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THE INTERACTION BETWEEN A RIGID CIRCULAR PUNCH ON AN ELASTIC HALFSPACE AND A MINDLIN FORCE

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Introduction

This paper extends Boussinesq's classical problem relating to a rigid circular punch resting in smooth contact with an isotropic elastic half-space, to include the effects of a Mindlin force. Such a force acts at an interior point of the halfspace along the axis of symmetry. The rigid punch experiences a displacement due to the combined action of the internal load and an external load. It is found that the load-displacement relationship for the punch and the contact stress distribution at the interface can be obtained in exact closed forms.

Analysis

The axisymmetric problem of a rigid circular punch resting in smooth contact with an isotropic elastic halfspace was first considered by Boussinesq [1]. A subsequent treatment of this problem by Harding and Sneddon [2] uses Hankel transform techniques to reduce the mixed boundary value problem to a system of dual integral equations. A systematic study of contact problems is also presented by Sneddon [3].

In these classical treatments of the contact problem it is usually assumed that the elastic halfspace is subjected to loads that are applied at its plane surface. This paper is concerned with the interaction analysis of a rigid circular flat punch which is subjected, simultaneously, to axisymmetric external and internal loads. The particular internal loading of the halfspace corresponds to a concentrated force which acts at a point in the interior of the halfspace (Fig. 1). The two known solutions relating to an isotropic elastic halfspace, namely, those for a surface normal stress distribution and for an internal force, are combined to solve the title problem. The solution of the interior load problem for a halfspace with a traction free surface was developed separately by Mindlin [4] and Dean et al. [5]. As in the classical treatments of the punch problem, the contact surface is assumed to be smooth. In addition, it is assumed that the interface between the rigid punch and the elastic halfspace is capable of sustaining tensile tractions. The solution of the title problem is approached by making use of the complex potential function formulation developed by Green [6]. Using this method a normal contact stress distribution under the rigid circular punch is found such that (i) the normal surface displacement field resulting from the contact stresses when combined with (ii) the normal surface displacement due to the internal load, gives constant displacement within the punch region. It is found that both the load-displacement relationship and the contact stress distribution corresponding to the punch problem posed here can be obtained in exact closed forms.

The punch problem related to the internally loaded halfspace has several useful engineering applications. These include the 'cable method of in-situ testing' (see e.g. Zienkiewicz and Holister [7] and Jaeger and Cook [8]) currently used for the determination of in-situ properties of soil and rock media. Here, the Mindlin force represents approximately, the influence of the reaction at an anchor region used for the purpose of providing the load for the rigid test plate. Other engineering applications of these results related to the analysis and design of structural foundations are also given by Selvadurai [9].

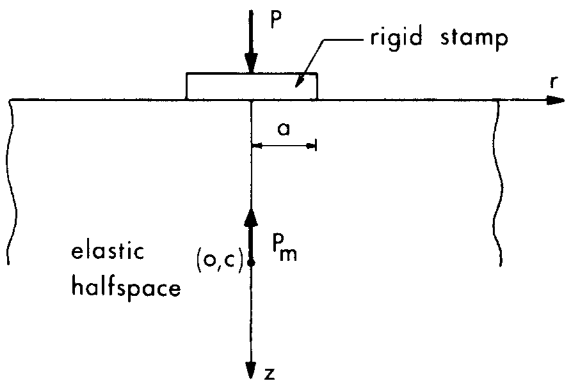

Fig. 1

Basic Equations

A complete account of the complex potential function approach together with its applications to crack and indentation problems is given in Green and Zerna [10]. For the purposes of the present discussion, only the relevant results will be briefly summarized. The class of problems in which the shearing stresses vanish at all points in a plane, say z=0, can be reduced to classical problems in potential theory. The displacement and stress components can be uniquely represented in terms of a single potential function $\phi(r,\theta,z)$; where (r,θ,z) represents the cylindrical polar co-ordinate system. The displacement and stress components of particular interest to the indentation problem are u_z, σ_{rz} and σ_{zz}. For axisymmetric problems, these components can be represented in the form

$$\begin{aligned} 2Gu_z(r,z) &= z\frac{\partial^2\phi}{\partial z^2} - 2(1-\nu)\frac{\partial\phi}{\partial z} \\ \sigma_{zz}(r,z) &= z\frac{\partial^3\phi}{\partial z^3} - \frac{\partial^2\phi}{\partial z^2} \\ \sigma_{rz}(r,z) &= z\frac{\partial^3\phi}{\partial r\partial z^2} \end{aligned} \tag{1}$$

where G and ν are the linear elastic shear modulus and Poisson's ratio respectively. In the frictionless indentation problem related to the halfspace region $(z > 0)$ it is necessary that (i) all the stresses and displacements should decay to zero as $z \to \infty$, and (ii) no shear stresses should act on the bounding plane $z = 0$. The third boundary condition is of a mixed type where

$$u_z(r,0) = \frac{-(1-\nu)}{G}\frac{\partial\phi}{\partial z} = u^*(r) \qquad \text{on} \qquad r < a \tag{2}$$

$$\sigma_{zz}(r,0) = \frac{-\partial^2\phi}{\partial z^2} = 0 \qquad \text{on} \qquad r > a \tag{3}$$

and a corresponds to the radius of the circular flat punch. To obtain a solution to the indentation problem we consider the representation (see e.g. Green and Zerna [10])

$$\frac{\partial\phi}{\partial z} = \frac{1}{2}\int_{-a}^{a} \frac{g(t)\ dt}{\sqrt{\{r^2 + (z+it)^2\}}} \tag{4}$$

which satisfies $\nabla^2\phi(r,z) = 0$, and the regularity conditions at infinity Then, from (4), we have

$$\begin{aligned} \frac{\partial\phi}{\partial z} &= \int_0^r \frac{g(t)\ dt}{\sqrt{r^2-t^2}} && \text{on } z = 0;\quad 0 < r < a \\ \frac{\partial^2\phi}{\partial z^2} &= 0 && \text{on } z = 0;\quad r > a \end{aligned} \tag{5}$$

and the boundary conditions (2) and (3) reduce to the single integral equation

$$u^*(r) = \int_0^r \frac{g(t)\ dt}{\sqrt{r^2-t^2}} \tag{6}$$

in terms of the unknown function $g(t)$. The Abel integral equation (6) may be inverted to complete the solution. Assuming that $u^*(r)$ is continuously differentiable in $0 \leqslant r \leqslant a$, the solution of (6) is given by

$$g(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{ru^*(r)\ dr}{\sqrt{t^2-r^2}} \tag{7}$$

The contact stress distribution at the circular punch interface can be expressed in terms of $g(t)$ in the form

$$\sigma_{zz}(r,0) = \frac{G}{(1-\nu)}\ \frac{1}{r}\ \frac{\partial}{\partial r}\int_r^a \frac{t\ g(t)\ dt}{\sqrt{t^2-r^2}} \tag{8}$$

It can also be shown that the total force exerted by the rigid circular punch can be represented in the form

$$P = \frac{2\pi G}{(1-\nu)}\int_0^a g(t)\,dt \tag{9}$$

The rigid punch problem

We consider the axisymmetric indentation of an internally loaded isotropic elastic halfspace by a circular rigid punch of radius a. The internal loading of the halfspace corresponds to a concentrated force of magnitude P_m acting at a distance c along the axis of symmetry and in the negative z direction. The total external force on the rigid circular punch is denoted by P. We assume that, due to the combined action of these force systems, the rigid circular punch experiences a displacement w_0 and that no separation occurs at the interface of the punch. The prescribed displacement field $u^*(r)$ corresponding to (2) is given by

$$u^*(r) = w_0 + \frac{k_1 c^2}{(r^2 + c^2)^{1/2}} + \frac{k_2 c^4}{(r^2 + c^2)^{3/2}} \tag{10}$$

where k_1 and k_2 are constants. For the Mindlin force problem

$$k_1 = \frac{P_m(1-\nu)}{2\pi G c^2} \quad , \quad k_2 = \frac{P_m}{4\pi G c^2} \tag{11}$$

Using the expression (10) in (7) we obtain the following result for the function g(t):

$$g(t) = \frac{2}{\pi}\left[w_0 + \frac{(k_1 + k_2)c^3}{(t^2 + c^2)} - \frac{2k_2 c^3 t^2}{(t^2 + c^2)^2} \right] \tag{12}$$

The displacement of the rigid circular punch due to the combined action of the external force P and the Mindlin force P_m can be obtained by evaluating the expression (9) for the total resultant force on the punch. Using (12) in (9) and evaluating the resulting integrals, it can be shown that

$$w_0 = \frac{P(1-\nu)}{4aG}\left[1 - \frac{P_m}{P}\left\{ \frac{2}{\pi}\tan^{-1}\left(\frac{a}{c}\right) + \frac{ac}{\pi(1-\nu)(a^2+c^2)} \right\} \right] \tag{13}$$

We note that as $P_m \to 0$ or as $c \to \infty$, (13) reduces to Boussinesq's classical result for the axisymmetric indentation of an isotropic elastic halfspace by a rigid circular punch. Also, when $P_m = P$ and $c \to 0$, the rigid circular punch is subjected to a doublet of concentrated forces; as such $w_0 = 0$

Similarly, using (12) in (8), we obtain an expression for the compressive contact stress distribution at the rigid circular punch - elastic halfspace interface. By combining this result with (13), we obtain

$$\sigma_{zz}(r,0) = \frac{-P}{2\pi a\sqrt{a^2-r^2}}\Bigg[1+\frac{P_m}{P}\Bigg\{-\frac{2}{\pi}\tan^{-1}\Big(\frac{a}{c}\Big)-\frac{ac}{\pi(1-\nu)(a^2+c^2)}+$$

$$+\frac{ac}{\pi(1-\nu)(r^2+c^2)^2}\Bigg[(1-2\nu)\ (r^2+c^2)+c^2+$$

$$+\{(1-2\nu)\ (r^2+c^2)+3c^2\}\sqrt{\frac{a^2-r^2}{r^2+c^2}}\tan^{-1}\sqrt{\frac{a^2-r^2}{r^2+c^2}}+$$

$$+\frac{c^2\{2a^2+c^2-r^2\}}{(a^2+c^2)}\Bigg]\Bigg\}\Bigg] \quad (14)$$

When either $P_m \to 0$ or $c \to \infty$ the contact stress distribution (14) reduces to Boussinesq's classical result. As $c \to 0$, the contact stresses reduce to zero at all locations of the interface except at the origin $r = 0$. When both r and c tend to zero, simultaneously, the contact stress (14) exhibits singular behaviour. The form of the expression for the contact stress (14) suggests that for certain values of (c/a), the contact stresses become tensile at certain locations within the contact area. The Fig.2 shows a typical range of values of c_0/a for which tensile contact stresses are developed at the circular punch - elastic halfspace interface due to the action of the internal force. (c_0 denotes the limiting value of c.) The influence of the internal force on the displacement of the rigid circular punch (w_0) is shown in Fig. 3. The dotted lines indicate the range of values of c/a for which tensile tractions are developed at the interface. Since the theoretical treatment of the indentation problem assumes both a smooth contact and no separation at the interface, the practical applicability of the solution is limited to those values of c/a which render the contact stress distribution compressive at every location of the contact region $r < a$.

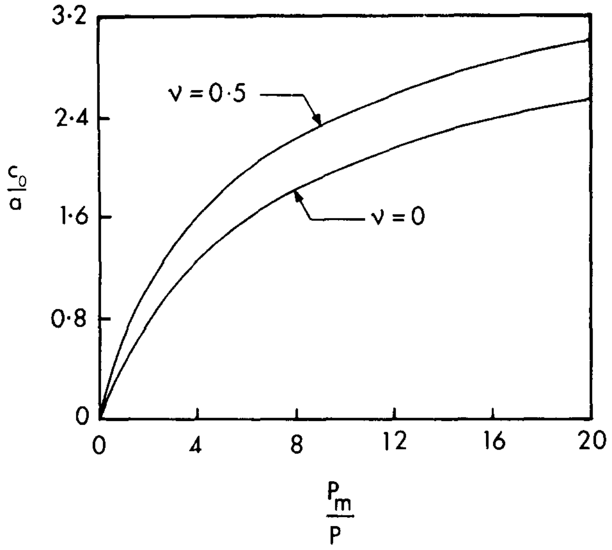

Fig. 2. Values of (c/a) for which the contact stress at the interface is tensile.

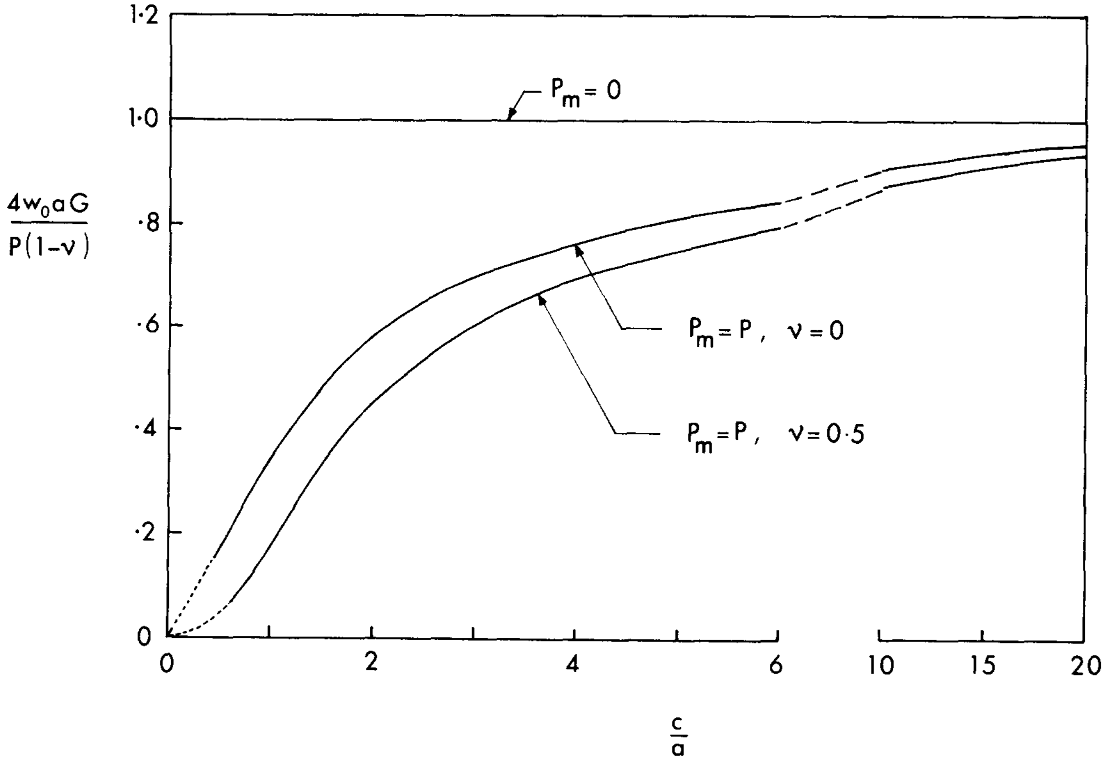

Fig. 3. The influence of the location of the Mindlin force on the deflection of the rigid circular punch.

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