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THE SETTLEMENT OF A RIGID CIRCULAR FOUNDATION RESTING ON A HALF-SPACE EXHIBITING A NEAR SURFACE ELASTIC NON-HOMOGENEITY

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SUMMARY

The present paper examines the elastostatic problem pertaining to the axisymmetric loading of a rigid circular foundation resting on the surface of a non-homogeneous elastic half-space. The non-homogeneity corresponds to a depth variation in the linear elastic shear modulus according to the exponential form $G(z) = G_1 + G_2 e^{-\zeta z}$. The equations of elasticity governing this type of non-homogeneity are solved by employing a Hankel transform technique. The mixed boundary value problem associated with the indentation of the half-space by the rigid circular foundation is reduced to a Fredholm integral equation which is solved via a numerical technique. The numerical results presented in the paper illustrate the influence of the near-surface elastic non-homogeneity on the settlement of the foundation.

KEY WORDS: elastic non-homogeneity; circular foundation; elastic indentation; integral equations; weathered crust; contact problem; surface non-homogeneity

1. INTRODUCTION

The study of the mechanics of non-homogeneous elastic media has always occupied a prominent position in the literature in mechanics. Quite apart from the intrinsic mathematical interest, the non-homogeneity problem in elasticity has applications to many problems of technological importance. In the context of geomechanics, the inhomogeneous medium serves as a model for the study of soil and rock media which exhibit spatial variations in the elastic properties. A large body of the literature pertaining to such elastostatic problems deals primarily with *depth variations* in the elastic properties of half-space regions. Even with such simplifications, the formulation and solution of problems involving non-homogeneous elastic media are certainly non-routine. It is convenient to discuss the developments in this area by focussing attention on specific categories of elastic inhomogeneity. Problems associated with a non-homogeneous elastic half-space region in which the Poisson ratio is constant and the shear modulus varies with depth according to a *power law* have been treated by a number of researchers. Klein[1] considered the stress analysis of such a medium and Korenev[2] examined the contact problem related to the smooth indentation of such a non-homogeneous half-space region. Rakov and Rvachev[3] have examined the contact problem for a flexible beam on an elastic half-space with a power law non-homogeneity in the variation of the shear modulus and Popov[4] has examined the problem of the unbounded plate on a non-homogeneous elastic half-space. A two-dimensional contact

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problem for a half-plane with a power law elastic non-homogeneity has been treated by Belik and Protsenko.[5] Kassir and Chuapreset[6] considered the contact problem for a cylindrical punch on a non-homogeneous half-space where the shear modulus varied with depth according to a *generalized power law*. Similar applications to two-dimensional contact problems are given by Stachowicz[7] and Stachowicz and Szefer.[8] A further class of elastic non-homogeneity deals with the case where the elastic modulus varies exponentially with depth. Mossakovskii[9] examined the contact problem for rigid circular punch on a half-space. Other contact problems related to the exponential non-homogeneity have been examined by Ter-Mkrtich'ian[10] and Szefer.[11] Plevako[12,13] presented a formulation of the theory of elasticity of a non-homogeneous elastic medium with a depth variation of the shear modulus. This formulation utilizes a single stress function approach and Plevako[13] has also demonstrated the application of harmonic function techniques to the solution of specific problems involving generalized non-homogeneities and localized loads on a half-space. Puro[14] examines a similar class of problems where attention is focussed on the analysis of the ordinary differential equations which stem from the formulation of problems with elastic non-homogeneity.

A third category of non-homogeneity relates to a *linear variation* in either the shear modulus or elastic modulus of the medium. The renewal of interest in the geomechanical modelling of non-homogeneous elastic media is largely due to a classic paper by Gibson.[15] Gibson obtained through analytical studies the solution to the problem of an incompressible elastic half-space with a linear variation in the shear modulus. This research was motivated by the applicability of such models to the study of the undrained elastic behaviour of over-consolidated saturated clays. In the particular instance when the surface shear modulus of the half-space reduces to zero (i.e. a Gibson soil), the surface deformation of the half-space resembles that of a Winkler medium which consists of a series of closely spaced independent spring elements. Other investigations pertaining to a non-homogeneous elastic medium with *linear* and *hyperbolic* variations in the shear modulus are given by Gibson *et al.*,[16] Awojobi and Gibson,[17] Awojobi,[18,19] Gibson and Sills[20] and Gibson.[21] The work of Awojobi[22] also indicates that the analogy between the incompressible medium with linearly varying non-homogeneity in the shear modulus and the Winkler model is also valid for a non-homogeneous elastic layer, provided it is underlain by a smooth rigid base. Calladine and Greenwood[23] also reconsidered the problem pertaining to the loading of an incompressible elastic half-space region with a linearly varying shear modulus. Using a much simpler analysis involving the radial stress state induced by a concentrated force, they were able to establish the equivalence between Gibson's result and the idealized Winkler foundation response. Booker *et al.*[24,25] have developed solutions for the surface loading of half-space regions (concentrated and distributed loads) where Young's modulus varies with depth according to a power law. Recently, Rajapakse and Selvadurai[26] have examined the class of problems related to the flexure of circular plate anchors which are embedded in such non-homogeneous elastic media. Rajapakse[27] has also examined the problem of the interior loading of a half-space region where the shear modulus varies with depth.

The classical problem of the torsional indentation of an isotropic elastic half-space by a rigid circular punch was examined by Reissner and Sagoci[28] and Sneddon.[29,30] This particular problem has also been modified to include effects of elastic non-homogeneity. The torsional indentation of a non-homogeneous half-space region which exhibits either *exponential* or *power law* variations in the shear modulus was examined by Protsenko,[31] Kassir,[32] Kolybikhin,[33] Singh[34] and Selvadurai *et al.*[35] Similar problems related to the non-homogeneous elastic layer were investigated by Protsenko,[36] Dhaliwal and Singh[37,38] and Hassan.[39] Gladwell and Coen[40] have examined the class of inverse problems which result from the torsional indentation of a non-homogeneous isotropic elastic half-space. Rajapakse and Selvadurai[41] have also examined the problem of the torsional loading of piles embedded in non-homogeneous soil media. Further references to contact problems associated with non-homogeneous elastic media are given in the studies by Olszak,[42] Golecki and Knops,[43] Selvadurai[44] and Gladwell.[45]

In this paper we examine the axisymmetric contact problem pertaining to a non-homogeneous elastic medium in which the shear modulus varies with depth in an exponential fashion. This is also referred to as a near surface elastic non-homogeneity where such effects are predominantly restricted to the surface region of the half-space and the depth variation in the shear modulus is specified in relation to a single variable parameter. Poisson's ratio of the medium is assumed to be constant. This type of elastic non-homogeneity ensures that the elastic shear modulus is bounded as the axial co-ordinate $z \to \infty$. This particular variation of elastic non-homogeneity has several potential geomechanics applications. In the context of problems in foundation engineering, elastic non-homogeneity in the vicinity of the surface of a brittle geomaterial can occur as a result of weathering processes induced by thermal loads, chemical transformations and other environmental effects. Such reductions in the near surface shear modulus can be traced to microcracking and damage of the geomaterial. Similarly, increases in the shear modulus can occur as a result of the formation of a weathered crust in silts and clay soils. In general, these processes can result in non-homogeneous elastic solids in which both the shear modulus and Poisson's ratio vary with depth. In this study, however, attention is restricted to non-homogeneous elastic solids where Poisson's ratio is assumed to be constant. The mathematical analysis focusses on the problem of the indentation of a non-homogeneous elastic half-space by a rigid circular foundation with a smooth base contact. A Hankel transform development of the governing equations is used to reduce the dual integral equations governing this problem to a Fredholm integral equation of the sescond kind. These equations are solved in a numerical fashion to generate results of geotechnical interest, particularly in reference to the load-settlement behaviour of a rigid circular foundation with a smooth base contact.

2. FUNDAMENTAL EQUATIONS

There are several approaches to the formulation of the elastostatic problem related to a non-homogeneous medium. Examples of these are given in the references cited previously. We consider the axisymmetric problem in elasticity referred to a half-space region where $r \in (0, \infty)$ and $z \in (0, \infty)$. The elastic material is assumed to be inhomogeneous such that the elastic constants $G(r, z)$ and $\nu(r, z)$ have specific variations of the forms

$$G(r, z) = G(z); \qquad \nu(r, z) = \nu = \text{constant} \tag{1}$$

The axisymmetric state of stress in the elastic medium is given the stress tensor σ_{ij} referred to a cylindrical polar co-ordinate system:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{rz} & 0 & \sigma_{zz} \end{bmatrix} = \sigma_{ij}(r, z) \tag{2}$$

The non-zero components of the displacement vector are

$$u_i = (u_r, 0, u_z) \tag{3}$$

The non-zero components of the strain tensor ε_{ij} are given by

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & 0 & \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ 0 & \frac{u_r}{r} & 0 \\ \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) & 0 & \frac{\partial u_z}{\partial z} \end{bmatrix} \tag{4}$$

Considering the linear elastic stress–strain relationship,

$$\sigma_{ij} = 2G(z)\varepsilon_{ij} + \frac{2\nu G(z)}{(1-2\nu)}\varepsilon_{kk}\delta_{ij} \tag{5}$$

where $\varepsilon_{kk} = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = e$ and δ_{ij} is Kronecker's delta function; the equations of equilibrium can be expressed in the form

$$\nabla^2 u_r + \frac{1}{(1-2\nu)}\frac{\partial e}{\partial r} - \frac{u_r}{r^2} + \frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}z}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0 \tag{6}$$

$$\nabla^2 u_z + \frac{1}{(1-2\nu)}\frac{\partial e}{\partial z} + \frac{2}{G}\frac{\mathrm{d}G}{\mathrm{d}z}\left(\frac{\partial u_z}{\partial z} + \frac{\nu}{(1-2\nu)}e\right) = 0 \tag{7}$$

where ∇^2 is the axisymmetric form of Laplace's operator referred to the cylindrical polar co-ordinate system, i.e.

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{8}$$

In the remainder of the paper we consider the specific exponential form of the depth variation of the shear modulus $G(z)$ such that

$$G(z) = G_1 + G_2 e^{-\zeta z}. \tag{9}$$

In order to ensure a bounded variation in $G(z)$ as $z \to \infty$, we require $\zeta > 0$. The values of G_1 and G_2 must be prescribed in such a way that the thermodynamic constraint which ensures positive definiteness of the elastic strain energy in the medium is satisfied, i.e.

$$G(z) > 0, \quad G_1 > 0, \quad z \in (0, \infty) \tag{10}$$

It is also convenient to interpret G_1 and G_2 in terms of the shear moduli that may be prescribed at $z = 0$ and $z \to \infty$. If we denote the surface shear modulus by G_s and the shear modulus at $z \to \infty$ by G_∞, then

$$G_1 = G_\infty, \quad G_2 = G_s - G_\infty \tag{11}$$

For the solution of the problem we impose the following constraints on the values of G_1 and G_2: i.e. $G_1 > 0$; $(G_2/G_1) < 1$. Alternatively, we have $G_s > 0$ and $G_\infty > 0$.

In order to solve equations (6) and (7) we introduce Hankel transform representations of the following form (see e.g. Reference 46):

$$u_r(r, z) = \int_0^\infty U(\xi, z) A(\xi) J_1(\xi r)\,\mathrm{d}\xi \tag{12}$$

$$u_z(r, z) = \int_0^\infty W(\xi, z)A(\xi)J_0(\xi r)\mathrm{d}\xi \tag{13}$$

where $A(\xi)$ is an arbitrary function and $J_n(\xi r)$ is the Bessel function of order n ($n = 0, 1$). The representations (12) and (13) can be used to reduce the displacement equations of equilibrium to the following forms:

$$\frac{\mathrm{d}^2U}{\mathrm{d}z^2} + q(z)\frac{\mathrm{d}U}{\mathrm{d}z} - \frac{2(1-\nu)}{(1-2\nu)}\xi^2 U - \frac{\xi}{(1-2\nu)}\frac{\mathrm{d}W}{\mathrm{d}z} - q(z)\xi W = 0 \tag{14}$$

$$\frac{\mathrm{d}^2W}{\mathrm{d}z^2} + q(z)\frac{\mathrm{d}W}{\mathrm{d}z} - \frac{(1-2\nu)}{2(1-\nu)}\xi^2 W + \frac{\xi}{2(1-\nu)}\frac{\mathrm{d}U}{\mathrm{d}z} + q(z)\,\xi\frac{\nu}{(1-\nu)}U = 0 \tag{15}$$

where

$$q(z) = \frac{1}{G(z)}\frac{\mathrm{d}G(z)}{\mathrm{d}z} \tag{16}$$

In order to formulate the mixed boundary value problem associated with the foundation problem we require expressions similar to (12) and (13) for the stress components σ_{zz} and σ_{rz}. Considering (5), (12) and (13) we obtain

$$\sigma_{zz} = \frac{2G(z)(1-\nu)}{(1-2\nu)}\int_0^\infty \left\{\frac{\mathrm{d}W}{\mathrm{d}z} + \frac{\nu\xi U}{(1-\nu)}\right\}A(\xi)J_0(\xi r)\,\mathrm{d}\xi \tag{17}$$

$$\sigma_{rz} = -\,G(z)\int_0^\infty \left\{-\frac{\mathrm{d}U}{\mathrm{d}z} + \xi W\right\}A(\xi)\,J_1(\xi r)\,\mathrm{d}\xi \tag{18}$$

Expressions for other stress components can be obtained by making use of (12) and (13) in (5).

3. THE RIGID FOUNDATION PROBLEM

We consider the problem of the axisymmetric loading of a rigid circular foundation resting in smooth contact at the surface of a non-homogeneous elastic half-space with an elastic non-homogeneity defined by (9) (see Figure 1). The relevant mixed boundary conditions are as follows:

$$u_z(r, 0) = w(r) = \Delta, \quad 0 \leqslant r \leqslant a \tag{19}$$

$$\sigma_{zz}(r, 0) = 0, \quad a \leqslant r < \infty \tag{20}$$

$$\sigma_{rz}(r, 0) = 0, \quad 0 \leqslant r < \infty \tag{21}$$

In (19), Δ is the uniform rigid settlement. Considering the boundary condition (21) and the regularity conditions applicable to u_r and u_z as $z \to \infty$, we obtain the following boundary conditions which are applicable to the differential equations (14) and (15):

$$-\frac{\mathrm{d}U}{\mathrm{d}z} + \xi W = 0, \quad z = 0 \tag{22}$$

$$U = W = 0, \quad z \to \infty \tag{23}$$

$$U = 1, \quad z = 0 \tag{24}$$

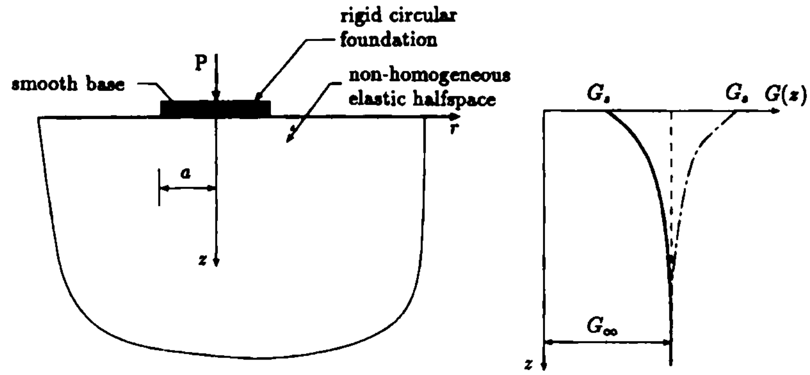

Figure 1. Indentation of a non-homogeneous elastic half-space by a rigid circular foundation

The governing equations (14) and (15) and the constraints (22)–(24) are sufficient to ensure a well-posed boundary value problem. From (19) and (20) we have

$$\int_0^\infty W(\xi, 0) A(\xi) J_0(\xi r)\, d\xi = \Delta, \quad 0 \leqslant r \leqslant a \tag{25}$$

$$\int_0^\infty \left[\frac{dW}{dz} + \frac{\nu}{(1-\nu)}\xi U\right]_{z=0} A(\xi) J_0(\xi r)\, d\xi = 0, \quad a < r < \infty \tag{26}$$

Let

$$\xi R(\xi) = \left[\frac{dW}{dz} + \frac{\nu}{(1-\nu)}\xi U\right]_{z=0} \tag{27}$$

and

$$A(\xi) R(\xi) = B(\xi) \tag{28}$$

Using these substitutions, equations (25) and (26) can be reduced to the forms

$$\int_0^\infty \frac{W(\xi, 0)}{R(\xi)} B(\xi)\, J_0(\xi r)\, d\xi = \Delta, \quad 0 \leqslant r \leqslant a \tag{29}$$

$$\int_0^\infty \xi B(\xi) J_0(\xi r)\, d\xi = 0, \quad a < r < \infty \tag{30}$$

Introducing the finite Fourier transform[43]

$$B(\xi) = \frac{2\Delta}{\pi}\int_0^a \phi(\alpha)\cos(\alpha\xi)\, d\alpha \tag{31}$$

it is evident that (30) is identically satisfied and equation (29) gives rise to a Fredholm integral equation of the second kind

$$\phi(\alpha) + \frac{2}{\pi}\int_0^\infty \phi(s)\, ds \int_0^\infty K(\xi)\cos(\alpha\xi)\cos(\alpha s)\, d\xi = 1, \quad 0 \leqslant \alpha \leqslant a \tag{32}$$

where

$$K(\xi) = \frac{W(\xi, 0)}{R(\xi)} - 1 \tag{33}$$

The solution of (32) can be used to evaluate the displacement and stress fields within the non-homogeneous elastic half-space region. A result of particular interest to geotechnical engineering applications concerns the influence of the elastic non-homogeneity on the load-settlement relationship for the rigid circular foundation, which can be evaluated by computing the total load (P) on the foundation. Considering the equilibrium of the rigid circular foundation we have

$$P = 2\pi \int_0^a r\sigma_{zz}(r, 0)\,\mathrm{d}r = 8\Delta \frac{(1-\nu)}{(1-2\nu)} [G(z)]_{z=0} \int_0^a \phi(\alpha)\,\mathrm{d}\alpha \tag{34}$$

It is convenient to normalize this load-settlement relation for the rigid foundation in relation to the load-settlement behaviour of a rigid circular foundation resting on a *homogeneous* elastic half-space with shear modulus G_∞; i.e.

$$\bar{P} = \frac{P(1-\nu)}{4\Delta G_\infty a} = \frac{2G_s}{G_\infty} \frac{(1-\nu)^2}{(1-2\nu)} \int_0^a \phi(\alpha)\,\mathrm{d}\alpha \tag{35}$$

4. NUMERICAL ANALYSIS

The numerical solution of the Fredholm integral equation (32) relies on how accurately the kernel function $K(\xi)$ can be evaluated. Returning to (14) and (15) and using the transformation $X = \xi z$ we obtain

$$\frac{\mathrm{d}^2 U}{\mathrm{d}X^2} + \frac{q(z)}{\xi}\frac{\mathrm{d}U}{\mathrm{d}X} - \frac{2(1-\nu)}{(1-2\nu)} U - \frac{1}{(1-2\nu)}\frac{\mathrm{d}W}{\mathrm{d}X} - \frac{q(z)}{\xi} W = 0 \tag{36}$$

$$\frac{\mathrm{d}^2 W}{\mathrm{d}X^2} + \frac{q(z)}{\xi}\frac{\mathrm{d}W}{\mathrm{d}X} - \frac{(1-2\nu)}{2(1-\nu)} W + \frac{1}{2(1-\nu)}\frac{\mathrm{d}U}{\mathrm{d}X} + \frac{q(z)}{\xi}\frac{\nu}{(1-\nu)} U = 0 \tag{37}$$

Consider the special case when $(q(z)/\xi) = 0$. The solutions of (36) and (37) which satisfy the boundary conditions (22)–(24) take the forms

$$U = (1 + bX)\mathrm{e}^{-X} \tag{38}$$

$$W = (c + dX)\mathrm{e}^{-X} \tag{39}$$

where

$$b = d = \frac{-1}{(1-2\nu)}, \qquad c = \frac{-2(1-\nu)}{(1-2\nu)} \tag{40}$$

Using (38), (39) and (27) in (33) we obtain

$$K(\xi) = \left[\frac{W}{\dfrac{\mathrm{d}W}{\mathrm{d}X} + \dfrac{U}{(1-\nu)}}\right]_{X=0} - 1 = \left\{ -\frac{2(1-\nu)^2}{(1-2\nu)} - 1 \right\} \tag{41}$$

Avoiding details of the analysis it can be shown that in the limit $\zeta \to \infty$, $\bar{P} = G_s/G_\infty$ and as $\zeta \to 0$; $K(0) = -1$ and $\bar{P} = 1$.

The differential equations (14) and (15) subject to the boundary conditions (22), (23) and (24) can be solved by a trial and error or multiple shooting technique (see e.g. Reference 47). A special technique is necessary to accommodate the numerical treatment of the boundary conditions (23). The limit $z \to \infty$ is replaced by a finite limit $z = L$. Considering the variation of $G(z)$ given by (9) we can write

$$q(z) = \frac{1}{G(z)}\frac{\mathrm{d}G(z)}{\mathrm{d}z} = \frac{-\eta\zeta \mathrm{e}^{-\zeta z}}{(1+\eta \mathrm{e}^{-\zeta z})} \tag{42}$$

where $\eta = (G_s - G_\infty)/G_\infty$. Since $q(z) \to 0$ when $z \to \infty$, U and W will have solutions of order $\mathrm{e}^{-\zeta z}$. Consequently, we shall choose L such that

$$|\mathrm{e}^{-\zeta L}| < \varepsilon \quad \text{and} \quad |\mathrm{e}^{-\xi L}| < \varepsilon \tag{43}$$

and for each ζ and ξ and the permissible error ε

$$L = \max\left[-\frac{\log\varepsilon}{\xi}, -\frac{\log\varepsilon}{\zeta}\right] \tag{44}$$

A solution of (14) and (15) gives the kernel function $K(\xi)$. Considering the limiting expressions for $K(\xi)$ evaluated previously (see e.g. equation (41)) we consider a series expansion for $K(\xi)$ of the form

$$K(\xi) = \frac{-2(1-\nu)^2}{(1-2\nu)} - 1 + \frac{B}{(1+\xi^2)} + \sum_{j=1}^{N} C_j \mathrm{e}^{-\xi j} \tag{45}$$

where B and C_j are constants which are determined by a least-squares technnique. The value of j can be assigned an integer order. In the present study, computations indicate that $N = 4$ yields sufficiently accurate results. Prior to solving the Fredholm integral equation (32), we note that the integrations involving the kernel function representation (45) take the following forms:

$$\int_0^\infty \cos(\alpha\xi)\cos(\alpha s)\,\mathrm{d}\alpha = \frac{\pi}{2}\delta(\xi - s) \tag{46}$$

$$\int_0^\infty \frac{\cos(\alpha\xi)\cos(\alpha s)}{(1+\alpha^2)}\,\mathrm{d}\alpha = \frac{\pi}{4}[\mathrm{e}^{-(\xi+s)} + \mathrm{e}^{-(|\xi-s|)}] \tag{47}$$

$$\int_0^\infty \mathrm{e}^{-j\alpha}\cos(\alpha\xi)\cos(\alpha s)\,\mathrm{d}\alpha = \frac{1}{2}\left[\frac{j}{j^2+(\xi+s)^2} + \frac{j}{j^2+(\xi-s)^2}\right] \tag{48}$$

where $\delta(\xi - s)$ is the Dirac delta function. Using (45) and the results (46)–(48), the integral equation (32) can be transformed into the form

$$(1+A^*)\phi(\xi) + \int_0^a \phi(s)K^*(s,\xi)\,\mathrm{d}s = 1 \tag{49}$$

where

$$A^* = -\left\{1 + \frac{2(1-\nu)^2}{(1-2\nu)}\right\} \tag{50}$$

and

$$K^*(s,\xi) = \frac{B}{2}\{\mathrm{e}^{-(\xi+s)} + \mathrm{e}^{-(|\xi-s|)}\} + \frac{1}{\pi}\sum_{j=1}^{N} C_j\left[\frac{j}{j^2+(\xi+s)^2} + \frac{j}{j^2+(\xi-s)^2}\right] \tag{51}$$

The Fredholm integral equation of the second kind (49) can be solved numerically by employing a standard technique (see e.g. References 48 and 49). For the interval $[0, a]$, we consider M segments with $h = \alpha/M$, and $\xi_i = (i - \frac{1}{2})h$ with $i = 1, 2, \ldots, M$. A matrix equation of (49) can be written in the generalized form

$$[A_{ij}]\{\phi(\xi_j)\} = \{\Delta_i\} \tag{52}$$

where $\{\Delta_i\}$ is implied from (49) and

$$A_{ij} = (1 + A^*)\delta_{ij} + K^*(\xi_j, \xi_i)h \tag{53}$$

Considering the result (35) for non-dimensional load–displacement relationship for the rigid foundation, the integral in (35) can be represented by its discretized equivalent

$$\bar{P} = \frac{2G_s}{G_\infty}\frac{(1 - \nu)^2}{(1 - 2\nu)}\left[\frac{1}{M}\sum_i \phi(\xi_i)\right] \tag{54}$$

The numerical procedure outlined in this section was applied to evaluate the influence of the near surface non-homogeneity on the axial stiffness of the rigid circular foundation resting on a non-homogeneous isotropic elastic half-space. The non-homogeneity is characterized by two non-dimensional parameters. The first parameter G_s/G_∞ describes either an elastic medium with stiffer surface region ($G_s > G_\infty$) or an elastic medium with a softened surface region ($G_s < G_\infty$) and the second parameter $\bar{\zeta}$ where

$$\zeta = \frac{\bar{\zeta}}{a} \tag{55}$$

describes the depth–dependent growth or decay of the shear modulus. Figures 2–4 illustrate the manner in which both $\bar{\zeta}$ and G_s/G_∞ influence the non-dimensional axial stiffness of the rigid foundation. It is evident that the load-settlement relationship is significantly influenced by both these parameters. As would be expected, the settlement of the foundation decreases as the surface shear modulus exceeds the shear modulus at infinity. As $\bar{\zeta}$ increases the stiffness is relatively uninfluenced by the shear modulus mismatch G_s/G_∞. In the limit as $\bar{\zeta} \to \infty$, the solution for the stiffness of the foundation reduces to that for a homogeneous elastic half-space of shear modulus G_∞.

It is of interest to relate the results presented in this study to those obtained by Calladine and Greenwood[23] for the incompressible elastic half-space with a linear variation of $G(z)$. This can be achieved by setting $G_s = 0$, $\nu = 1/2$ and matching the two variations in $G(z)$ (i.e. linear and exponential) at the origin of z. The comparison of the results can be made by evaluating the non-dimensional parameter

$$\Gamma(\bar{\zeta}) = \frac{\{P(1 - \nu)/4G_\infty \Delta a\}}{\{\pi\bar{\zeta}/4\}} \tag{56}$$

which effectively represents the ratio of the displacement of the rigid circular foundation on an incompressible elastic half-space with an *exponential* non-homogeneity in the shear modulus to that of the displacement of a rigid circular foundation on an incompressible elastic half-space with a *linear* variation of shear modulus. It also follows that G_∞ must be made to behave as $(\bar{\zeta})^{-1}$. Figure 5 illustrates the variation of Γ with the non-dimensional parameter $\bar{\zeta}$ which characterizes the form of the exponential non-homogeneity. It is evident that as $\bar{\zeta}$ becomes small, the two types

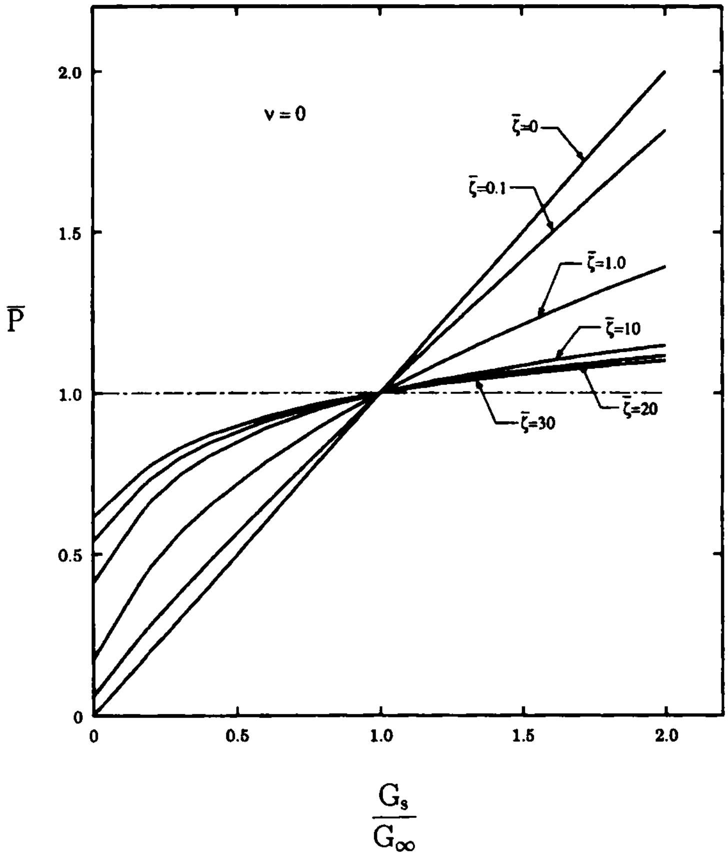

Figure 2. Axial stiffness of a rigid circular foundation on a non-homogeneous elastic half-space

of elastic non-homogeneity give identical estimates for the displacement of the rigid foundation. As $\bar{\zeta}$ becomes large, the variation of shear modulus near the surface of the half-space region has a significant influence on the displacement of the rigid circular foundation. An explanation for the deviation in the results can be obtained by expanding $G(z)$ in a power series form. As $G_s \to 0$, (13) gives the following:

$$G(z) = G_\infty\left[\overline{\zeta z} - \left\{\frac{(\overline{\zeta z})^2}{2}\right\} + \left\{\frac{(\overline{\zeta z})^2}{6}\right\} - \cdots\right] \tag{57}$$

where $\bar{z} = z/a$. As is evident, a good correlation will exist between results derived for the *linear* and *exponential* variations in the shear modulus (with $\nu = 1/2$; $G_s = 0$) provided $\overline{\zeta z}$ is small, or in a weak sense we can impose the condition $\bar{\zeta} \ll 1$.

An important assumption in the development of the analysis relates to the smooth base contact between the rigid circular foundation and the elastic half-space. This idealization is, of course, rarely achieved in practice, where frictional effects or adhesive contact conditions are expected to influence the load–displacement behaviour of the rigid circular foundation. This assertion can be

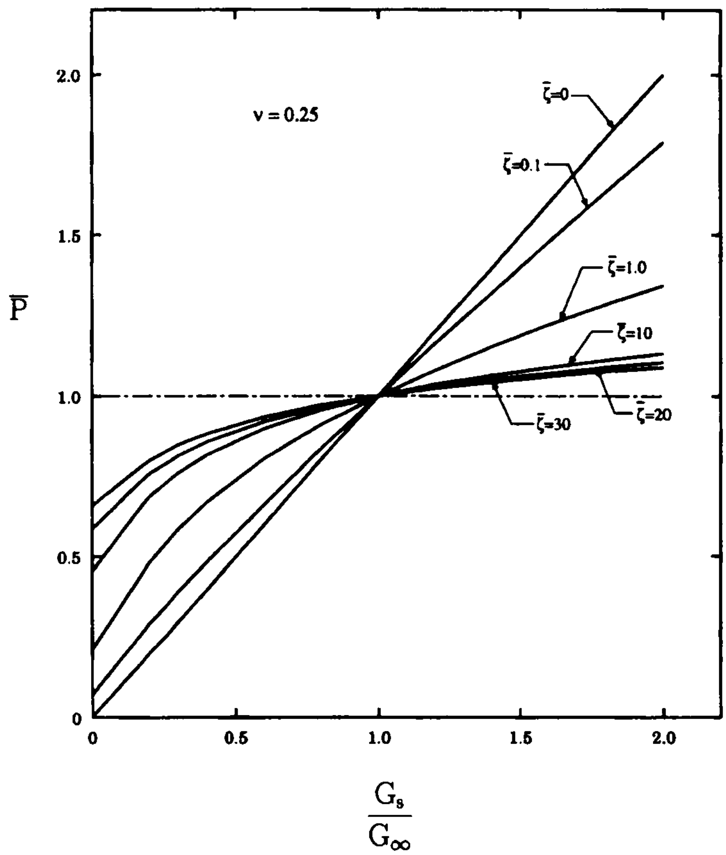

Figure 3. Axial stiffness of a rigid circular foundation on a non-homogeneous elastic half-space

conjectured from the generalized results developed by Shield and Anderson[50] and Shield.[51] In general, the stiffness of the rigid circular foundation is expected to increase in the presence of adhesion or frictional effects at the base of the foundation. From the point of application of the results to engineering practice, the assumption of smooth contact conditions at the soil–foundation will always result in an underestimation of the elastic stiffness of the foundation.

5. CONCLUSIONS

The exponential form of the non-homogeneity in the linear elastic shear modulus considered in the paper represents a realistic variation which gives bounded values for the shear modulus at any point within the half-space region. The specific problem examined in the paper deals with axial settlement of a foundation on the surface of the non-homogeneous half-space region. It is shown that the integral equations governing this problem can be reduced to a single Fredholm integral equation of the second kind. A numerical solution of the integral equation is used to generate results of interest to geomechanics. It is shown that the load-settlement behaviour of the rigid circular foundation is influenced by the shear modulus ratio of the medium characterized by G_s/G_∞ and the spatial change of shear modulus with depth characterized by the parameter $\bar{\zeta}$. The numerical results presented in the paper for the displacement of the rigid circular foundation also

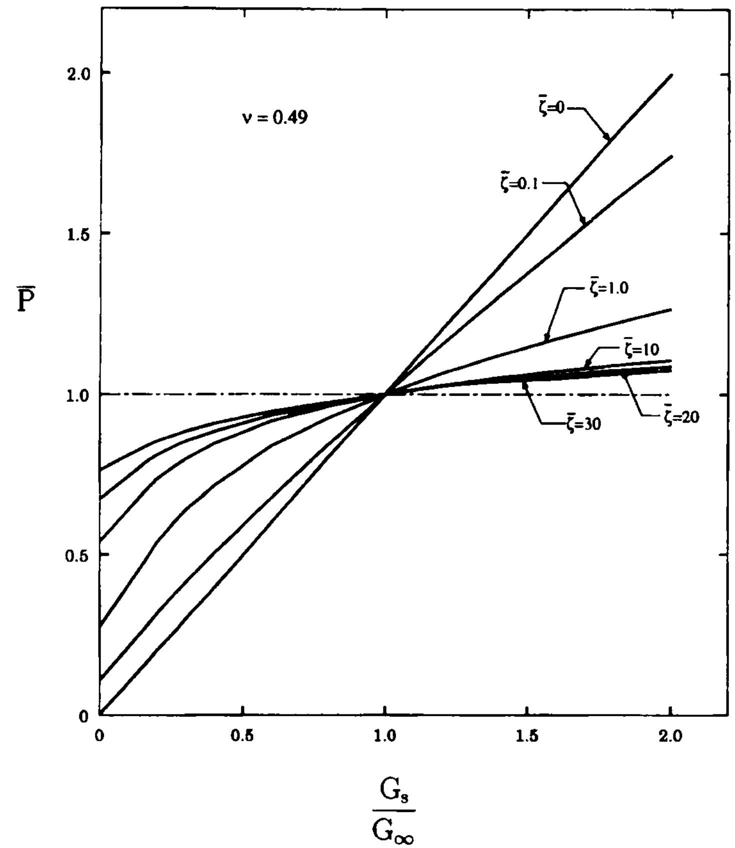

Figure 4. Axial stiffness of a rigid circular foundation on a non-homogeneous elastic half-space

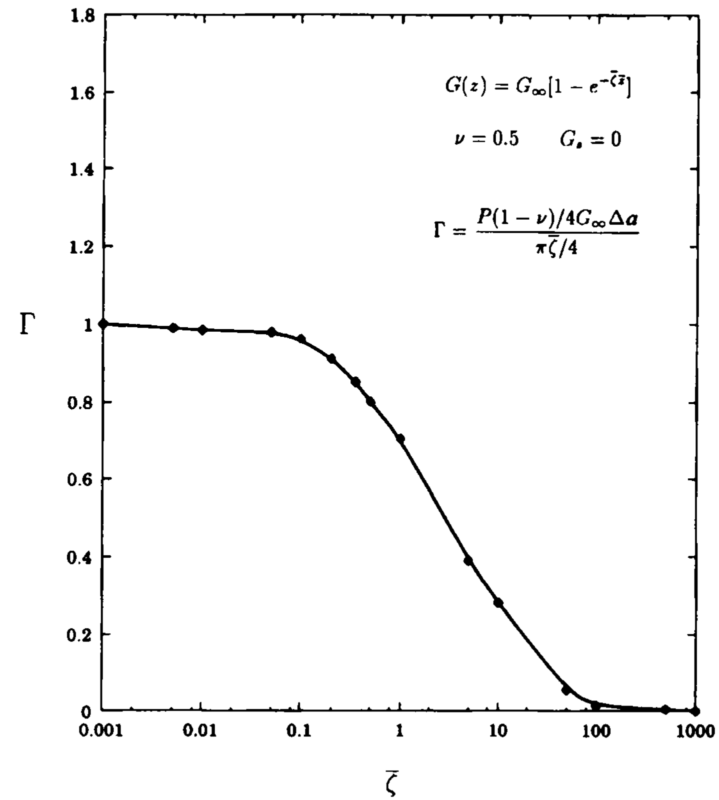

Figure 5. Displacement of a rigid circular foundation resting on a non-homogeneous incompressible elastic half-space. Comparison between linear and exponential variations in shear modulus ($G_s = 0$; $G(z) = G_\infty[1 - e^{\bar{\zeta}\bar{z}}]$)

make comparisons with equivalent results derived for the case of a rigid circular foundation resting on an incompressible elastic half-space with a linear variation of shear modulus and zero shear modulus at the surface. It is shown that the displacement of the rigid circular foundation is influenced by the far-field distribution in $G(z)$ and that the deviation in $G(z)$ between the *linear* and *exponential* variations (even though they may be matched at the surface of the half-space region) contributes to the observed differences. The condition $G(z) \to \infty$ as $z \to \infty$ is recognized as a strong constraint on the regularity and decay in the displacement and stress fields and such a constraint could contribute to the observed discrepancies. The basic mathematical procedure outlined in this paper can also be applied to evaluate the influences of other forms of elastic non-homogeneities associated with half-space regions. In such instances the axisymmetric problem basically reduces to the evaluation of a characteristic set of ordinary differential equations which are governed by the appropriate form of the elastic non-homogeneity.

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