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The Matrix-Fiber Crack in an Elastic Solid

In the presence of strong interface adhesion, the fracture of an embedded fiber can also result in the cracking of the surrounding matrix. While the orientation of such matrix cracks can be varied, the flat penny-shaped crack represents a critical crack orientation which is of particular interest to the study of the micromechanics of fracture processes in fiber-reinforced solids. This paper considers the axisymmetric problem of the uniform straining of a composite elastic solid which contains a penny-shaped crack occupying both the fiber and matrix regions. The isolated cracked fiber-matrix crack interaction is formulated as a mixed boundary value problem related to a two-domain half-space region. The resulting integral equations are solved in a numerical fashion to evaluate the stress intensity factor at the boundary of the penny-shaped crack. The numerical results presented in the paper illustrate the influence of the elasticity mismatch between the fiber and the matrix on the stress intensity factor at the crack-tip located in the matrix. The numerical results are presented for typical fiber-reinforced composites consisting of epoxy and ceramic matrices reinforced with silicon, glass, and kevlar fibers.

Introduction

The interaction of cracks and fibers is an important consideration in the modeling of micromechanics of damage in fiber-reinforced solids. Such damage can take a variety of forms depending upon the constitutive properties of the matrix, the reinforcing fibers and the conditions at the fiber-matrix interface (Backlund, 1981). With predominantly brittle elastic fiber-reinforced solids, the damage takes the form of matrix cracking, fiber-matrix interface delamination and fiber fracture (see, e.g., Dvorak, 1991). For example, matrix cracking in the presence of intact fibres results in the process of flaw and crack bridging which has been investigated by a number of researchers including Aveston et al. (1971), Kelly (1970), Sih (1979), and Selvadurai (1981). In recent years a number of investigators including Marshall et al. (1985), Budiansky et al. (1986), McCartney (1987), Stang (1987), and Rose (1987) have extended and improved the flaw bridging model to include various categories of linear and nonlinear flaw bridging processes. While matrix cracking and bridging usually results from loading of the composites in the direction of the fibers, interface delaminations can result from thermal loadings and loadings normal to the fiber directions and in-plane shearing. In many instances, interface delamination is usually treated as a two-dimensional plane-strain problem which considers the cross section of a unidirectional fiber-reinforced composite normal to the fiber direction. In this case, the classical solutions related to inclusion arrays in a matrix with delaminations at the fiber-matrix interface can be used to examine the problem. In addition to the classical studies by England (1965) and Dundurs and Mura (1964), references to developments in this area are given by Sendeckyi (1974), Chen and Sih (1985), and Mura (1988, 1989). These studies have also been extended to include cases where the interface debonded region is subjected to antiplane shear (Sendeckyi, 1974; Atkinson, 1979; Karihaloo and Viswanathan, 1985). The complete interface delamination problem is three-dimensional where regions of delamination can occur in a random fashion. Özbek and Erdogan (1969) and Erdogan and Özbek (1969) (see also Cherepanov, 1979) have examined the axisymmetric problem of an infinite elastic fiber embedded into an infinite elastic matrix where interface debonding occurs over a cylindrical region. These problems simulate the influence of thermal loadings of the debonded composite region. A majority of interface delamination problems involving composite regions assume loss of contact at the delaminated region. Processes such as interface frictional locking, slip, closure, etc., have an important influence on the mechanical behavior of the interface crack (Kelly, 1970; Comninou and Schmueser, 1979).

When the fiber-matrix interface exhibits strong adhesive toughness characteristics, fiber fracture usually results in the development of matrix fracture. This mode of fiber-matrix failure has been observed in fragmentation tests conducted to determined the adhesive toughness of fiber-matrix interfaces. Examples of this mode of failure are given by Chamis (1974) and van den Berg (1990), Busschen and Selvadurai (1995), and Selvadurai and Busschen (1995). The fracture topography of such combined fiber-matrix failure is usually quite complex. The matrix crack resulting in the presence of strong interface adhesion can be conoidal or a combination of conoidal and flat penny-shaped cracks. The interaction of cracks and fibers have been investigated by a number of researchers. Pacella and Erdogan (1974) and Narayan and Erdogan (1975) have examined the axisymmetric elastic problems where a penny-shaped matrix crack is surrounded by regular arrays of elastic fibers which are located in the vicinity of the crack. Dhaliwal et al. (1979) have examined the problem of the internal pressurization of a penny-shaped crack which is located at the interior of an embedded elastic fiber. These studies have also been extended to include the mechanics of an external matrix crack surrounding an elastic fiber (Singh et al. 1979). Recently, Wijeywickrema et al. (1990) have examined the interaction between an elastic fiber and an annular crack located in the matrix surrounding the fiber. Of related interest is the class of problems which deal with the interaction of inclusions and cracks. Taya and Mura (1981) and Taya and Chou (1983) have examined the problem of a penny-shaped crack which is located at the extremity of an elongated ellipsoidal elastic inclusion embedded in an elastic medium. Selvadurai (1991a) has utilized the boundary integral equation technique to examine the behavior of penny-shaped cracks

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which are located at the boundary of cylindrical elastic inclusions.

In this paper, we restrict attention to matrix cracking which accompanies the complete fracture of an isolated fiber. It is assumed that this fracture configuration results in a defect which can be modeled as a penny-shaped crack located in a single fibre-matrix composite solid. The treatment of the matrix as a domain of infinite extent is of course a highly idealized approximation to an actual situation of local damage in a fiber-reinforced composite. A more rigorous and realistic treatment of the cracked fiber problem related to a composite material should also take into consideration the influence of neighboring uncracked fibers, random locations of matrix fiber cracks, off-axis location of the matrix crack, and possible obliquity of the penny-shaped crack. The analysis of the single matrix-fiber crack presented in the paper is applicable to situations where low fiber volume concentrations results in fiber spacings which are large in comparison with the diameter of the fibers. The resulting idealization deals with the problem of an isolated fiber which is located in an isotropic elastic matrix of infinite extent. The composite region is weakened by a penny-shaped crack which occupies both the matrix and the single fiber regions (Fig. 1). The composite region is subjected to a state of uniaxial strain along the fiber direction. Owing to the symmetry of matrix-fiber crack region about the plane $z = 0$, the analysis of the problem can be formulated as a mixed boundary value problem related to a two-domain half-space region. The resulting system of Fredholm integral equations of the second kind are solved in a numerical fashion to generate the stress intensity factor at the boundary of the penny-shaped crack.

The Matrix-Fiber Crack Problem

An isotropic elastic matrix of infinite extent contains an elastic fibre which is perfectly bonded to it. The composite region is weakened by a penny-shaped crack which is generated by the fracture of the fiber and that of the matrix. The radius of the fiber and that of the matrix crack are denoted by a and b, respectively. The composite region is subjected to a uniform axial strain ϵ_0 at regions remote from the penny-shaped crack. The solution of the crack problem follows the conventional approach of combining two solutions; the first solution corresponds to the uniform straining of an uncracked composite region and a corrective solution which renders the region occupying the crack, free of surface tractions.

For the solution of the uncracked composite region we consider Lamé-type stress and displacement fields in the matrix (suffix m) and fiber (suffix f) regions which take the following forms (Timoshenko and Goodier, 1970):

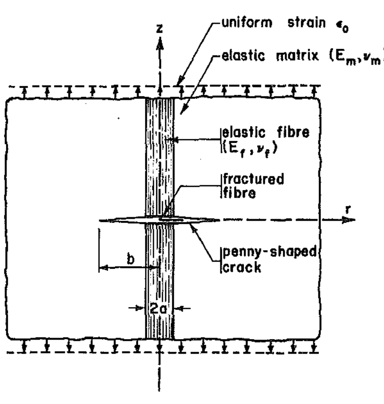

Fig. 1 The matrix-fiber crack in an elastic solid

$$\sigma_{rr}^{(m)}(r, z) = \frac{A_m}{r^2} + 2B_m \tag{1a}$$

$$\sigma_{\theta\theta}^{(m)}(r, z) = -\frac{A_m}{r^2} + 2B_m \tag{1b}$$

$$u_r^{(m)}(r, z) = \frac{r}{E_m}\left\{2B_m - \frac{A_m}{r^2} - \nu_m\left[2B_m + 4\nu_m B_m + E_m\epsilon_0 + \frac{A_m}{r^2}\right]\right\} \tag{1c}$$

and

$$\sigma_{rr}^{(f)}(r, z) = \sigma_{\theta\theta}^{(f)}(r, z) = 2C_f \tag{2a}$$

$$u_r^{(f)}(r, z) = \frac{r}{E_f}\{2C_f - \nu_f[2C_f + 4\nu_f C_f + E_f\epsilon_0]\} \tag{2b}$$

where E_i, ν_i are the elastic constants of the fiber ($i = f$) and matrix ($i = m$) regions and A_m, B_m, and C_f are constants which are to be determined by making use of the traction and displacement continuity conditions at the fiber-matrix boundary and the far-field stress state: i.e.,

$$\sigma_{rr}^{(f)}(a, z) = \sigma_{rr}^{(m)}(a, z) \tag{3a}$$

$$u_r^{(f)}(a, z) = u_r^{(m)}(a, z) \tag{3b}$$

$$\sigma_{rr}^{(m)}(r, z) \to 0 \text{ as } (r^2 + z^2)^{1/2} \to \infty. \tag{3c}$$

Considering the above results it can be shown that the axial stresses present in the fiber and matrix regions in the presence of complete interface bonding and uniaxial straining in the z-direction take the forms

$$\sigma_{zz}^{(f)}(r, z) = \Omega\sigma_0 \tag{4a}$$

$$\sigma_{zz}^{(m)}(r, z) = \sigma_0 \tag{4b}$$

where

$$\sigma_0 = E_m\epsilon_0 \tag{5a}$$

$$\Omega = \frac{\mu_f}{\mu_m}\left\{\frac{(1 + \nu_f)}{(1 + \nu_m)} + \frac{2\nu_f(\nu_f - \nu_m)}{(1 + \nu_m)\left[1 - 2\nu_f + \frac{\mu_f}{\mu_m}\right]}\right\} \tag{5b}$$

and μ_i are the linear elastic shear moduli of the fiber ($i = f$) and matrix ($i = m$) regions, respectively.

In the corrective solution, the composite region is subjected to equal and opposite axial stresses defined by (4a) and (4b) in the region corresponding to the crack. Considering the symmetry of the cracked fiber-matrix region about the plane $z = 0$, the corrective solution can be formulated as a mixed boundary value problem associated with a two-domain half-space region occupying the region $z \geq 0$. The boundary and interface conditions corresponding to the corrective solution are as follows:

$$u_z^{(f)}(a, z) = u_z^{(m)}(a, z); \quad 0 \leq z < \infty \tag{6}$$

$$u_r^{(f)}(a, z) = u_r^{(m)}(a, z); \quad 0 \leq z < \infty \tag{7}$$

$$\sigma_{rr}^{(f)}(a, z) = \sigma_{rr}^{(m)}(a, z); \quad 0 \leq z < \infty \tag{8}$$

$$\sigma_{rz}^{(f)}(a, z) = \sigma_{rz}^{(m)}(a, z); \quad 0 \leq z < \infty \tag{9}$$

$$\sigma_{rz}^{(f)}(r, 0) = 0; \quad 0 < r < a \tag{10}$$

$$\sigma_{rz}^{(m)}(r, 0) = 0; \quad a < r < \infty \tag{11}$$

$$\sigma_{zz}^{(f)}(r, 0) = -\Omega\sigma_0; \quad 0 < r < a \tag{12}$$

$$\sigma_{zz}^{(m)}(r, 0) = -\sigma_0; \quad a < r < \mathrm{b} \tag{13}$$

$$u_z^{(m)}(r, 0) = 0; \quad b \leq r < \infty. \tag{14}$$

In addition, the displacement and stress fields in the two-domain half-space region should satisfy the regularity conditions

$$u_{\alpha}^{(i)},\quad \sigma_{\alpha\beta}^{(i)} \to 0 \text{ as } (r^2+z^2)^{1/2} \to \infty;\quad i=f, m.$$

The solution of the mixed boundary value problem defined by (6) to (14) yields the appropriate singular stress field at the boundary of the penny-shaped crack which can be used to evaluate the stress intensity factor at the crack-tip.

The Governing Equations

For the solution of the axisymmetric problem governing the corrective stress state, it is convenient to employ the strain potential approach proposed by Love (1944). It can be shown that in the absence of body forces the displacement and stress fields can be expressed in terms of a function $\chi^{(i)}(r, z)$, ($i = f, m$) which satisfies

$$\nabla^2\nabla^2\chi^{(i)}(r, z) = 0 \tag{15}$$

where ∇^2 is Laplace's operator referred to the cylindrical polar coordinate system: i.e.,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{16}$$

The expressions for the displacements and stresses take the form

$$2\mu_i u_r^{(i)}(r, z) = -\frac{\partial^2\chi^{(i)}}{\partial r\partial z} \tag{17}$$

$$2\mu_i u_z^{(i)}(r, z) = \left[(1-2\nu_i)\nabla^2 + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\chi^{(i)} \tag{18}$$

and

$$\sigma_{rr}^{(i)}(r, z) = \frac{\partial}{\partial z}\left\{\nu_i\nabla^2 - \frac{\partial^2}{\partial r^2}\right\}\chi^{(i)} \tag{19}$$

$$\sigma_{\theta\theta}^{(i)}(r, z) = \frac{\partial}{\partial z}\left\{\nu_i\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right\}\chi^{(i)} \tag{20}$$

$$\sigma_{zz}^{(i)}(r, z) = \frac{\partial}{\partial z}\left\{(2-\nu_i)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\chi^{(i)} \tag{21}$$

$$\sigma_{rz}^{(i)}(r, z) = \frac{\partial}{\partial r}\left\{(1-\nu_i)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\chi^{(i)}, \tag{22}$$

respectively.

To formulate the mixed boundary value problem defined by (6) to (14) we seek solutions of (15) applicable to the fiber and matrix regions, respectively. A suitable biharmonic function applicable to the fiber region ($r\epsilon(0, a)$; $z\epsilon(0, \infty)$) can be taken as (Sneddon, 1951)

$$\chi^{(f)}(r, z) = -2\mu_f\int_0^\infty s^{-3}F_1(s)[2\nu_f + sz]e^{-sz}J_0(sr)ds$$
$$- 2\mu_f\int_0^\infty s^{-2}[\{C(s) + 4(1-\nu_f)D(s)\}I_0(sr)$$
$$- srD(s)I_1(sr)]\sin(sz)ds \tag{23}$$

where $F_1(s)$, $C(s)$, and $D(s)$ are unknown functions which are to be determined and $J_p(\alpha)$ and $I_p(\alpha)$ denote, respectively, Bessel functions of the first kind and modified Bessel functions of the second kind of order $p(\ge 0)$. The components of the displacement vector and the stress tensor for the fiber region can be obtained with the aid of Eqs. (17) to (22). A biharmonic function $\chi^{(m)}(r, z)$ applicable to the matrix region ($r\epsilon(a, \infty)$; $z\epsilon(0, \infty)$) takes the form

$$\chi^{(m)}(r, z) = -2\mu_m\int_0^\infty s^{-3}F_2(s)[2\nu_m + sz]e^{-sz}J_0(sr)ds$$
$$- 2\mu_m\int_0^\infty s^{-2}[\{A(s) - 4(1-\nu_m)B(s)\}K_0(sr)$$
$$- srK_1(sr)B(s)]\sin(sz)ds \tag{24}$$

where $A(s)$, $B(s)$, and $F_2(s)$ are unknown functions and $K_p(\alpha)$ are modified Bessel functions of the first kind of order $p(\ge 0)$. The components of the displacement vector and the stress tensor for the matrix region can be obtained by making use of (24) and expressions (17) to (22).

Integral Equations Governing the Matrix-Fiber Crack Problems

The integral expressions for $u_r^{(i)}$, $u_z^{(i)}$, $\sigma_{rr}^{(i)}$, $\sigma_{zz}^{(i)}$, and $\sigma_{rz}^{(i)}$ ($i = f, m$) obtained from (23) and (24) can be used to formulate the integral equations governing the matrix-fiber crack problem. Considering the matrix region we observe the boundary condition (11) is identically satisfied and the boundary conditions (13) and (14) yield the following integral equations:

$$\int_0^\infty F_2(s)J_0(sr)ds + \int_0^\infty s\{A(s)K_0(sr)$$
$$+ [2\nu_m K_0(sr) - srK_1(sr)]B(s)\}ds = \frac{\sigma_0}{2\mu_m};$$
$$a < r < b \tag{25}$$

$$\int_0^\infty s^{-1}F_2(s)J_0(sr)ds = 0;\quad b \le r < \infty. \tag{26}$$

Consider a trial solution of $F_2(s)$ of the form

$$F_2(s) = s\left[\int_a^b h(t)\cos(st)dt + \int_b^\infty g(t)\cos(st)dt\right] \tag{27}$$

where $h(t)$ and $g(t)$ are arbitrary functions and $g(\infty) = 0$. Substituting the above representation in (26) we obtain an Abel-type integral equation

$$\int_b^r \frac{g(t)dt}{(r^2-t^2)^{1/2}} + \int_a^b \frac{h(t)dt}{(r^2-t^2)^{1/2}} = 0;\quad b \le r < \infty. \tag{28}$$

The solution of (28) can be written as

$$g(t) = -\frac{2}{\pi}\frac{t}{(t^2-b^2)^{1/2}}\int_a^b \frac{h(u)(b^2-u^2)^{1/2}}{(t^2-u^2)}du;$$
$$b \le t < \infty. \tag{29}$$

Similarly, using the representation (27), the integral Eq. (25) can be reduced to the form

$$h(t) - \frac{2}{\pi}(b^2-t^2)^{1/2}\int_b^\infty \frac{ug(u)du}{(u^2-t^2)(u^2-b^2)^{1/2}}$$
$$+ \frac{2}{\pi}\int_0^\infty s\{A(s)L_1(s, t) + [2\nu_m L_1(s, t)$$
$$- sL_2(s, t)]B(s)\}ds = \frac{\sigma_0(b^2-t^2)^{1/2}}{\pi\mu_m};$$
$$a < t < b \tag{30}$$

where

$$L_1(s, t) = \int_t^b \frac{rK_0(sr)dr}{(r^2 - t^2)^{1/2}} \quad (31a)$$

$$L_2(s, t) = \int_t^b \frac{r^2K_1(sr)dr}{(r^2 - t^2)^{1/2}}. \quad (31b)$$

Considering the fiber-matrix interface continuity conditions (6) to (9), we obtain via a Fourier inversion the following:

$$saI_1(sa)D(s) - I_0(sa)C(s) - saK_1(sa)B(s) + K_0(sa)A(s)$$

$$= \frac{2}{\pi}\int_0^\infty u^{-1}[\{2(1-\nu_m)f_1 + uf_2\}F_2(u) - \{2(1-\nu_f)f_1 + uf_2\}F_1(u)]J_0(ua)du = X_1(s); \quad 0 \le s < \infty \quad (32)$$

$$I_1(sa)C(s) + \{4(1-\nu_f)I_1(sa) - saI_0(sa)\}D(s) + K_1(sa)A(s) - \{4(1-\nu_m)K_1(sa) + saK_0(sa)\}B(s)$$

$$= \frac{2}{\pi}\int_0^\infty u^{-1}[\{(2\nu_m - 1)f_3 + uf_4\}F_2(u) - \{(2\nu_f - 1)f_3 + uf_4\}F_1(u)]J_1(ua)du = X_2(s); \quad 0 \le s < \infty \quad (33)$$

$$\mu_m\{[K_1(sa) + saK_0(sa)]A(s) - [\{4(1-\nu_m) + s^2a^2\}K_1(sa) + (3-2\nu_m)saK_0(sa)]B(s)\} + \mu_f\{[I_1(sa) - saI_0(sa)]C(s) + [\{4(1-\nu_f) + s^2a^2\}I_1(sa) - (3-2\nu_f)saI_0(sa)]D(s)\}$$

$$= \frac{2}{\pi}\mu_f a\int_0^\infty\left[\{-f_3 + uf_4\}J_0(ua) + \{(1-2\nu_f)f_3 - uf_4\}\frac{J_1(ua)}{ua}\right]F_1(u)du - \frac{2}{\pi}\mu_m a\int_0^\infty\left[\{-f_3 + uf_4\}J_0(ua) + \{(1-2\nu_m)f_3 - uf_4\}\frac{J_1(ua)}{ua}\right]F_2(u)du$$

$$= \mu_f X_3(s); \quad 0 \le s < \infty \quad (34)$$

$$\mu_m\{A(s)K_1(sa) - [2(1-\nu_m)K_1(sa) + saK_0(sa)]B(s)\} + \mu_f\{C(s)I_1(sa) + [2(1-\nu_f)I_1(sa) - saI_0(sa)]D(s)\}$$

$$= \frac{2}{\pi}\frac{\mu_m}{s}\int_0^\infty uF_2(u)f_2J_1(ua)du - \frac{2}{\pi}\frac{\mu_f}{s}\int_0^\infty uF_1(u)f_2J_1(ua)du$$

$$= \frac{\mu_f}{s}X_4(s); \quad 0 \le s \le \infty \quad (35)$$

where

$$f_1 = \int_0^\infty e^{-uz}\sin(sz)dz = \frac{s}{(s^2 + u^2)} \quad (36)$$

$$f_2 = \int_0^\infty ze^{-uz}\sin(sz)dz = \frac{2su}{(s^2 + u^2)^2} \quad (37)$$

$$f_3 = \int_0^\infty e^{-uz}\cos(sz)dz = \frac{u}{(s^2 + u^2)} \quad (38)$$

$$f_4 = \int_0^\infty ze^{-uz}\cos(sz)dz = \frac{(u^2 - s^2)}{(u^2 + s^2)^2}. \quad (39)$$

In the sequel, it is convenient to adopt the following convention:

$$I_0(sa) = I_0; \quad I_1(sa) = I_1 \quad (40)$$

$$K_0(sa) = K_0; \quad K_1(sa) = K_1. \quad (41)$$

The Eqs. (32) to (35) can now be solved to evaluate the arbitrary functions $A(s)$, $B(s)$, $C(s)$, and $D(s)$ in terms of $X_i(s)$ ($i = 1, 2, 3, 4$) or consequently in terms of $F_1(s)$ and $F_2(s)$. These expressions are given in Appendix A.

Considering the boundary conditions (10) and (12) it can be shown that the boundary condition (10) is identically satisfied by $\sigma_{rz}^{(f)}$ and that the expression for $\sigma_{zz}^{(f)}$ yields the following integral equation:

$$\int_0^\infty F_1(s)J_0(sr)ds + \int_0^\infty s\{C(s)I_0(sr) - [2\nu_f I_0(sr) + srI_1(sr)]D(s)\}ds = \frac{\Omega\sigma_0}{2\mu_f}; \quad 0 < r < a. \quad (42)$$

Consider the representation

$$F_1(s) = s\int_0^a m(t)\sin(st)dt. \quad (43)$$

We note that the Eq. (42) can be rewritten in the form

$$\frac{\partial}{\partial r}\left\{r\int_0^\infty \frac{F_1(s)}{s}J_1(sr)ds\right\} + r\int_0^\infty s\{C(s) - 2\nu_f D(s)\}I_0(sr)ds - r^2\int_0^\infty s^2I_1(sr)D(s)ds = \frac{\Omega\sigma_0}{2\mu_f}; \quad 0 < r < a. \quad (44)$$

Using (43), (44) can be expressed in the form

$$\frac{\partial}{\partial r}\int_0^r \frac{tm(t)dt}{(r^2 - t^2)^{1/2}} + r\int_0^\infty s\{C(s) - 2\nu_f D(s)\}I_0(sr) - r^2\int_0^\infty s^2I_1(sr)D(s)ds = \frac{\Omega\sigma_0 r}{2\mu_f}; \quad 0 < r < a. \quad (45)$$

Making use of the following results

$$\int_0^t \frac{rI_0(sr)ds}{(t^2 - r^2)^{1/2}} = \frac{\sinh(st)}{s} \quad (46)$$

$$\int_0^t \frac{r^2I_1(sr)dr}{(t^2 - r^2)^{1/2}} = \frac{st\cosh(st) - \sinh(st)}{s^2}, \quad (47)$$

the integral Eq. (45) can be expressed in the form

$$m(t) + \frac{2}{\pi}\int_0^\infty [X_1(s)R_1(s, t) + X_2(s)R_2(s, t) + X_3(s)R_3(s, t) + X_4(s)R_4(s, t)]ds = \frac{\Omega\sigma_0 t}{\pi\mu_f}; \quad 0 < t < a \quad (48)$$

where

$$\eta R_1(s, t) = \sinh(st)(a_7a_{13} - a_8a_{12}) - (a_{13}I_0 + a_8I_1)\{\sinh(st)(2\nu_f - 1) + st\cosh(st)\} \quad (49)$$

$$\eta R_2(s, t) = \sinh(st)(a_7a_{14} - a_9a_{12}) - (a_{14}I_0 + a_9I_1)\{\sinh(st)(2\nu_f - 1) + st\cosh(st)\} \quad (50)$$

$$\eta R_3(s,t) = \sinh(st)(a_7a_{15} - a_{10}a_{12}) - (a_{15}I_0 + a_{10}I_1)\{\sinh(st)(2\nu_f - 1) + st\cosh(st)\} \quad (51)$$

$$\eta R_4(s,t) = \sinh(st)(a_7a_{16} - a_{11}a_{12}) - (a_{16}I_0 + a_{11}I_1)\{\sinh(st)(2\nu_f - 1) + st\cosh(st)\}. \quad (52)$$

Considering the integrals of Eqs. (32) to (35) containing $F_1(u)$, the integral expressions (36) to (39) and the representation (43) it can be shown that

$$-\frac{2}{\pi}\int_0^\infty \{2(1-\nu_f)f_1 + uf_2\}\frac{F_1(u)}{u}J_0(ua)du = -\frac{2}{\pi}\int_0^a [(3-2\nu_f)\sinh(su)K_0(sa) - saK_1(sa)\sinh(su) + su\cosh(su)K_0(sa)]m(u)du; \quad 0 < s < \infty \quad (53)$$

$$-\frac{2}{\pi}\int_0^\infty \{(2\nu_f - 1)f_3 + uf_4\}\frac{F_1(u)}{u}J_1(ua)du = -\frac{2}{\pi}\int_0^a [(2\nu_f-1)K_1(sa)\sinh(su) - \{saK_0(sa)\sinh(su) - suK_1(sa)\cosh(su)\}]m(u)du; \quad 0 < s < \infty \quad (54)$$

$$\frac{2}{\pi}a\mu_f\int_0^\infty \left\{(-f_3 + uf_4)J_0(ua) + [(1-2\nu_f)f_3 - uf_4]\frac{J_1(ua)}{ua}\right\}F_1(u)du = \frac{2}{\pi}\mu_f\int_0^a \{sa[sa\sinh(su)K_1(sa) - su\cosh(su)K_0(sa)] + (1-2\nu_f)K_1(sa)\sinh(su) - suK_1(sa)\cosh(su)\}m(u)du; \quad 0 < s < \infty \quad (55)$$

$$-\frac{2}{\pi}\frac{\mu_f}{s}\int_0^\infty uF_1(u)f_2J_1(ua)du = -\frac{2\mu_f}{\pi}\int_0^a [\sinh(su)K_1(sa) + su\cosh(su)K_1(sa) - sa\sinh(su)K_0(sa)]m(u)du; \quad 0 < s < \infty. \quad (56)$$

In a similar manner, the integrals of Eqs. (32) to (35) containing $F_2(u)$ can be expressed in terms of the functions $h(u)$ and $g(u)$ (defined by (27)) by the following expressions:

$$\frac{2}{\pi}\int_0^\infty u^{-1}[\{2(1-\nu_m)f_1 + uf_2\}F_2(u)J_0(ua)du = [(3-2\nu_m)I_0(sa) + saI_1(sa)] \times \left[\int_a^b h(u)e^{-su}du + \int_b^\infty g(u)e^{-su}du\right] - sI_0(sa)\left[\int_a^b uh(u)e^{-su}du + \int_b^\infty ug(u)e^{-su}du\right]; \quad 0 < s < \infty \quad (57)$$

$$\frac{2}{\pi}\int_0^\infty u^{-1}[\{(2\nu_m - 1)f_3 + uf_4\}F_2(u)]J_1(ua)du = -\left\{\frac{sa}{2}[I_0(sa) + I_2(sa)] + 2\nu_mI_1(sa)\right\} \times \left[\int_a^b h(u)e^{-su}du + \int_b^\infty g(u)e^{-su}du\right] + sI_1(sa)\left[\int_a^b uh(u)e^{-su}du + \int_b^\infty ug(u)e^{-su}du\right]; \quad 0 < s < \infty \quad (58)$$

$$-\frac{2}{\pi}a\mu_m\int_0^\infty \left[(-f_3 + uf_4)J_0(ua) + \{(1-2\nu_m)f_3 - uf_4\}\frac{J_1(ua)}{ua}\right]F_2(u)du = \mu_m\left[\int_a^b h(t)e^{-st}\left[sa\left\{\frac{1}{2}I_0(sa) + saI_1(sa) - stI_0(sa)\right\} + I_1(sa)(st - 2\nu_m) - \frac{sa}{2}I_2(sa)\right]dt\right] + \mu_m\left[\int_b^\infty g(t)e^{-st}\left[sa\left\{\frac{1}{2}I_0(sa) + saI_1(sa) - stI_0(sa)\right\} + I_1(sa)(st - 2\nu_m) - \frac{sa}{2}I_2(sa)\right]dt\right]; \quad 0 < s < \infty \quad (59)$$

$$\frac{2}{\pi}\mu_m\int_0^\infty uF_2(u)f_2J_1(ua)du = -s\mu_m\left[2I_1(sa) + \frac{sa}{2}(I_0(sa) + I_2(sa))\right] \times \left[\int_a^b h(u)e^{-su}du + \int_b^\infty e^{-su}g(u)du\right] + s^2\mu_mI_1(sa)\left[\int_a^b uh(u)e^{-su}du + \int_b^\infty ug(u)e^{-su}du\right]; \quad 0 < s < \infty. \quad (60)$$

Making use of the expressions (53) to (60) it is possible to obtain expressions for $X_i(s)$ ($i = 1, 2, 3, 4$) in terms of the functions $h(s)$, $g(s)$, and $m(s)$. These expressions are given in Appendix A. Using the results given in Eqs. (A24) to (A27), the integral Eq. (30) can be expressed in the form

$$H(t) - \frac{2}{\pi}(b^2 - t^2)^{1/2}\int_b^\infty \frac{uG(u)du}{(u^2-t^2)(u^2-b^2)^{1/2}} + \frac{2}{\pi}\int_a^b H(u)S_1(u,t)du + \frac{2}{\pi}\int_b^\infty G(u)S_1(u,t)du + \frac{4}{\pi^2}\int_0^a M(u)S_2(u,t)du = (b^2-t^2)^{1/2}; \quad a < t < b \quad (61)$$

where

$$H(t) = \frac{\pi\mu_m}{\sigma_0}h(t) \quad (62)$$

$$G(t) = \frac{\pi\mu_m}{\sigma_0}g(t) \quad (63)$$

$$M(t) = \frac{\pi \mu_m}{\sigma_0} m(t), \tag{64}$$

and the kernel functions S_1 and S_2 are given by

$$S_1(u, t) = \int_0^\infty e^{-su}[\{(3 - 2\nu_m)I_0(sa) + saI_1(sa) - suI_0(sa)\}P_1(s, t) - \left\{\frac{sa}{2}(I_0(sa) + I_2(sa)) + 2\nu_m I_1(sa) - suI_1(sa)\right\}P_2(s, t) + \Gamma\left\{sa\left[\frac{1}{2}I_0(sa) + saI_1(sa) - suI_0(sa)\right] + I_1(sa)(su - 2\nu_m) - \frac{sa}{2}I_2(sa)\right\}P_3(s, t) - \Gamma\left\{2sI_1(sa) + \frac{s^2a}{2}[I_0(sa) + I_2(sa)] - s^2uI_1(sa)\right\}P_4(s, t)\Bigg]ds \tag{65}$$

$$S_2(u, t) = -\int_0^\infty [\{(3 - 2\nu_f)\sinh(su)K_0(sa) - saK_1(sa)\sinh(su) + su\cosh(su)K_0(sa)\}P_1(s, t) + \{(2\nu_f - 1)K_1(sa)\sinh(su) - [sa\sinh(su)K_0(sa) - su\cosh(su)K_1(sa)]\}P_2(s, t) - \{sa[sa\sinh(su)K_1(sa) - su\cosh(su)K_0(sa)] + (1 - 2\nu_f)K_1(sa)\sinh(su) - suK_1(sa)\cosh(su)\}P_3(s, t) + s\{\sinh(su)K_1(sa) + su\cosh(su)K_1(sa) - sa\sinh(su)K_0(sa)\}P_4(s, t)]ds. \tag{66}$$

The expressions for the functions $P_i(s, t)$ $(i = 1, 2, 3, 4)$ occurring in Eqs. (65) and (66) are also given in Appendix A.

Using a similar procedure, the integral Eq. (48) can be reduced to the form

$$M(t) + \frac{2}{\pi}\int_a^b H(u)S_3(u, t)du + \frac{2}{\pi}\int_b^\infty G(u)S_3(u, t)du + \frac{4}{\pi^2}\int_0^a M(u)S_4(u, t)du = t\Gamma\Omega; \quad 0 < t < a \tag{67}$$

where

$$S_3(u, t) = \int_0^\infty e^{-su}\Bigg[\{(3 - 2\nu_m)I_0(sa) + saI_1(sa) - suI_0(sa)\}R_1(s, t) - \left\{\frac{sa}{2}[I_0(sa) + I_2(sa)] + 2\nu_m I_1(sa) - suI_1(sa)\right\}R_2(s, t) + \Gamma\left\{sa\left[\frac{1}{2}I_0(sa) + saI_1(sa) - suI_0(sa)\right] + I_1(sa)[su - 2\nu_m] - \frac{sa}{2}I_2(sa)\right\}R_3(s, t) - \Gamma\left\{2sI_1(sa) + \frac{s^2a}{2}[I_0(sa) + I_2(sa)] - s^2uI_1(sa)\right\}R_4(s, t)\Bigg]ds. \tag{68}$$

$$S_4(u, t) = -\int_0^\infty [\{(3 - 2\nu_f)\sinh(su)K_0(sa) - saK_1(sa)\sinh(su) + su\cosh(su)K_0(sa)\}R_1(s, t) + \{(2\nu_f - 1)K_1(sa)\sinh(su) - [sa\sinh(su)K_0(sa) - su\cosh(su)K_1(sa)]\}R_2(s, t) - \{sa[sa\sinh(su)K_1(sa) - su\cosh(su)K_0(sa)] + (1 - 2\nu_f)K_1(sa)\sinh(su) - suK_1(sa)\cosh(su)\}R_3(s, t) + s\{\sinh(su)K_1(sa) + su\cosh(su)K_1(sa) - sa\sinh(su)K_0(sa)\}R_4(s, t)]ds \tag{69}$$

where the expressions for the functions R_i $(i = 1, 2, 3, 4)$ occurring in the kernel functions $S_3(u, t)$ and $S_4(u, t)$ are given in Appendix A.

Finally, by using the transformations given by (62) and (63), Eq. (28) can be rewritten in the form

$$G(t) + \frac{2}{\pi}\frac{t}{(t^2 - b^2)^{1/2}}\int_a^b \frac{H(u)(b^2 - u^2)^{1/2}}{(t^2 - u^2)}du = 0; \quad b \le t < \infty. \tag{70}$$

Equations (61), (67), and (70) represent the system of coupled integral equations for the unknown functions $H(t)$, $G(t)$, and $M(t)$ which govern the mixed boundary value problem for the penny-shaped matrix-fiber crack for an elastic solid which is subjected to uniform straining. It is evident that owing to the complexity of the kernel functions $S_i(u, t)$ $(i = 1, 2, 3, 4)$ occurring in these equations recourse must be made to solve these coupled integral equations in a numerical fashion. The details of the numerical scheme are given in Appendix B.

The Stress Intensity Factor

A result of primary importance to the examination of brittle elastic fracture processes of the matrix region is the stress intensity factor at the boundary, $r = b$, of the matrix-fiber crack. A critical value of the stress intensity factor is an indicator of the fracture toughness of the matrix region. In the case of the penny-shaped matrix-fiber crack, the magnitude of the stress intensity factor will be influenced by the elasticity mismatch between the fiber and matrix regions. Developing an expression for $\sigma_{zz}^{(m)}$ we have

$$\frac{\sigma_{zz}^{(m)}(r, 0)}{2\mu_m} = -\int_0^\infty F_2(s)J_0(sr)ds - \int_0^\infty s\{A(s)K_0(sr) + [2\nu_m K_0(sr) - srK_1(sr)]B(s)\}ds; \quad a < r < b. \tag{71}$$

The stress intensity factor k_1 at the crack tip is given by the result

$$k_1 = \lim_{r \to b^+} [2(b - r)]^{1/2}\sigma_{zz}^{(m)}(r, 0). \tag{72}$$

Omitting details it can be shown that

$$\frac{k_1}{k_0} = -\frac{[H(b) - G(b)]}{b} \tag{73}$$

where k_0 is the stress intensity factor for a penny-shaped crack of radius b which is located in a homogeneous elastic solid and subjected to a uniform strain ϵ_0 (or uniform stress $E_m\epsilon_0$) (Sneddon, 1946), i.e.,

$$k_0 = \frac{2\sigma_0\sqrt{b}}{\pi}. \quad (74)$$

Numerical Results

The numerical procedure described in Appendix B is used to determine the stress intensity factor at the tip of the penny-shaped crack which is located in the matrix region. The accuracy of the numerical scheme was verified by comparison with exact analytical results. For example the result for the stress intensity factor for a penny-shaped crack located in an isotropic elastic solid can be recovered in the limit $E_f = E_m = E$; $\nu_f = \nu_m = \nu$. This reduction can also be confirmed from the result obtained from the general solution in the case where $a \to 0$. The numerical result for this particular case compares very accurately, to within 0.01 percent, with the exact closed-form result (74). The material characteristics influencing the stress intensity factor k_1 include, the fiber-matrix modular ratio (E_f/E_m), the Poisson's ratios (ν_f, ν_m) and the fiber-crack radii ratio (a/b). It is of interest to evaluate the amplification of the stress intensity factors at the matrix crack for typical fiber-reinforced composites encountered in engineering applications. Results for several fiber-reinforced composites consisting of E-Glass, Kevlar, Stainless Steel, fibers and matrices consisting of Epoxy Resin, reaction bonded Silicon Nitride (Si_3N_4) and aluminum have been computed using the basic elastic properties of the fiber and matrix planes indicated in Table 1. The elastic properties for these constituents are given by Lawn and Wilshaw (1975), Davidge (1979), Sih and Chen (1981), Hull (1981), and Watt and Perov (1985). Figure 2 indicates the variation of k_1/k_0 for typical fiber-reinforced composites. These results also emphasize the influence of E_f/E_m on the amplification of k_1/k_0.

In all computations, the geometrical parameter a/b is restricted to the range $(a/b)\epsilon(0, 0.90)$. In the limit as $a/b \to 1$ the matrix-fiber crack problem reduces to that of a cracked fiber which is embedded in an undamaged matrix. In this instance, the character of the crack problem changes due to the fact that the crack tip now terminates at an interface (Zak and Williams, 1963). The stress singularity can now exhibit an order different from $1/r^{1/2}$ ... depending upon the magnitudes of the elastic properties of the fiber and matrix regions. In order to solve the resulting problem, it is of course necessary to reformulate the integral equation scheme where the specific character of the singular stress fields are incorporated in the analysis. This analysis will be examined in detail in a subsequent paper.

Conclusions

Fiber breakage in particular will occur in a fiber-reinforced solid and such discontinuities can introduce complex stress states in the matrix region. In the presence of strong fiber-matrix interface adhesion, the crack extension will occur into the matrix. Although the analysis of crack extension should take into consideration the influence of other intact fibers in the vicinity of the cracked fiber location, some insight into the micromechanical processes of matrix crack extension can be gained by examining a *reduced* problem related to an isolated fiber. Such an idealized isolated fiber configuration is also relevant to the study of matrix crack extension in "fragmentation tests" which are developed for the study of fiber-matrix interface adhesion. The paper has developed a mathematical formulation of a fiber-matrix crack which is subjected to a homogeneous strain field. In this study attention has been restricted to the idealized problem of a penny-shaped matrix crack which is located at a cracked fiber. It is shown that this idealized fiber-matrix crack problem can be formulated within the context of the classical theory of elasticity to evaluate results of interest to microfracture mechanics of composite materials.

The numerical results derived for typical fiber-reinforced composite materials with $E_f > E_m$ indicate that the stress intensity factor at the matrix crack boundary is always higher than that for a penny-shaped crack of equal radius located in a homogeneous matrix (without a fiber). This stress intensity factor amplification increases dramatically with increase in the fiber-matrix modular ratio. The condition for brittle elastic crack extension can be obtained by comparing the stress intensity factor for the fiber-matrix crack with the critical value of the stress intensity factor k_{Ic} applicable to the matrix material. Alternatively, the results given in this study can be used to assess the stable geometries of fiber-matrix cracks that can occur at cracked fiber locations.

Table 1 Elastic constants for components of fiber-reinforced composites

Material	Young's modulus (GN/m^2)	Poisson's ratio
Epoxy resin	3.0	0.40
Si_3N_4	100.0	0.20
Silicon carbide	400.0	0.20
E-glass	70.0	0.26
Kevlar	130.0	0.32
Polyester	2.0	0.40
Glass fibre	70.0	0.20
Graphite	240.0	0.30
Stainless steel	207.0	0.30
Aluminum	69.0	0.33

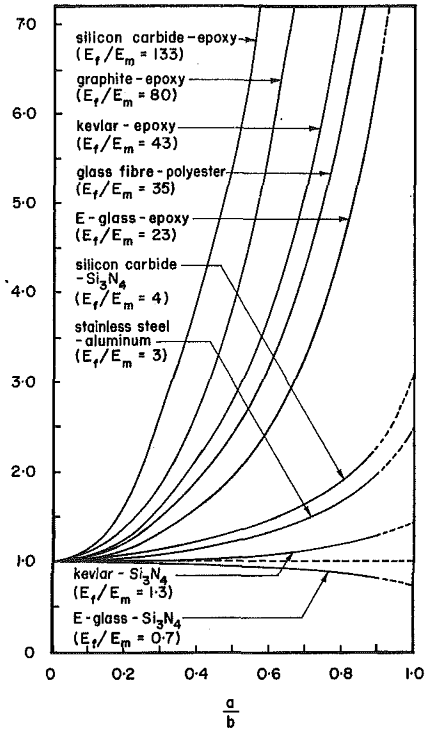

Fig. 2 Stress intensity factor for the fiber-matrix crack. Results for typical fiber-reinforced composites.

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APPENDIX A

The expressions for the functions $A(s)$, $B(s)$, $C(s)$, and $D(s)$, take the following forms:

$$\zeta A(s) = -\Bigg[X_1(s)\{(a_{13}I_0 + a_8I_1)(a_1a_6 - a_3a_4) + a_3I_1(a_{12}I_0 + a_7I_1)\} + X_2(s)\{(a_9I_1 + a_{14}I_0)(a_1a_6 - a_3a_4) + a_3I_0(a_{12}I_0 + a_7I_1)\} + X_3(s)\{(a_{10}I_1 + a_{15}I_0)(a_1a_6 - a_3a_4) - a_6I_1(a_{12}I_0 + a_7I_1)\} + X_4(s)\Bigg\{(a_{11}I_1 + a_{16}I_0) \times (a_1a_6 - a_3a_4) + \frac{a_6}{s}(I_1 - asI_0)(a_{12}I_0 + a_7I_1)\Bigg\}\Bigg] \quad \text{(A1)}$$

$$\zeta B(s) = X_1(s)\{a_2I_1(a_{12}I_0 + a_7I_1) - (a_2a_4 - a_1a_5) \times (a_{13}I_0 + a_8I_1)\} + X_2(s)\{a_2I_0(a_{12}I_0 + a_7I_1) - (a_2a_4 - a_1a_5)(a_{14}I_0 + a_9I_1)\} - X_3(s) \times \{a_5I_1(a_{12}I_0 + a_7I_1) + (a_2a_4 - a_1a_5)(a_{15}I_0 + a_{10}I_1)\} + X_4(s)\Bigg\{\frac{a_5}{s}(I_1 - saI_0)(a_{12}I_0 + a_7I_1) - (a_2a_4 - a_1a_5)(a_{16}I_0 + a_{11}I_1)\Bigg\} \quad \text{(A2)}$$

$$\eta C(s) = X_1(s)\{a_7a_{13} - a_8a_{12}\} + X_2(s)\{a_7a_{14} - a_{12}a_9\} + X_3(s)\{a_{15}a_7 - a_{10}a_{12}) + X_4(s)\{a_7a_{16} - a_{11}a_{12}\} \quad \text{(A3)}$$

$$\eta D(s) = X_1(s)\{a_{13}I_0 + a_8I_1\} + X_2(s)\{a_9I_1 + a_{14}I_0\} + X_3(s)\{a_{10}I_1 + a_{15}I_0\} + X_4(s)\{a_{11}I_1 + a_{16}I_0\} \quad \text{(A4)}$$

where

$$\zeta = (a_2a_6 - a_3a_5)(a_{12}I_0 + a_7I_1) \quad \text{(A5)}$$

$$\eta = (a_7I_1 + a_{12}I_0) \quad \text{(A6)}$$

$$a_1 = s^2a^2(I_1^2 - I_0^2) + 2(1 - \nu_f)I_1^2 \quad \text{(A7)}$$

$$a_2 = sa\Gamma\{K_0I_1 + K_1I_0\} \quad \text{(A8)}$$

$$a_3 = -\Gamma[K_1I_1\{2 - 2\nu_m + s^2a^2\} + 2(1 - \nu_m)saI_0K_1 + 2sa(1 - \nu_m)K_0I_1 + K_0I_0s^2a^2] \quad \text{(A9)}$$

$$a_4 = sa(I_1^2 - I_0^2) + 4(1 - \nu_f)I_0I_1 \quad \text{(A10)}$$

$$a_5 = \frac{a_2}{sa\Gamma} \quad \text{(A11)}$$

$$a_6 = -\{sa(I_1K_1 + I_0K_0) + 4(1 - \nu_m)I_0K_1\} \quad \text{(A12)}$$

$$a_7 = saI_1 + \left\{\frac{saK_1(a_2a_4 - a_5a_1) + K_0(a_3a_4 - a_6a_1)}{a_2a_6 - a_3a_5}\right\} \quad \text{(A13)}$$

$$a_8 = 1 + \frac{I_1(a_2saK_1 + a_3K_0)}{(a_2a_6 - a_3a_5)} \quad \text{(A14)}$$

$$a_9 = \frac{I_0(saa_2K_1 + a_3K_0)}{(a_2a_6 - a_3a_5)} \quad \text{(A15)}$$

$$a_{10} = -\frac{I_1(a_6K_0 + saa_5K_1)}{(a_2a_6 - a_3a_5)} \quad \text{(A16)}$$

$$a_{11} = \left\{\frac{a_6K_0 + saa_5K_1}{s}\right\}\left\{\frac{I_1 - saI_0}{a_2a_6 - a_3a_5}\right\} \quad \text{(A17)}$$

$$a_{12} = 2(1 - \nu_f)I_1 - saI_0 + \frac{\Gamma K_1(a_3a_4 - a_1a_6)}{(a_2a_6 - a_3a_5)} + \frac{\Gamma(a_2a_4 - a_1a_5)}{(a_2a_6 - a_3a_5)}\{2(1 - \nu_m)K_1 + saK_0\} \quad \text{(A18)}$$

$$a_{13} = \frac{\Gamma}{(a_2a_6 - a_3a_5)}[a_3K_1I_1 + a_2I_1\{2(1 - \nu_m)K_1 + saK_0\}] \quad \text{(A19)}$$

$$a_{14} = \frac{\Gamma}{(a_2a_6 - a_3a_5)}[a_3K_1I_1 + a_2I_0\{2(1 - \nu_m)K_1 + saK_0\}] \quad \text{(A20)}$$

$$a_{15} = \frac{-\Gamma I_1}{(a_2a_6 - a_3a_5)}[a_6K_1 + a_5\{2(1 - \nu_m)K_1 + saK_0\}] \quad \text{(A21)}$$

$$a_{16} = \frac{1}{s}\left[1 + \frac{\Gamma(I_1 - saI_0)}{(a_2a_6 - a_3a_5)} \times \{a_6K_1 + [2(1 - \nu_m)K_1 + saK_0]a_5\}\right] \quad \text{(A22)}$$

and

$$\Gamma = \frac{\mu_m}{\mu_f}. \quad \text{(A23)}$$

By making use of the expressions (53) to (60) and Eqs. (32) to (35) we can obtain expressions for the functions $X_i(s) = (i = 1, 2, 3, 4)$ in terms of $h(s)$, $g(s)$, and $m(s)$. These expressions are

$$X_1(s) = [(3 - 2\nu_m)I_0(sa) + saI_1(sa)]\left[\int_a^b h(u)e^{-su}du + \int_b^\infty g(u)e^{-su}du\right] - sI_0(sa)\left[\int_a^b uh(u)e^{-su}du + \int_b^\infty ug(u)e^{-su}du\right] - \frac{2}{\pi}\int_0^a [(3 - 2\nu_f)\sinh(su)K_0(sa) - saK_1(sa)\sinh(su) + su\cosh(su)K_0(sa)]m(u)du \quad \text{(A24)}$$

$$X_2(s) = -\left\{\frac{sa}{2}[I_0(sa) + I_2(sa)] + 2\nu_mI_1(sa)\right\} \times \left\{\int_a^b h(u)e^{-su}du + \int_b^\infty g(u)e^{-su}du\right\} + sI_1(sa)\left[\int_a^b uh(u)e^{-su}du + \int_b^\infty ug(u)e^{-su}du\right] - \frac{2}{\pi}\int_0^a \{(2\nu_f - 1)K_1(sa)\sinh(su) - [sa\sinh(su)K_0(sa) - su\cosh(su)K_1(sa)]\}m(u)du \quad \text{(A25)}$$

$$X_3(s) = \frac{2}{\pi}\left[\int_0^a m(u)\{sa[sa\sinh(su)K_1(sa) - su\cosh(su)K_0(sa)] + (1 - 2\nu_f)K_1(sa)\sinh(su) - suK_1(sa)\cosh(su)\}du\right] + \Gamma\int_a^b h(t)e^{-st} \times \left\{sa\left[\frac{1}{2}I_0(sa) + saI_1(sa) - stI_0(sa)\right] + I_1(sa)(st - 2\nu_m) - \frac{sa}{2}I_2(sa)\right\}dt + \Gamma\int_b^\infty g(t)e^{-st}\left[sa\left\{\frac{1}{2}I_0(sa) + saI_1(sa) - stI_0(sa)\right\} + I_1(sa)\{st - 2\nu_m\} - \frac{sa}{2}I_2(sa)\right]dt \quad \text{(A26)}$$

$$X_4(s) = -\Gamma\left[2sI_1(sa) + \frac{s^2a}{2}\{I_0(sa) + I_2(sa)\}\right] \times \left[\int_a^b h(u)e^{-su}du + \int_b^\infty g(u)e^{-su}du\right] + \Gamma s^2I_1(sa)\left[\int_a^b uh(u)e^{-su}du + \int_b^\infty ug(u)e^{-su}du\right] - \frac{2s}{\pi}\int_0^a m(u)[\sinh(su)K_1(sa) + su\cosh(su)K_1(sa) - sa\sinh(su)K_0(sa)]du. \quad \text{(A27)}$$

The expressions for $P_i(s, t)$ ($i = 1, 2, 3, 4$) occurring in the kernel functions S_1 and S_2 defined by Eqs. (65) and (66) are given by

$$\zeta P_1(s, t) = -[\{(a_{12}I_0 + a_7 I_1)a_3 I_1 + (a_8 I_1 + a_{13} I_0) \times (a_1 a_6 - a_3 a_4)\}L_1(s, t) - \{a_2 I_1(a_{12}I_0 + a_7 I_1) - (a_2 a_4 - a_1 a_5)(a_{13}I_0 + a_8 I_1)\} \times \{2\nu_m L_1(s, t) - sL_2(s, t)\}] \quad (A28)$$

$$\zeta P_2(s, t) = -[\{(a_{12}I_0 + a_7 I_1)a_3 I_0 + (a_9 I_1 + a_{14} I_0) \times (a_1 a_6 - a_3 a_4)\}L_1(s, t) - \{a_2 I_0(a_{12}I_0 + a_7 I_1) - (a_2 a_4 - a_1 a_5)(a_{14}I_0 + a_9 I_1)\} \times \{2\nu_m L_1(s, t) - sL_2(s, t)\}] \quad (A29)$$

$$\zeta P_3(s, t) = [\{(a_{12}I_0 + a_7 I_1)a_6 I_1 - (a_{10}I_1 + a_{15}I_0) \times (a_1 a_6 - a_3 a_4)\}L_1(s, t) - \{a_5 I_1(a_{12}I_0 + a_7 I_1) + (a_2 a_4 - a_1 a_5)(a_{15}I_0 + a_{10}I_1)\} \times \{2\nu_m L_1(s, t) - sL_2(s, t)\}] \quad (A30)$$

$$\zeta P_4(s, t) = -\left[\left\{\frac{a_6}{s}(a_{12}I_0 + a_7 I_1)(I_1 - saI_0) + (a_1 a_6 - a_3 a_4)(a_{11}I_1 + a_{16}I_0)\right\}L_1(s, t) - \left\{\frac{a_5}{s}(a_{12}I_0 + a_7 I_1)(I_1 - saI_0) - (a_2 a_4 - a_1 a_5) \times (a_{16}I_0 + a_{11}I_1)\right\} \cdot \{2\nu_m L_1(s, t) - sL_2(s, t)\}\right]. \quad (A31)$$

The expressions for $R_i(s, t)$ ($i = 1, 2, 3, 4$) occurring in the kernel functions S_3 and S_4 defined by Eqs. (68) and (69) are given by

$$\eta R_1(s, t) = [\sinh(st)\{a_7 a_{13} - a_8 a_{12}\} - (a_{13}I_0 + a_8 I_1) \times \{\sinh(st)(2\nu_f - 1) + st\cosh(st)\}] \quad (A32)$$

$$\eta R_2(s, t) = [\sinh(st)\{a_7 a_{14} - a_9 a_{12}\} - (a_{14}I_0 + a_9 I_1) \times \{\sinh(st)(2\nu_f - 1) + st\cosh(st)\}] \quad (A33)$$

$$\eta R_3(s, t) = [\sinh(st)\{a_7 a_{15} - a_{10} a_{12}\} - (a_{15}I_0 + a_{10} I_1) \times \{\sinh(st)(2\nu_f - 1) + st\cosh(st)\}] \quad (A34)$$

$$\eta R_4(s, t) = [\sinh(st)\{a_7 a_{16} - a_{11} a_{12}\} - (a_{16}I_0 + a_{11} I_1) \times \{\sinh(st)(2\nu_f - 1) + st\cosh(st)\}]. \quad (A35)$$

APPENDIX B

The system of integral Eqs. (61), (67), and (70) can be represented in the generalized form

$$\mathbf{X}(t) + \int_0^\infty \mathbf{X}(u)\mathbf{K}(u, t)du = \mathbf{B}(t); \quad 0 \le t < \infty \quad (B1)$$

where the unknown functions $X(t)$ are defined as

$$\mathbf{X}(t) = \begin{cases} M(t); & 0 \le t < a \\ H(t); & a \le t < b \\ G(t); & b \le t < \infty. \end{cases} \quad (B2)$$

The right-hand side of the prescribed function $\mathbf{B}(t)$ in (B1) takes the form

$$\mathbf{B}(t) = \begin{cases} t\Gamma\Omega; & 0 \le t < a \\ (b^2 - t^2)^{1/2}; & a \le t < b \\ 0; & b \le t < \infty \end{cases} \quad (B3)$$

and the kernel functions $\mathbf{K}(u, t)$ are defined as follows:

$$\mathbf{K}(u,t) = \begin{cases} \frac{4}{\pi^2}S_4(u,t); & (0<u<a; 0<t<a) \\ \frac{2}{\pi}S_3(u,t); & (a<u<\infty; 0<t<a) \\ \frac{4}{\pi^2}S_2(u,t); & (0<u<a; a<t<b) \\ \frac{2}{\pi}S_1(u,t); & (a<u<b; a<t<b) \\ \frac{2}{\pi}S_1(u,t) - \frac{2}{\pi}\frac{(b^2-t^2)^{1/2}}{(u^2-t^2)(u^2-b^2)^{1/2}}; & (b<u<\infty; a<t<b) \\ -\frac{2}{\pi}\frac{t(b^2-u^2)^{1/2}}{(t^2-b^2)^{1/2}(t^2-u^2)}; & (a<u<b; b<t<\infty), \end{cases} \quad (B4)$$

with $\mathbf{K}(u, t)$ being zero for all other intervals of $u\epsilon(0, \infty)$, $t\epsilon(0, \infty)$.

The system of integral Eqs. (B1) are systems of coupled Fredholm integral equations of the second kind. The solution of this class of integral equations via numerical techniques has been extensively documented in the texts by Atkinson (1976), Baker (1977), and Delves and Mohamed (1985). Examples of applications of these procedures are also given by Selvadurai et al. (1991).

For the numerical solution of the Eq. (B1) we discretize the intervals $[0, a]$, $[a, b]$ and $[b, \infty]$ by N_1, N_2, and N_3 segments, respectively. The corresponding end points of the segments are given by

$$x_i = (i - 1)h_x; \quad i = 1, 2, \ldots (N_1 + 1) \quad (B5)$$

$$y_i = a + (i - 1)h_y; \quad i = 1, 2, \ldots (N_2 + 1) \quad (B6)$$

$$z_i = z_{i-1} + C_r(z_{i-1} - z_{i-2}); \quad i = 3, 4, \ldots (N_3 + 1) \quad (B7)$$

with $z_1 = b$ and $z_2 = b + h_y$ where h_x and h_y are given by a/N_1 and $(b - a)/N_2$, respectively, and C_r is a constant greater than unity such that the recurrence relation (B7) can generate a large number at z_{N_3+1}. For the collocation points of the integral equations we have

$$t_i = \begin{cases} (x_i + x_{i+1})/2; & 1 \le i \le N_1 & (B8) \\ (y_j + y_{j+1})/2; & N_1 < i < N_1 + N_2; \quad j = i - N_1 & (B9) \\ (z_j + z_{j+1})/2; & N_1 + N_2 < i < N; \quad j = i - N_1 - N_2 & (B10) \end{cases}$$

where $N = N_1 + N_2 + N_3$. Using the above procedures and representations, the discretized form of (B1) can be written as

$$\mathbf{A}_{ij}\mathbf{X}_j = \mathbf{B}_j \qquad \text{(B11)}$$

where the unknowns are

$$\mathbf{X}_i = \mathbf{X}(t_i). \qquad \text{(B12)}$$

The right-hand side of the vector is

$$\mathbf{B}_i = \mathbf{B}(t_i) \qquad \text{(B13)}$$

and the coefficients of the matrix $\mathbf{A}_{ij}$ are

$$\mathbf{A}_{ij} = \delta_{ij} + K(t_i, t_j)\Delta t, \qquad \text{(B14)}$$

where δ_{ij} is the Kronecker delta and Δt is given as h_x, h_y or $(z_{j+1} - z_j)$ with $1 < j \le N_1$, $N_1 < j \le (N_1 + N_2)$ or $(N_1 + N_2) < j \le N$, respectively.

Upon solution of the matrix Eq. (B11), the stress intensity factor at the tip of the crack in the matrix region can be obtained from (73); i.e.,

$$\frac{k_1}{k_0} = \frac{\{X_{N_1+N_2+1} - X_{N_1+N_2}\}}{b}. \qquad \text{(B15)}$$