Pergamon

Mechanics Research Communications, Vol. 23, No. 6, pp. 621–624, 1996

0093-6413/96 $12.00 + .00

PII S0093-6413(96)00067-5

ON THE PROBLEM OF AN ELECTRIFIED DISC LOCATED AT THE CENTRAL OPENING OF A COPLANAR EARTHED SHEET

A.P.S. Selvadurai
Department of Civil Engineering and Applied Mechanics
McGill University, Montreal, QC., Canada, H3A 2K6

(Received 28 February 1996; accepted for print 19 August 1996)

ABSTRACT

This paper develops an approximate solution for the capacity of an electrified circular disc located at the central opening of a coplanar earthed sheet of infinite extent.

INTRODUCTION

The determination of the electrostatic potential of a circular disc which is charged to a prescribed potential constitutes a fundamental problem both in electrostatics and applied mathematics. The problem has been investigated by a number of celebrated physicists and applied mathematicians and extended by other researchers to include interaction between the electrified boundaries and other earthed boundaries. Extensive accounts of these developments are given among others by Jeans [1], Smythe [2] and Sneddon [3]. In this paper we focus attention on the problem of an electrified disc which is placed within the central opening of an earthed co-planar sheet (Figure 1). The mixed boundary value problem associated with the problem is reduced to the solution of two coupled integral equations. These equations are solved in an approximate fashion to generate a series expansion solution for the capacity C of the system.

THE ELECTRIFIED DISC PROBLEM

We consider the axisymmetric problem related to the electrified disc of radius a which is raised to unit potential and placed centrally within the opening (radius b) of a co-planar earthed sheet of infinite extent. The problem can be formulated as a mixed boundary value problem applicable to a halfspace region. Considering a Hankel transform development of Laplace's equation, it can be shown that the harmonic function applicable to the region

$z \geq 0$ is

$$\Phi(r,z) = \int_0^\infty \varphi(\xi)\, e^{-\xi z} J_0(\xi r)\, d\xi \tag{1}$$

where $\varphi(\xi)$ is an arbitrary function which needs to be determined by satisfying the mixed boundary conditions

$$\Phi(r,0) = 1 \quad ; \quad 0 \leq r \leq a \tag{2}$$

$$\left[\frac{\partial \Phi}{\partial z}\right]_{z=0} = 0 \quad ; \quad a < r < b \tag{3}$$

$$\Phi(r,0) = 0 \quad ; \quad b \leq r < \infty \tag{4}$$

Using (1), the mixed boundary conditions (2) to (4) yield the following system of integral equations

$$\int_0^\infty \varphi(\xi)\, J_0(\xi r)\, d\xi = 1 \quad ; \quad 0 \leq r \leq a \tag{5}$$

$$\int_0^\infty \xi \varphi(\xi)\, J_0(\xi r)\, d\xi = 0 \quad ; \quad a < r < b \tag{6}$$

$$\int_0^\infty \varphi(\xi)\, J_0(\xi r)\, d\xi = 0 \quad ; \quad b \leq r < \infty \tag{7}$$

We assume that $\varphi(\xi)$ admits a representation

$$\int_0^\infty \xi \varphi(\xi)\, J_0(\xi r)\, d\xi = \begin{cases} \varphi_1(r) & ; \quad 0 < r < a \\ \varphi_3(r) & ; \quad b < r < \infty \end{cases} \tag{8}$$

Using the Hankel inversion theorem, the integral equations can be reduced to the forms

$$\int_0^a \lambda \varphi_1(\lambda)\, L(\lambda, r)\, d\lambda + \int_0^\infty \lambda \varphi_3(\lambda)\, L(\lambda, r)\, d\lambda = 1 \quad ; 0 \leq r \leq a \tag{9}$$

$$\int_0^a \lambda \varphi_1(\lambda)\, L(\lambda, r)\, d\lambda + \int_0^\infty \lambda \varphi_3(\lambda)\, L(\lambda, r)\, d\lambda = 0 \quad ; b \leq r < \infty \tag{10}$$

where

$$\begin{aligned} L(\lambda, r) &= \int_0^\infty J_0(\xi\lambda)\, J_0(\xi r)\, d\xi \\ &= \int_0^{\min(\lambda,r)} \frac{ds}{[(\lambda^2 - s^2)(r^2 - s^2)]^{1/2}} = \int_{\max(\lambda,r)}^\infty \frac{ds}{[(s^2 - \lambda^2)(s^2 - r^2)]^{1/2}} \end{aligned} \tag{11}$$

Introducing the functions

$$F_1(s) = \int_s^a \frac{\lambda \varphi_1(\lambda)\, d\lambda}{(\lambda^2 - s^2)^{1/2}} \quad ; \quad 0 \leq s \leq a \tag{12}$$

$$F_3(s) = \int_s^b \frac{\lambda \varphi_3(\lambda)\, d\lambda}{(s^2 - \lambda^2)^{1/2}} \quad ; \quad b \leq s \leq \infty \tag{13}$$

and the substitutions

$$s = as_1 \quad ; \quad u = bu_1 \quad ; \quad c = \frac{a}{b} \tag{14}$$

the integral equations governing the problem are reduced to the following

$$F_1(s_1) + \frac{2}{\pi} \int_1^\infty \frac{u_1 F_3(u_1)\, du_1}{(u_1^2 - c^2 s_1^2)} = 1 \quad ; \quad 0 \leq s_1 < 1 \tag{15}$$

$$F_3(s_1) + \frac{2 s_1 c}{\pi} \int_0^1 \frac{F_1(u_1)\, du_1}{(s_1^2 - c^2 u_1^2)} = 0 \quad ; \quad 1 \leq s_1 < \infty \tag{16}$$

The coupled integral equations can be solved in an approximate manner using a variety of numerical techniques (see e.g. Delves and Mohamed [4]). In this paper, however, we present a solution scheme which utilizes power series approximations for $F_1(s_1)$ and $F_3(s_1)$ in terms of a small non-dimensional parameter. In this case, the parameter is chosen as $c (= a/b) < 1$. We assume that $F_1(s_1)$ and $F_3(s_1)$ admit representations

$$[F_1(s_1) \quad ; F_3(s_1)] = \sum_{i=0}^{N} c^i \left[\chi_i^{(1)}(s_1) \quad ; \chi_i^{(3)}(s_1)\right] \tag{17}$$

Expanding $(u_1^2 - c^2 s_1^2)^{-1}$ and $(s_1^2 - c^2 u_1^2)^{-1}$ also in power series in terms of the parameter c and substituting these into the coupled integral equations (15) and (16) we can determine the functions $\chi_i^{(1)}(s_1)$ and $\chi_i^{(3)}$ up to any required order in c.

Considering both sides of the electrified disc, the capacity of the disc can be obtained from the result

$$C = -\frac{2a}{\pi} \int_0^1 F_1(s_1)\, ds_1 \tag{18}$$

Avoiding details of calculations it can be shown that

$$\begin{aligned} C \ &= \ \frac{2a}{\pi} \left[1 + \left(\frac{4}{\pi^2}\right) c + \left(\frac{16}{\pi^4}\right) c^2 + \left(\frac{64}{\pi^6} + \frac{8}{9\pi^2}\right) c^3 \right. \\ &+ \ \left(\frac{64}{9\pi^4} + \frac{256}{\pi^8}\right) c^4 + \left(\frac{92}{225\pi^2} + \frac{384}{9\pi^6} + \frac{1024}{\pi^{10}}\right) c^5 \\ &+ \ \left. 0\left(c^6\right) \right] \end{aligned} \tag{19}$$

As is evident when $b \to \infty$, the result (19) reduces to the classical result for the capacity of a electrified disc in an unbounded region [1-3]. The approximation (19) is meaningful for $(a/b) \ll 1$.

The result for the three-part boundary value problem examined in this paper also has applications to the study of a contact problem in elasticity. The solution for the load (P) - displacement (Δ) relationship for a rigid circular punch (radius a) indenting the surface of an incompressible (rubber-like) elastic halfspace which is restrained at the surface over the region $r\epsilon\ (b,\infty)$ can be obtained in the form $P = 4\pi G\Delta C$ where G is the linear elastic shear modulus of the incompressible materials and C is defined by (19).

REFERENCES

[1] J. Jeans, (1908) *The Mathematical Theory of Electricity and Magnetism*, Cambridge University Press, Cambridge.

[2] W.R. Smythe, (1968) *Static and Dynamic Electricity*, McGraw-Hill, New York.

[3] I.N. Sneddon, (1966) *Mixed Boundary Value Problems in Potential Theory*, North Holland Publ. Co., Amsterdam.

[4] J.M. Delves and J.L. Mohamed, (1985) *Computational Methods for Integral Equations*, Cambridge University Press, Cambridge.

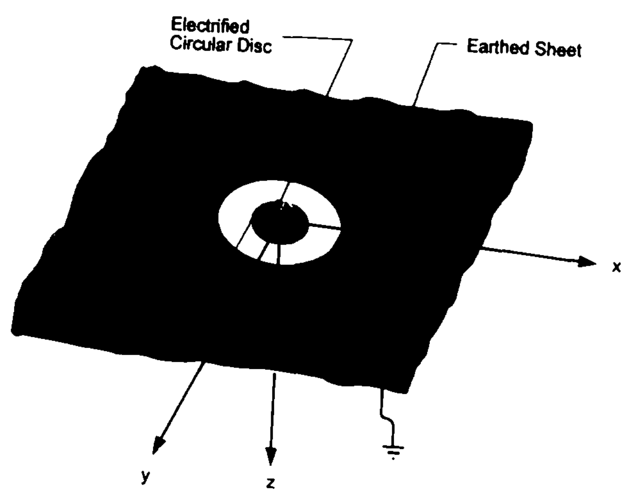

Figure 1. Electrified disc at the central circular opening of a co-planar earthed sheet.