\ell/a = 2$ and $b/a = 1$.

4.2.1. *The stationary cylindrical crack*

Figure 7 illustrates the typical finite element discretization used in the numerical modeling associated with the fracture with dimensions $\ell/a = 2$ and $b/a = 1$. The skeleton of the geomaterial and the pore fluid are assumed to be incompressible. Suitable mesh refinements are incorporated in the vicinity of the crack tip and in the loaded region in order to improve the accuracy of computational estimates. The results of primary interest relate to singularity at the crack tip and the geometrical characteristics of the crack on the time-dependent degree of consolidation settlement.

The degree of consolidation (U) which is based on settlement of rigid indentor can be defined as follows

$$U = \frac{u_z^0(t) - u_z^0(0)}{u_z^0(\infty) - u_z^0(0)}, \tag{15}$$

where $u_z^0(t) = u_z(0, 0, t)$. The time factor associated with the consolidation process is selected as

$$T_v = \frac{C_v t}{a^2}, \tag{16a}$$

where

$$C_v = \frac{2\mu(1-\nu)\kappa}{(1-2\nu)}. \tag{16b}$$

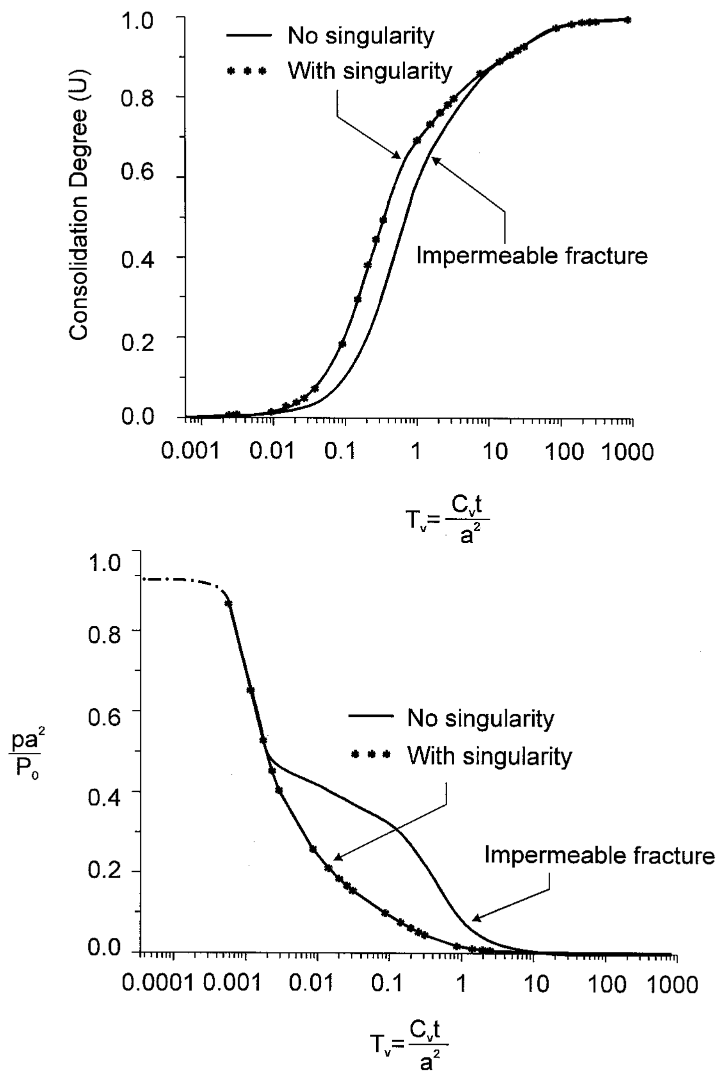

Figure 8. Effect of crack tip singularity and permeability of crack face for a cylindrical crack of $\ell/a = 1$ and $b/a = 1$ on; (a) Consolidation settlement of indentor; and (b) Pore pressure at crack tip.

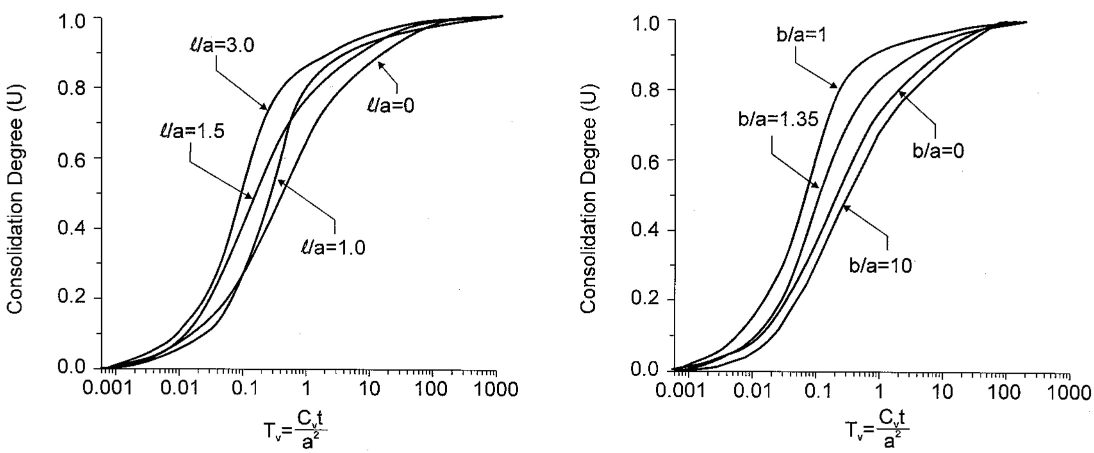

Figure 9. Consolidation response of a poroelastic half-space contaning a cylindrical crack; (a) Effect of crack length for $b/a = 1$; and (b) Effect of location of crack for $\ell/a = 3$.

Figure 8 illustrates the influence of the crack tip singularity and the permeability of crack surface on the degree of consolidation and the pore pressure at the crack tip for the typical case when $\ell/a = 1$ and $b/a = 1$. These results demonstrate that the singularity of the crack tip does not greatly influence the results, whereas the permeability of the crack surface has significant influence on the local and global poroelastic behaviour.

Figure 9 illustrates the variations in the degree of consolidation settlement of indentor and the manner in which the poroelastic response is influenced by the length and location of the cylindrical crack. The consolidation rate decreases for a relative crack length of $\ell/a < 1$ and starts to increase for a relative crack length of $\ell/a > 1$ for the case of $b/a = 1$. This shows that the influence of small cracks $(\ell/a < 1)$ on alteration of stiffness characteristics is greater than that of hydraulic behaviour of the medium. However, for longer cracks $(\ell/a > 1)$ the variation of consolidation rate can be attributed to the permeability of crack surfaces. The greatest change in poroelastic response is observed when a crack is located right at the edge of the rigid indentor $(b/a = 1)$. For cracks located remote from the edge of the indentor $(b/a > 1$ or $b/a < 1)$ the effect of crack location on the consolidation rate decays with distance.

4.2.2. *The extending cylindrical crack*

This section focuses on the analysis of mixed mode quasi-static extension of a ‘cylindrical’ crack emanating from the edge of the indentor on the poroelastic half-space. The pore pressure, displacement and traction boundary conditions are updated in relation to the extent of the crack. The permeable crack faces are considered for the opening mode of crack extension and impermeable crack faces are assumed for the shear mode crack extension. The overlapping of elements due to crack closure is prevented by provision of the rigid links between the faces of the crack.

The results of crack extension analyses associated with the indentation problem for a poroelastic medium with either a compressible soil skeleton or a compressible pore fluid $(\nu = 0.3)$ are shown Figure 10. A critical stress intensity factor $K_{IC} = 1.0$ MPa$\sqrt{\text{m}}$ is assumed for the geomaterial and a similar Heaviside step function type for constant total load $P(t)$ is considered. Two different finite element discretizations of the medium are used. A cylindrical starter crack with length $\ell/a = 0.1$ at the edge of the indentation is considered. The fracture patterns for two meshes, A and B, are found to be similar. The time-dependent consolidation settlement of the indentor shown in Figure 10b indicates that the effect of crack extension on the consolidation rate is nominal.

4.3. The crack-indentor interaction problem

This section presents the behaviour of a rigid anchor bonded to the surface of a poroelastic half-space containing a flat penny-shaped co-axial crack located at the depth h (Figure 2). The loading corresponds to a Heaviside step function type for a constant total load $P(t)$ given in (14). Figure 11 illustrates the variation of degree of consolidation for different depth ratios of h/b for both impermeable and permeable interfaces between the indentor and the poroelastic geomaterial. Influences of crack geometry characteristics, similar to those described in Section 4.2.1, are observed for the case of the poroelastic response of an anchor type indentor.

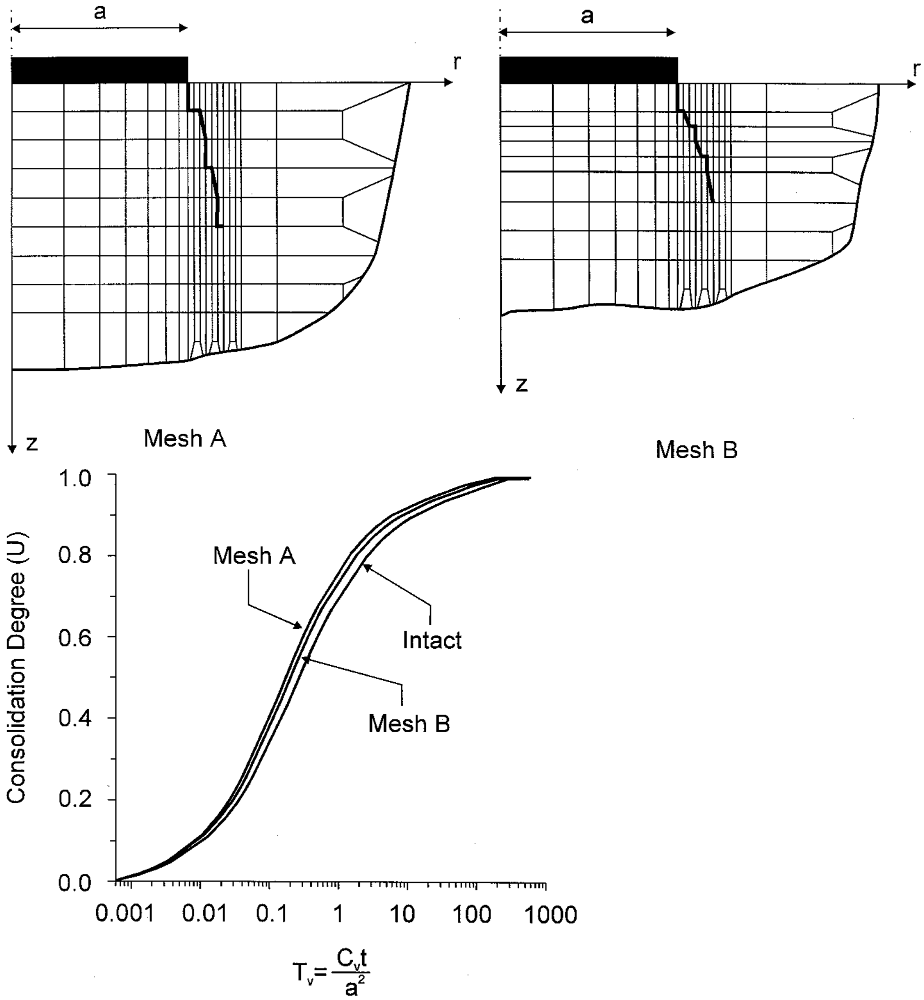

Figure 10. (a) Finite element discretization of quasi-static crack extension problem in a poroelastic medium with $\nu_u = 0.3$ and $\nu = 0.1$; and (b) Consolidation settlement of indentor.

5. Concluding remarks

Applications of the theory of poroelasticity have largely focused on problems related to conventional geomechanical applications where the emphasis has been on the evaluation of deformations and consolidation rates in such media subjected primarily to external loads. As the applications of the theory diversify, attention needs to be focused on other aspects of importance. With porous brittle geomaterials which are saturated with fluids, the initiation and extension cracks are recognized as an area of both practical and fundamental interest. This paper has documented the application of finite element techniques to the study of fractures in poroelastic media. It is shown that the evaluation of time-dependent variation of stress intensity factors at stable cracks can be computed quite conveniently using the time-dependent analysis of the associated finite element problem. The methodologies that are developed for the study of stable cracks can also be extended to examine the problem of crack extension. The mixed mode crack extension criteria applicable to the porous material can be directly utilized in the examination of the crack extension, via an incremental scheme. The incremental nature of the iterative scheme allows the time-dependent analysis of the problem where the pore pressure and tractions in the geomaterial fabric and displacements are appropriately adjusted. The searching scheme for the identification of orientation of crack extension is achieved through either a node splitting technique at inter-element boundaries or through an element. This procedure also prevents overlapping at crack boundaries throughout the time-dependent analysis and accounts for the continual updating of the boundary conditions. The numerical scheme is used to obtain solutions to a number of problems of interest to geomechanics and the accuracy of the computational scheme has been verified with known analytical solutions and limiting cases recovered through analogous problems in classical elasticity.

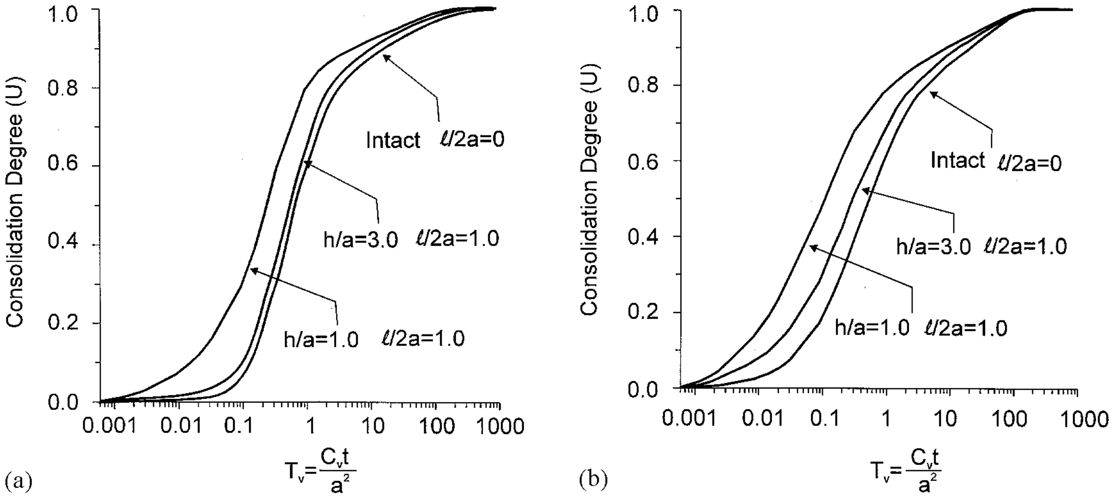

Figure 11. Consolidation response of a poroelastic half-space containing a flat penny-shaped co-axial crack; (a) impermeable bonded interface; and (b) permeable bonded interface.

Appendix: Formulation of the finite element matrices

Assume that $\boldsymbol{u} = \boldsymbol{N}^u \bar{\boldsymbol{u}}$; $\boldsymbol{p} = \boldsymbol{N}^p \bar{\boldsymbol{p}}$; and $\varepsilon = B\bar{\boldsymbol{u}}$ where $\boldsymbol{N}^u$ and $\boldsymbol{N}^p$ are the nodal shape functions for displacements and pore pressures, respectively. Therefore

$$[\mathbf{K}] = \int_\Omega B^T D B \, \mathrm{d}\Omega, \tag{17}$$

where D is the elasticity matrix for the geomaterial skeleton. Also

$$[\mathbf{C}] = \alpha \int_\Omega N^u \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} N^p \, \mathrm{d}\Omega, \tag{18}$$

$$[\mathbf{H}] = \kappa \int_\Omega \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} N^p \Bigg\}^T \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} N^p \Bigg\} \, \mathrm{d}\Omega, \tag{19}$$

$$[\mathbf{E}] = \alpha \int_\Omega N^{p^T} \frac{1}{M} N^p \, \mathrm{d}\Omega, \tag{20}$$

where

$$M = \frac{2\kappa\mu B^2(1-2\nu)(1+\nu_u)^2}{9(1-2\nu_u)(\nu_u-\nu)}.$$

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