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PII: S0955-7997(96)00029-X

Engineering Analysis with Boundary Elements 17 (1996) 287–294

0955-7997/96/$15.00

On integral equation approaches to the mechanics of fibre–crack interaction

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The paper examines the problem of a penny-shaped crack which is formed by the development of a crack in both the fibre and the matrix of a composite consisting of an isolated elastic fibre located in an elastic matrix of infinite extent. The composite region is subjected to a uniform strain field in the direction of the fibre. The paper presents two integral-equation based approaches for the analysis of the problem. The first approach considers the formulation of the complete integral equations governing the associated elasticity problem for a two material region. The second approach considers the boundary integral equation formulation of the problem. Both methods entail the numerical solution of the governing integral equations. The solutions to these integral equations are used to evaluate the stress intensity factor at the boundary of the penny-shaped crack. Copyright © 1996 Elsevier Science Ltd

Key words: Fibre–crack interaction, integral equations, fractured fibre, composite material, fragmentation tests, stress intensity factors, boundary element method.

1 INTRODUCTION

The bulk behaviour of fibre reinforced elastic composites is strongly influenced by processes at the micro-mechanical level. In particular, the fracture behaviour of the composite can be influenced by local fracture of fibres, the fracture of the matrix and delamination at the fibre–matrix interface. Examples of defects occurring in fibre reinforced composites are given by Kelly,[1] Sih,[2] Backlund,[3] Sih and Chen,[4] Hull,[5] Selvadurai,[6,7] Marshall *et al.*,[8] Stang[9] and Dvorak.[10] This paper examines the idealized problem of the interaction between an isolated fractured cylindrical elastic fibre and the surrounding matrix in the particular instance where the perfect bond between the fibre and the matrix initiates a matrix fracture in the form of a penny-shaped crack. The problem is of importance to the examination of 'fragmentation tests' which are conducted to determine the effectiveness of coupling agents which enhance the adhesive bond between the fibre and the matrix.[11,12] Examples of this mode of failure are also given by Chamis.[13] The mechanics of interaction between fibres and cracks has been investigated by a number of researchers. The axisymmetric problems related to a penny-shaped crack which is surrounded by regular arrays of fibres located in the vicinity of the crack have been examined by Pacella and Erdogan[14] and Narayan and Erdogan.[15] Dhaliwal *et al.*[16] and Singh *et al.*[17] have examined problems related to interior penny-shaped cracks located in the fibre and external matrix cracks for isolated elastic fibres embedded in an elastic medium. The study by Wijeywickrema *et al.*[18] considered the interaction between an intact elastic fibre and an annular crack located in the matrix surrounding the elastic fibre.

In this paper we examine the problem of the interaction between a fractured cylindrical elastic fibre and the surrounding matrix in the particular instance where the perfect bond between the fibre and the matrix initiates matrix fracture in the form of a penny-shaped crack. The mathematical analysis of the problem is reduced to the solution of a mixed boundary value problem for a halfspace region which can be further reduced to three coupled integral equations. The numerical analysis of these integral equations is used to evaluate the crack opening mode stress intensity factor at the boundary of the crack for the particular instance when the composite region is subjected to a state of uniaxial strain. The results derived from the numerical solution of the integral equations are used to check the accuracy of the boundary integral equation solution of the analogous problem, where the stress singularities at the boundary of the matrix crack are appropriately modelled.

2 THE FIBRE–MATRIX CRACK PROBLEM: INTEGRAL EQUATION APPROACH

The problem under consideration refers to an elastic fibre of radius a which is located in an isotropic elastic matrix of infinite extent. At the cracked fibre location a penny-shaped crack of radius b extends to the matrix region (Fig. 1). The composite region containing the fibre matrix crack is subjected to a uniform axial strain ϵ_0.

We first consider the auxiliary problem where an intact fibre–matrix region is subjected to a uniform axial strain ϵ_0. It can be shown that in the presence of complete bonding at the fibre–matrix interface, the axial stresses in the fibre (superscript 'f') on matrix (superscript 'm') are given by

$$\sigma_{zz}^{(\mathrm{f})}(r,z) = \Omega\sigma_0 \tag{1}$$

$$\sigma_{zz}^{(\mathrm{m})}(r,z) = \sigma_0 \tag{2}$$

where

$$\sigma_0 = E_{\mathrm{m}}\epsilon_0 \tag{3}$$

$$\Omega = \frac{\mu_{\mathrm{f}}}{\mu_{\mathrm{m}}}\left\{\frac{(1+\nu_{\mathrm{f}})}{(1+\nu_{\mathrm{m}})} + \frac{2\nu_{\mathrm{f}}(\nu_{\mathrm{f}}-\nu_{\mathrm{m}})}{(1+\nu_{\mathrm{m}})[1-2\nu_{\mathrm{f}}+\mu_{\mathrm{f}}/\mu_{\mathrm{m}}]}\right\} \tag{4}$$

where μ_i and ν_i are, respectively, the linear elastic shear moduli and Poisson's ratios for the fibre ($i = \mathrm{f}$) and matrix ($i = \mathrm{m}$) regions. In order to render the region $r \in (0,b)$ free of normal tractions, corresponding to a traction free fibre–matrix crack region, a corrective solution must be incorporated. Considering the symmetry of the cracked fibre ($r \in (0,a)$)-matrix crack ($r \in (a,b)$) region about the plane $z = 0$ the corrective solution can be formulated as a mixed boundary value problem associated with a two-domain halfspace region occupying $z \geq 0$. The boundary and interface conditions corresponding to the corrective solution are as follows:

$$\begin{aligned}
u_z^{(\mathrm{f})}(a,z) &= u_z^{(\mathrm{m})}(a,z); & 0 \leq z < \infty\\
u_r^{(\mathrm{f})}(a,z) &= u_r^{(\mathrm{m})}(a,z); & 0 \leq z < \infty\\
\sigma_{rr}^{(\mathrm{f})}(a,z) &= \sigma_{rr}^{(\mathrm{m})}(a,z); & 0 \leq z < \infty\\
\sigma_{rz}^{(\mathrm{f})}(a,z) &= \sigma_{rz}^{(\mathrm{m})}(a,z); & 0 \leq z < \infty
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\sigma_{rz}^{(\mathrm{f})}(r,0) &= 0; & 0 < r < a\\
\sigma_{rz}^{(\mathrm{m})}(r,0) &= 0; & a < r < \infty\\
\sigma_{zz}^{(\mathrm{f})}(r,0) &= -\Omega\sigma_0; & 0 < r < a\\
\sigma_{zz}^{(\mathrm{m})}(r,0) &= -\sigma_0; & a < r < b\\
u_z^{(\mathrm{m})}(r,0) &= 0; & b \leq r < \infty
\end{aligned} \tag{6}$$

In addition, the displacement and stress fields in the two domain halfspace region should satisfy the regularity conditions

$$u_\alpha^{(i)}, \quad \sigma_{\alpha\beta}^{(i)} \to 0 \text{ as } (r^2+z^2)^{1/2} \to \infty; \qquad i = \mathrm{f}, \mathrm{m}$$

For the solution of the axisymmetric problem governing the corrective state of stress required to render the crack region traction free, it is convenient to employ the strain potential approach proposed by Love.[19] It can be shown that in the absence of body forces, the displacement and stress fields can be expressed in terms of a function $\chi^{(i)}(r,z)$, ($i = \mathrm{f}, \mathrm{m}$), which satisfies

$$\nabla^2\nabla^2\chi^{(i)}(r,z) = 0 \tag{7}$$

where ∇^2 is Laplace's operator referred to the cylindrical polar coordinate system, i.e.

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{8}$$

Relevant solutions of (7) applicable to the fibre and matrix regions take the forms (see, for example, Sneddon[20])

$$\begin{aligned}
\chi^{(\mathrm{f})}(r,z) = &-2\mu_{\mathrm{f}}\int_0^\infty s^{-3}A_1(s)[2\nu_{\mathrm{f}}+sz]\mathrm{e}^{-sz}J_0(sr)\,\mathrm{d}s\\
&-2\mu_{\mathrm{f}}\int_0^\infty s^{-2}[\{A_2(s)+4(1-\nu_{\mathrm{f}})A_3(s)\}\\
&\times I_0(sr) - srA_3(s)I_1(sr)]\sin(sz)\,\mathrm{d}z
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
\chi^{(\mathrm{m})}(r,z) = &-2\mu_{\mathrm{m}}\int_0^\infty s^{-3}B_1(s)[2\mu_{\mathrm{m}}+sz]\mathrm{e}^{-sz}J_0(sr)\,\mathrm{d}s\\
&-2\mu_{\mathrm{m}}\int_0^\infty s^{-2}[\{B_2(s)-4(1-\nu_{\mathrm{m}})B_3(s)\}\\
&\times K_0(sr) - srB_3(s)K_1(sr)]\sin(sz)\,\mathrm{d}z
\end{aligned} \tag{10}$$

where $A_1(s), A_2(s)\ldots$ etc., are unknown functions and $J_\rho(\alpha), K_\rho(\alpha)$ and $I_\rho(\alpha)$ are, respectively Bessel functions

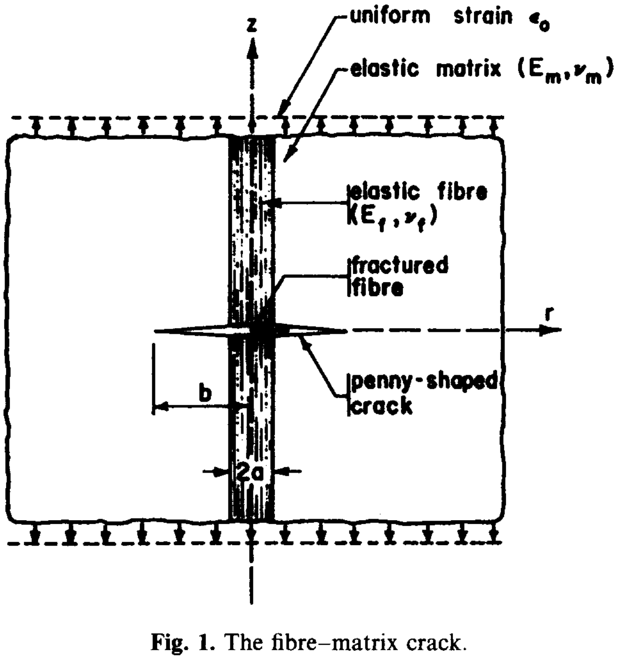

Fig. 1. The fibre–matrix crack.

of the first kind, modified Bessel functions of the first kind and modified Bessel functions of the second kind all of order $\rho(\geq 0)$. Using the continuity conditions (5) for the fibre–matrix interface $r = a$; $z(0, \infty)$ and the mixed boundary conditions (6), on the plane $z = 0$, the analysis of the correction problem can be reduced to the solution of a set of coupled Fredholm-type integral equations of the form

$$F_1(t) - \frac{2}{\pi}(b^2 - t^2)^{1/2}\int_b^\infty \frac{uF_2(u)\,\mathrm{d}u}{(u^2 - t^2)(u^2 - b^2)^{1/2}} + \frac{2}{\pi}\int_a^b F_1(u)\,S_1(u,t)\,\mathrm{d}u + \frac{2}{\pi}\int_b^\infty F_2(u)\,S_1(u,t)\,\mathrm{d}u + \frac{4}{\pi^2}\int_0^a F_3(u)\,S_2(u,t)\,\mathrm{d}u = (b^2 - t^2)^{1/2}; \quad a < t < b \tag{11}$$

$$F_2(u) + \frac{2}{\pi}\frac{t}{(t^2 - b^2)^{1/2}}\int_a^b \frac{F_1(u)(b^2 - u^2)^{1/2}}{(t^2 - u^2)}\,\mathrm{d}u = 0; \quad b < t < \infty \tag{12}$$

$$F_3(t) + \frac{2}{\pi}\int_a^b F_1(u)\,S_3(u,t)\,\mathrm{d}u + \frac{2}{\pi}\int_b^\infty F_2(u)\,S_3(u,t)\,\mathrm{d}u + \frac{4}{\pi^2}\int_0^a F_3(u)\,S_4(u,t)\,\mathrm{d}u = t\Gamma\Omega; \quad 0 < t < a \tag{13}$$

for the unknown functions $F_i(t)$; $(i = 1,2,3)$ and S_n $(n = 1,2,3,4)$ are kernel functions given by

$$\begin{aligned} S_1(u,t) = \int_0^\infty \mathrm{e}^{-su}\Big[&\{(3 - 2\nu_{\mathrm{m}})I_0(sa) + saI_1(sa) \\ &- suI_0(sa)\}P_1(s,t) - \Big\{\frac{sa}{2}(I_0(sa) + I_2(sa)) \\ &+ 2\nu_{\mathrm{m}}I_1(sa) - suI_1(sa)\Big\}P_2(s,t) \\ &+ \Gamma\Big\{sa[\tfrac{1}{2}I_0(sa) + saI_1(sa) - suI_0(sa)] \\ &+ I_1(sa)(su - 2\nu_{\mathrm{m}}) - \frac{sa}{2}I_2(sa)\Big\}P_3(s,t) \\ &- \Gamma\Big\{2sI_1(sa) + \frac{s^2a}{2}[I_0(sa) + I_2(sa)] \\ &- s^2uI_1(sa)\Big\}P_4(s,t)\Big]\,\mathrm{d}s \end{aligned} \tag{14}$$

$$\begin{aligned} S_2(u,t) = -\int_0^\infty \Big[&\{(3 - 2\nu_{\mathrm{f}})\sinh(su)K_0(sa) \\ &- saK_1(sa)\sinh(su) + su\cosh(su)K_0(sa)\} \\ &\times P_1(s,t) + \{(2\nu_{\mathrm{f}} - 1)K_1(sa)\sinh(su) \\ &- [sa\sinh(su)K_0(sa) - su\cosh(su)K_1(sa)]\} \\ &\times P_2(s,t) - \{sa[sa\sinh(su)K_1(sa) \\ &- su\cosh(su)K_0(sa)] + (1 - 2\nu_{\mathrm{f}})K_1(sa) \\ &\times \sinh(su) - suK_1(sa)\cosh(su)\}P_3(s,t) \\ &+ s\{\sinh(su)K_1(sa) + su\cosh(su)K_1(sa) \\ &- sa\sinh(su)K_0(sa)\}P_4(s,t)\Big]\,\mathrm{d}s \end{aligned} \tag{15}$$

$$\begin{aligned} S_3(u,t) = \int_0^\infty \mathrm{e}^{-su}\Big[&\{(3 - 2\nu_{\mathrm{m}})I_0(sa) + saI_1(sa) \\ &- suI_0(sa)\}R_1(s,t) - \Big\{\frac{sa}{2}[I_0(sa) + I_2(sa)] \\ &+ 2\nu_{\mathrm{m}}I_1(sa) - suI_1(sa)\Big\}R_2(s,t) \\ &+ \Gamma\Big\{sa[\tfrac{1}{2}I_0(sa) + saI_1(sa) - suI_0(sa)] \\ &+ I_1(sa)[su - 2\nu_{\mathrm{m}}] - \frac{sa}{2}I_2(sa)\Big\}R_3(s,t) \\ &- \Gamma\Big\{2sI_1(sa) + \frac{s^2a}{2}[I_0(sa) + I_2(sa)] \\ &- s^2uI_1(sa)\Big\}R_4(s,t)\Big]\,\mathrm{d}s \end{aligned} \tag{16}$$

$$\begin{aligned} S_4(u,t) = -\int_0^\infty \Big[&\{(3 - 2\nu_{\mathrm{f}})\sinh(su)K_0(sa) \\ &- saK_1(sa)\sinh(su) + su\cosh(su)K_0(sa)\} \\ &\times R_1(s,t) + \{(2\nu_{\mathrm{f}} - 1)K_1(sa)\sinh(su) \\ &- [sa\sinh(su)K_0(sa) - su\cosh(su)K_1(sa)]\} \\ &\times R_2(s,t) - \{sa[sa\sinh(su)K_1(sa) \\ &- su\cosh(su)K_0(sa)] + (1 - 2\nu_{\mathrm{f}})K_1(sa) \\ &\times \sinh(su) - suK_1(sa)\cosh(su)\}R_3(s,t) \\ &+ s\{\sinh(su)K_1(sa) + su\cosh(su)K_1(sa) \\ &- sa\sinh(su)K_0(sa)\}R_4(s,t)\Big]\,\mathrm{d}s \end{aligned} \tag{17}$$

The functions $P_i(s,t)$ and $R_i(s,t)$, $(i = 1,2,3,4)$ occurring in the kernel functions $S_i(u,t)$ are also functions of $I_0, I_1, K_0, K_1, \nu_{\mathrm{f}}, \nu_{\mathrm{m}}$ and $\Gamma = \mu_{\mathrm{m}}/\mu_{\mathrm{f}}$.

The coupled integral equations (11)–(13) can be numerically evaluated to generate results of engineering interest. These equations can be represented in the generalized form

$$\mathbf{F}(t) + \int_0^\infty \mathbf{F}(u)\mathbf{S}(u,t)\,\mathrm{d}u = \mathbf{B}(t); \quad 0 \leq t \leq \infty \tag{18}$$

where

$$\mathbf{F}(t) = \begin{cases} F_1(t); & 0 \leq t \leq a \\ F_2(t); & a \leq t < b \\ F_3(t); & b \leq t < \infty \end{cases} \tag{19}$$

and the prescribed function $\mathbf{B}(t)$ takes the form:

$$\mathbf{B}(t) = \begin{cases} t\Gamma\Omega; & 0 \le t < a \\ (b^2 - t^2)^{1/2}; & a \le t < b \\ 0; & b \le t < \infty \end{cases} \tag{20}$$

The kernel functions $\mathbf{S}(u,t)$ are defined as follows.

$$\mathbf{S}(u,t) = \begin{cases} \frac{4}{\pi^2} S_4(u,t); & (0 < u < a;\ 0 < t < a) \\ \frac{2}{\pi} S_3(u,t); & (a < u < \infty;\ 0 < t < a) \\ \frac{4}{\pi^2} S_2(u,t); & (0 < u < a;\ a < t < b) \\ \frac{2}{\pi} S_1(u,t); & (a < u < b;\ a < t < b) \\ \frac{2}{\pi} S_1(u,t) & \\ \quad - \frac{2}{\pi} \frac{(b^2 - t^2)^{1/2}}{(u^2 - t^2)(u^2 - b^2)^{1/2}}; & (b < u < \infty;\ a < t < b) \\ - \frac{2}{\pi} \frac{t(b^2 - u^2)^{1/2}}{(t^2 - b^2)^{1/2}(t^2 - u^2)}; & (a < u < b;\ b < t < \infty) \end{cases} \tag{21}$$

with $\mathbf{S}(u,t)$ being zero for all other intervals of $u \in (0,\infty)$ and $t \in (0,\infty)$. The numerical solution of integral equations of the type (18) has been discussed in the texts and articles by Atkinson,[21] Baker,[22] Delves and Mohamed[23] and Selvadurai *et al.*[24,25] Equation (18) is reduced to a discretized form

$$\mathbf{A}_{ij}\mathbf{F}_j = \mathbf{B}_j \tag{22}$$

where

$$\mathbf{F}_i = \mathbf{F}(t_i); \qquad \mathbf{B}_i = \mathbf{B}(t_i) \tag{23}$$

and the coefficients of the matrix $\mathbf{A}_{ij}$ are

$$\mathbf{A}_{ij} = \mathbf{\Delta}_{ij} + \mathbf{S}(t_i, t_j)\Delta t \tag{24}$$

where $\mathbf{\Delta}_{ij}$ is Kronecker's delta function and Δt is the discretization interval. The results of the numerical analysis can be used to determine results of particular interest to the behaviour of the fibre–matrix crack. In particular the stress intensity factor at the boundary of the penny-shaped crack can be obtained from the result

$$k_1 = \lim_{r \to b^+} [2(b - r)]^{1/2} \sigma_{zz}^{(m)}(r, 0) \tag{25}$$

It can be shown that

$$\frac{k_1}{k_0} = -\left[\frac{F_1(b) - F_2(b)}{b}\right] \tag{26}$$

where k_0 is the stress intensity factor for a penny-shaped crack of radius b which is situated in a homogeneous elastic solid which is subjected to a uniform axial strain ϵ_0 (or uniform axial stress $E_m\epsilon_0$) (see, for example, Sneddon[26]), i.e.

$$k_0 = \frac{2\sigma_0\sqrt{b}}{\pi} \tag{27}$$

3 THE FIBRE–MATRIX CRACK PROBLEM: BOUNDARY INTEGRAL EQUATION APPROACH

We focus attention on the application of a boundary integral equation technique to the solution of the fibre–matrix crack problem. The boundary integral equation applicable to axisymmetric deformations of the elastic medium can be written in the form (see, for example, Brebbia[27])

$$c_{lk} u_k^{(\alpha)} + \int_{\Gamma_{(\alpha)}} \{P_{lk}^{*(\alpha)} u_k^{(\alpha)} - u_{lk}^{*(\alpha)} P_k\} \frac{r}{r_i}\, d\Gamma = 0 \tag{28}$$

where $P_{lk}^{*(\alpha)}$ and $u_{lk}^{*(\alpha)}$ are the traction and displacement fundamental solutions. The superscript α refers to the fibre (f) and matrix (m) regions. The displacement fundamental solutions take the form

$$u_{rr}^{(\alpha)} = C_1 \left[\left\{ \frac{4(1 - \nu_a)(p^2 + z^{-2}) - p^2}{2r\bar{R}} \right\} K(m) - \left\{ \frac{(7 - 8\nu_\alpha)}{4r} \bar{R} - \left(\frac{e^4 - z^{-4}}{4r\bar{R}^3 m_1} \right) \right\} E(m) \right] \tag{29}$$

$$u_{rz}^{*(\alpha)} = C_1 \bar{z} \left[\frac{(e^2 + z^{-2})}{2\bar{R}^3 m_1} E(m) - \frac{1}{2\bar{R}} K(m) \right] \tag{30}$$

$$u_{zr}^{(\alpha)} = C_1 r_i \bar{z} \left[\frac{(e^2 - z^{-2})}{2r\bar{R}^3 m_1} E(m) + \frac{1}{2r\bar{R}} K(m) \right] \tag{31}$$

$$u_{zz}^{(\alpha)} = C_1 r_i \left[\frac{(3 - 4\nu_\alpha)}{\bar{R}} K(m) + \frac{z^{-2}}{\bar{R}^3 m_1} E(m) \right] \tag{32}$$

where

$$\bar{z} = z - z_i; \qquad \bar{r} = r + r_i; \qquad \rho^2 = r^2 + r_i^2 \tag{33}$$

$$e^2 = r^2 - r_i^2; \qquad \bar{R}^2 = \bar{r}^2 + \bar{z}^2;$$

$$C_1 = 1/4\pi\mu_\alpha(1 - \nu_\alpha) \tag{34}$$

and $K(m)$ and $E(m)$ are, respectively, the complete elliptic integrals of the first and second kind where $m = 4rr_i/\bar{R}^2$ and $m_1 = 1 - m$. The corresponding components of the fundamental solution for the traction $P_{lk}^{*(\alpha)}$ can be obtained by the proper manipulation of the results (29)–(32). In (28) c_{lk} is a constant ($= 0$ if the point is outside the body; $= \delta_{ij}$ if the point is inside the body and $= \delta_{ij}/2$ if the point is located at a smooth boundary).

By discretizing the boundaries Γ_α into boundary elements, the boundary integral equation (28) can be replaced by its discretized equivalent. For an isoparametric boundary element, the geometric, displacement and traction variations can be represented in the form

$$x_i = \sum_\beta N^\beta(\xi) x_i^\beta = [N(\xi)]\{x_i\} \tag{35}$$

and

$$u_i = N(\xi)\{u_i\}; \qquad P_i = N(\xi)\{P_i\} \tag{36}$$

In the case of a quadratic element $\beta = 3$ and the shape functions $N(\xi)$ are given

$$N^{(1)}(\xi) = \frac{\xi(\xi-1)}{2}; \qquad N^{(2)}(\xi) = (1-\xi^2);$$
$$N^{(3)}(\xi) = \frac{\xi(\xi+1)}{2} \tag{37}$$

with $-1 \le \xi \le 1$. The discretized version of (28) can be written as

$$c_{lk}u_k^{(\alpha)} = \sum_e \int_1^{-1} P_{lk}^{*(\alpha)}[N(\xi)]\,|J|\,\frac{r}{r_i}\,\mathrm{d}\xi\{u_k\}^e$$
$$= \sum_e \int_{-1}^{1} u_{lk}^{*(\alpha)}[N(\xi)]\,|J|\,\frac{r}{r_i}\,\mathrm{d}\xi\{P_k\}^e \tag{38}$$

where e is the element number and $|J|$ is the boundary Jacobian, which, for the axisymmetric case reduces to

$$|J| = \left[\left(\frac{\partial r}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2\right]^{1/2} \tag{39}$$

Upon completion of the integrations and summations, eqn (38) can be written in matrix form

$$[\mathbf{H}^{(\alpha)}\ \mathbf{H}_\mathrm{I}^{(\alpha)}]\begin{bmatrix}\mathbf{u}^{(\alpha)}\\ \mathbf{u}_\mathrm{I}^{(\alpha)}\end{bmatrix} = [\mathbf{M}^{(\alpha)}\ \mathbf{M}_\mathrm{I}^{(\alpha)}]\begin{bmatrix}\mathbf{P}^{(\alpha)}\\ \mathbf{P}_\mathrm{I}^{(\alpha)}\end{bmatrix} \tag{40}$$

where $\mathbf{u}_\mathrm{I}^{(\alpha)}$ and $\mathbf{P}_\mathrm{I}^{(\alpha)}$ are respectively the displacements and tractions at the interface between the fibre and the matrix. For complete bonding at the interface

$$\mathbf{u}_\mathrm{I}^{(\mathrm{f})} = \mathbf{u}_\mathrm{I}^{(\mathrm{m})} = \mathbf{u}_\mathrm{I}; \qquad \mathbf{P}_\mathrm{I}^{(\mathrm{f})} = -\mathbf{P}_\mathrm{I}^{(\mathrm{m})} = \mathbf{P}_\mathrm{I} \tag{41}$$

Using the above constraints, the boundary element matrix equation (40) can be written as

$$\begin{bmatrix}\mathbf{H}^{(\mathrm{f})} & \mathbf{H}_\mathrm{I}^{(\mathrm{f})} & 0\\ 0 & \mathbf{H}_\mathrm{I}^{(\mathrm{m})} & \mathbf{H}^{(\mathrm{m})}\end{bmatrix}\begin{bmatrix}\mathbf{u}^{(\mathrm{f})}\\ \mathbf{u}_\mathrm{I}\\ \mathbf{u}^{(\mathrm{m})}\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{M}^{(\mathrm{f})} & \mathbf{M}^{(\mathrm{f})} & 0\\ 0 & -\mathbf{M}_\mathrm{I}^{(\mathrm{m})} & \mathbf{M}^{(\mathrm{m})}\end{bmatrix}\begin{bmatrix}\mathbf{P}^{(\mathrm{f})}\\ \mathbf{P}_\mathrm{I}\\ \mathbf{P}^{(\mathrm{m})}\end{bmatrix} \tag{42}$$

which is the resulting bi-material boundary element matrix equation for the fibre–matrix system. The two regions can be subjected to the following types of boundary conditions:

(i) prescribed displacement boundary conditions

$$u_i^{(\alpha)} = \bar{u}_i^0 \qquad (i = r, z) \tag{43}$$

(ii) prescribed traction boundary conditions

$$P_i^{(\alpha)} = \bar{P}_i \qquad (i = r, z) \tag{44}$$

depending upon the nature of the loading of the composite and the restraining influences.

We consider the boundary element modelling of the matrix fibre crack problem in which the crack tip is located in the matrix region. To model the appropriate form of the singularities in the displacement and stress fields we employ a singular traction quarter point boundary element where the displacements vary according to

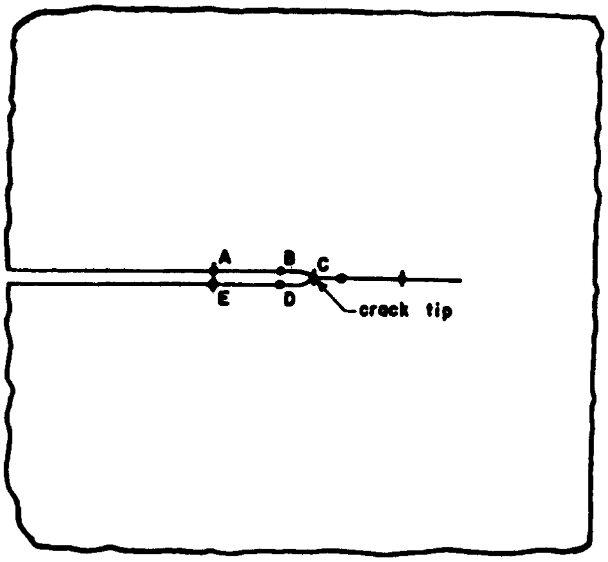

Fig. 2. Boundary element modelling of crack.

$$u_i^{(\alpha)} = b_0 + b_1\sqrt{r} + b_2 r \tag{45}$$

and the tractions vary according to

$$t_i^{(\alpha)} = \frac{c_0}{\sqrt{r}} + c_1 + c_2\sqrt{r} \tag{46}$$

where b_i and c_i $(i = 0, 1, 2)$ are constants. The accuracy and performance of crack tip elements have been investigated by Blandford *et al.*,[28] Smith and Mason[29] and Selvadurai and Au.[30] These studies indicate that the results derived via the singular traction quarter point element compare very accurately with known exact solutions.

To obtain the stress intensity factors k_1 and k_2 one could apply the displacement correlation method which utilizes the nodal displacements obtained from the crack tip elements at opposite sides of the crack. Referring to the Fig. 2, the stress intensity factors can be derived from the following general relationships

$$k_1 = \frac{\nu_\mathrm{m}}{(1-\nu_\mathrm{m})}\sqrt{\frac{\pi}{2l_0}}\{4[u_x(B) - u_x(D)]$$
$$+ u_x(E) - u_x(A)\} \tag{47}$$

$$k_2 = \frac{\nu_\mathrm{m}}{(1-\nu_\mathrm{m})}\sqrt{\frac{\pi}{2l_0}}\{4[u_y(B) - u_y(D)]$$
$$+ u_y(E) - u_y(A)\} \tag{48}$$

where l_0 is the length of the crack-tip element and A, B, C, D and E are the nodes of the two crack tip elements on either side of the crack. The boundary element scheme can be used to evaluate the relative magnitudes of the stress intensity factors k_I and k_II at

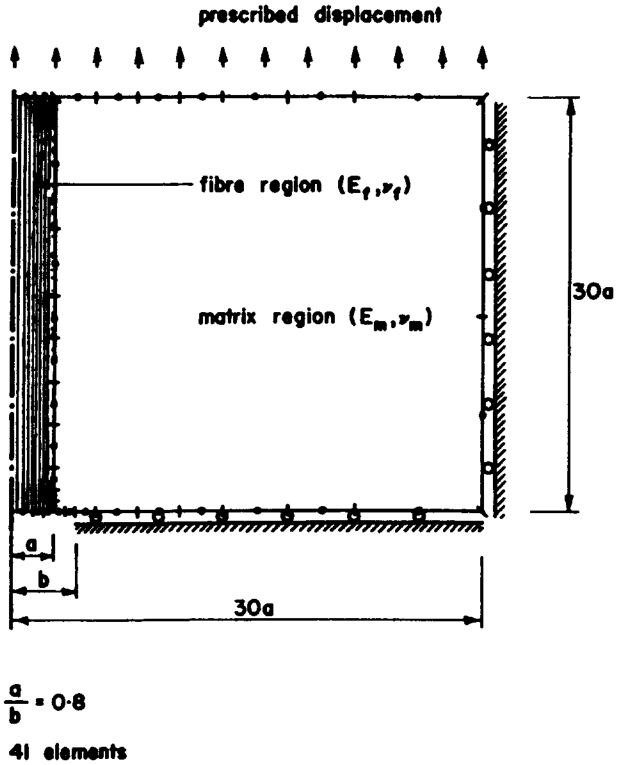

Fig. 3. Boundary element discretization of the fibre–matrix crack.

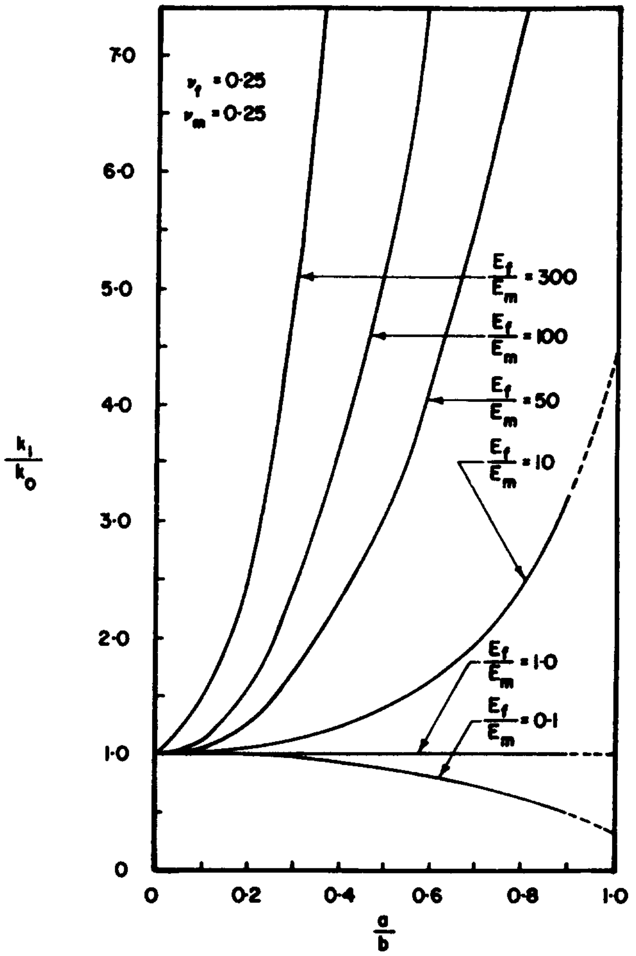

Fig. 4. Stress intensity factor for the matrix crack: analytical solution.

the crack tip. The boundary element mesh used in the computation of the stress intensity factor at the tip of the matrix–fibre crack is shown in Fig. 3.

4 NUMERICAL RESULTS

The numerical procedure for the solution of the integral equations, governing the analytical formulation of the fibre–matrix crack, given in section 2, was verified by comparison with analytical results. For example, the result for the stress intensity factor for a penny-shaped crack located in an isotropic elastic solid can be recovered as a limiting case of the general formulation when $E_f = E_m = E$; $\nu_f = \nu_m = \nu$. This reduction can also be confirmed from the result obtained from the general solution for the case when $a \to 0$. The numerical results for this particular case compares very accurately, to within 0·01%, with the exact closed form result (27). The material parameter influencing the stress intensity factor k_I include the fibre–matrix modular ratio (E_f/E_m), Poisson's ratios (ν_f, ν_m) and the fibre–crack radii ratio (a/b). For purposes of illustration, results for the stress intensity factors have been evaluated for a specified set of Poisson's ratios ($\nu_f = \nu_m = 0{\cdot}25$) and a range of values of the fibre–matrix modular ratio $\{(E_f/E_m) \in (0{\cdot}1, 300)\}$. Figure 4 illustrates the manner in which the normalized stress intensity factor k_I/k_0 is influenced by these material parameters and the fibre-crack aspect ratio. The boundary element procedure described in Section 4 was employed to evaluate the stress intensity factor at the tip of a crack located in the matrix region of fibre reinforced composite regions consisting of combinations of silicon carbide - epoxy, E-Glass - epoxy and stainless steel - aluminum (see Table 1). The comparison between results derived via the integral equation scheme and corresponding results obtained via the boundary element scheme are shown in Fig. 5.

In all computations, the geometrical parameter (a/b) is restricted to the range $(a/b) \in (0, 0{\cdot}9)$. In the limit as $(a/b) \to 1$, the fibre–matrix crack problem reduces to that of a cracked fibre which is embedded in an

Table 1. Elastic constants for components of fibre reinforced composites

Material	Young's modulus (GN/m^2)	Poisson's ratio
Epoxy resin	3·0	0·40
Silicon carbide	400·0	0·20
E-Glass	70·0	0·26
Stainless steel	207·0	0·30
Aluminium	69·0	0·33

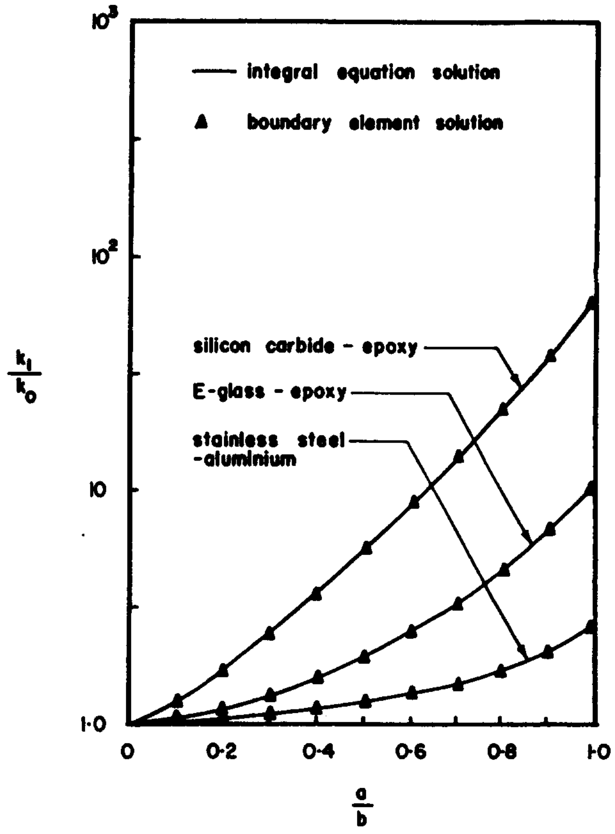

Fig. 5. Stress intensity factor for the matrix crack: comparison of analytical results and boundary element estimates.

undamaged matrix. In this instance, the character of the crack problem changes due to the fact that the crack tip is now located at the boundary of a bimaterial region. The stress singularity at the crack tip can be different from the conventional $1/\sqrt{r}$ type stress singularity applicable to the cases where the crack tip can be situated either in the fibre region or the matrix region. The order of the singularity appropriate for the completely cracked fibre problem is given by Zak and Williams[31] and this singular behaviour needs to be incorporated in the treatment of the fibre crack problem via both the analytical technique and the boundary element scheme.

5 CONCLUSIONS

The mechanics of an isolated cracked fibre problem is of some technological importance particularly in relation to the modelling of the mechanical response of damaged composites and in the assessment of the load transfer efficiency at the fibre–matrix interface. The mathematical modelling of the elastostatic problem related to the fibre–matrix crack can be examined by either considering an analytical approach which develops the integral equations governing an associated mixed boundary value problem or by considering a boundary integral equation approach which models all aspects of the fibre–matrix crack configuration. The numerical results derived for typical fibre reinforced composite materials with $E_f > E_m$ indicate that the stress intensity factor at the matrix crack boundary is always higher than that for a penny-shaped crack of equal radius located in a homogeneous matrix (without a fibre). This stress amplification increases as the fibre–matrix modular ratio increases. The stress intensity for the fibre–matrix crack can be compared with the critical value of the stress intensity factor k_{IC} applicable to the matrix material. Alternatively, the analysis can be used to establish the stable geometries of matrix cracks that can occur at cracked fibre locations.

The paper also develops a boundary integral equation approach to the study of this class of fibre–matrix crack problem. The boundary element approach is particularly suited to this type of problem since the result of specific interest, namely the stress intensity factor at the crack tip, can be evaluated directly from the boundary element scheme. The results obtained from the boundary element analysis of the fibre–matrix crack problem compare very accurately with equivalent results derived via the analytical formulation and the numerical solution of the associated integral equations.

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