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A NOTE ON THE TORSION OF A NON-HOMOGENEOUS SOLID BY AN ANNULAR DISC

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Abstract—The present paper examines the elasto-static problem concerning the axisymmetric rotation of a rigid annular circular punch perfectly bonded to the surface of a non-homogeneous isotropic, elastic half-space possessing a shear modulus, $\mu(z)$, varying with depth according to the relation $\mu(z) = \mu_1(c+z)^\alpha$, where c, μ_1 and α are constants.

1. INTRODUCTION

The problem of determining the torsional deformation of a semi-infinite isotropic, homogeneous, elastic solid when a circular cylinder is fully bonded to its planar surface and forced to rotate about its longitudinal axis was considered by Reissner and Sagoci (1944), Sagoci (1944) and Sneddon (1947). Later on, Sneddon (1966), Uflyand (1959), Gladwell (1969) and Freeman and Keer (1967) investigated the torsion problem for a cylinder and for a layer.

The Reissner–Sagoci problems for a vertically non-homogeneous, isotropic, elastic half-space were considered by Kassir (1970) and Chuaprasert and Kassir (1973). Moreover, Dhaliwal and Singh (1978) generalized the results of Kassir (1970) and Chuaprasert and Kassir (1973) assuming that the half-space consisted of two horizontal layers. In this paper we determine the deformation induced by rotating an annular rigid disc fully welded to the planar surface $z = 0$ of a vertically non-homogeneous half-space. The surfaces $z = 0$, $0 < r < a$ and $r > b$ with $a < b$ are stress-free. Finally, the moment required to produce rotation of the rigid disc is calculated and the results are shown graphically.

2. BASIC EQUATION AND ITS SOLUTION

In the case of an axisymmetric torsion problem, the displacement vector **U** assumes the form (o, v, o) in a cylindrical polar coordinate system (r, θ, z). The equation of equilibrium takes the form:

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}+\frac{1}{\mu}\frac{\partial v}{\partial z}\frac{\partial \mu}{\partial z}=0. \tag{1}$$

If one assumes that the shear modulus $\mu(z)$ is of the form:

$$\mu(z)=\mu_1(c+z)^{\alpha}, \quad c>0, \tag{2}$$

where μ_1, c and α are real constants, then the general solution of eqn (3) can be taken as

$$v(r,z)=(c+z)^p\int_0^\infty A(\xi)K_p[(c+z)\xi]J_1(r\xi)\,\mathrm{d}\xi, \tag{3}$$

where

$$p=\frac{1-\alpha}{2}. \tag{4}$$

$J_p(\,)$ and $K_p(\,)$ are, respectively, the Bessel function of the first kind and the modified Bessel function of the second kind of order p. $A(\xi)$ is an arbitrary function of ξ. With eqn (3) we find that

$$\sigma_{\theta z}(r,z)=-\mu_1(c+z)^{1-p}\int_0^\infty \xi A(\xi)K_{p-1}[(c+z)\xi]J_1(r\xi)\,\mathrm{d}\xi. \tag{5}$$

3. BOUNDARY CONDITIONS AND SOLUTION OF THE PROBLEM

The boundary conditions of the problem can be written as:

$$v(r,z)=f(r), \quad a<r<b, \tag{6}$$

$$\sigma_{\theta z}(r,z)=0, \quad 0<r<a, \quad r>b, \tag{7}$$

where $f(r)$ is a prescribed function of r. Making use of eqns (3) and (5) and conditions (6) and (7), we find the following triple integral equations:

$$\int_0^\infty \xi^{-1}B(\xi)(1+h(\xi))J_1(r\xi)\,\mathrm{d}\xi=c^{-p}f(r), \quad a<r<b, \tag{8}$$

$$\int_0^\infty B(\xi)J_1(r\xi)\,\mathrm{d}\xi=0, \quad 0<r<a, \quad r>b, \tag{9}$$

where

$$h(\xi)=\frac{K_p(c\xi)}{K_{p-1}(c\xi)}-1, \tag{10a}$$

$$B(\xi)=\xi A(\xi)K_{p-1}(c\xi). \tag{10b}$$

Making use of the paper of Cooke (1963), the solution of triple integral eqns (8) and (9) can be expressed in the following form:

$$B(\xi) = \xi \int_a^b ug(u)J_1(u\xi)\,du, \tag{11}$$

where $g(u)$ is determined from the integral equation

$$g(u) = -\frac{2}{\pi}\frac{d}{du}\int_u^b \frac{sG(s)\,ds}{(s^2-u^2)^{1/2}}, \tag{12}$$

$$s^2G(s) = F(s) + \int_a^b [K_1(s,t)+K_2(s,t)]G(t)\,dt, \quad a < s < b, \tag{13}$$

$$K_1(s,t) = \frac{-2st}{\pi^2[t^2-s^2]\sqrt{(s^2-a^2)(t^2-a^2)}}\left[s(a^2-s^2)\log\left|\frac{s+a}{s-a}\right| - t(a^2-t^2)\log\left|\frac{a+t}{a-t}\right|\right], \tag{14}$$

$$K_2(s,t) = -\frac{2}{\pi}\int_0^\infty h(\xi)I(s,\xi)I(t,\xi)\,d\xi, \tag{15}$$

$$I(s,\xi) = \left[\frac{saJ_1(\xi a)}{(s^2-a^2)^{1/2}} + s\xi e_1(s)\right], \tag{16}$$

$$e_1(s) = \int_a^s \frac{rJ_0(\xi r)\,dr}{(s^2-r^2)^{1/2}}, \tag{17}$$

and

$$F(s) = c^{-p}\frac{d}{ds}\int_a^s \frac{r^2 f(r)\,dr}{(s^2-r^2)^{1/2}}. \tag{18}$$

Equations (12) and (13) are an Abel integral equation and a Fredholm integral equation of the second kind, respectively. For determining $g(u)$, we first find $G(s)$ from eqn (13) and then finally $g(u)$ from eqn (12).

If $f(r) = \gamma r$, then we find from eqn (18) that

$$F(s) = \frac{\gamma s(2s^2-a^2)c^{-p}}{(s^2-a^2)^{1/2}}. \tag{19}$$

If $a \to 0$, then eqn (13) reduces in form to be in complete agreement with the result obtained by Chuaprasert and Kassir [1973: p. 709, eqn (20)]. The moment required of the disc as prescribed by the condition (7) is given by

$$T = -2\pi\int_a^b r^2\sigma_{\theta z}(r,0)\,dr = 2\pi\mu_1 c^{1-p}\int_a^b r^2 g(r)\,dr. \tag{20}$$

Making use of eqn (12), we find from eqn (20) that

$$T = 4\mu_1 c^{1-p}\left[a^2 \int_a^b \frac{sG(s)\,ds}{(s^2-a^2)^{1/2}} + 2\int_a^b s\sqrt{s^2-a^2}G(s)\,ds\right]. \tag{21}$$

If $a \to 0$ and $\alpha = 0$, we can easily find that

$$T_\infty = \frac{16\mu_1\gamma_1 b^3}{3}. \tag{22}$$

For computational purposes we assume that

$$s = a\sec\theta, \quad t = a\sec\phi, \quad G(a\sec\theta)\sec^2\theta = \gamma H_1(\theta)c^{-p}. \tag{23}$$

Making use of eqns (19) and (23), we write eqn (13) in the following form:

$$\sin\theta\cos^2\theta H_1(\theta) = 1+\sin^2\theta+\sin\theta\cos^2\theta\int_0^{\sec^{-1}(b/a)}[K_1(\theta,\phi)+K_2(\theta,\phi)] \times \sin\phi H(\phi)\,d\phi, \quad 0<\theta<\sec^{-1}(b/a), \tag{24}$$

where

$$K_1(\theta,\phi) = \frac{4\sec\theta\sec\phi}{\pi^2\tan\theta\tan\phi[\sec^2\phi-\sec^2\theta]}\left[\sec\phi\tan^2\phi\log\tan\left(\frac{\phi}{2}\right) - \sec\theta\tan^2\theta\log\tan\left(\frac{\theta}{2}\right)\right] \tag{25}$$

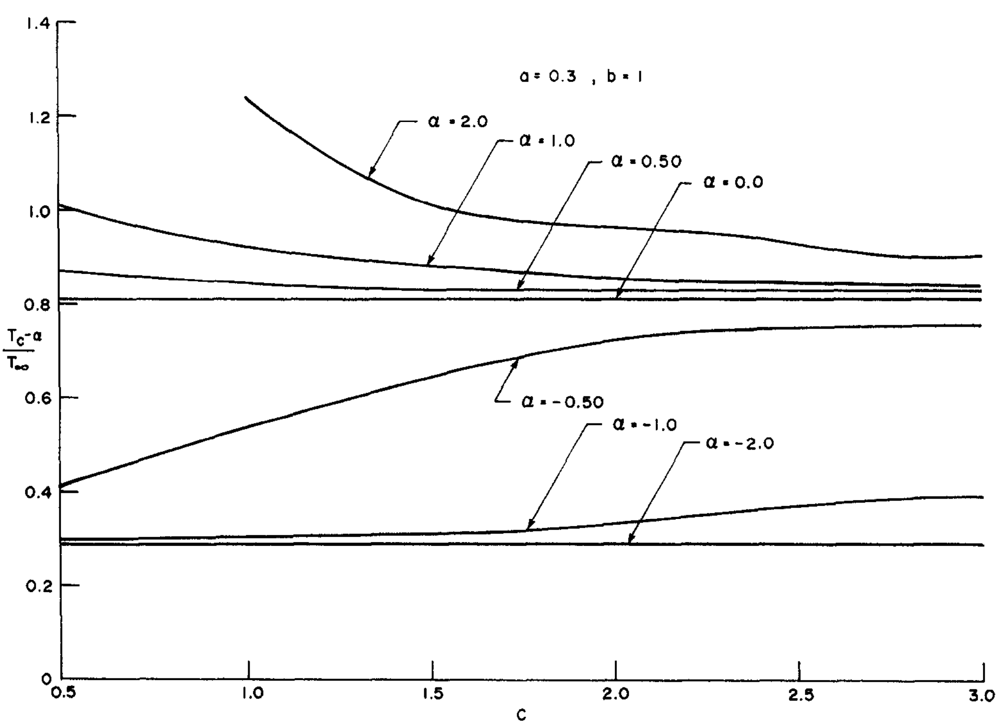

Fig. 1. Plot of $(T/T_\infty)c^{-\alpha}$ versus c for various α when $a = 0.3$ and $b = 1$.

and

$$K_2(\theta,\phi) = \frac{-2}{\pi a}\int_0^\infty h(\xi)I(a\sec\phi,\xi)I(a\sec\theta,\xi)\,d\xi. \tag{26}$$

With the help of eqns (21)–(23), we find that

$$\frac{T}{T_\infty} = \frac{3}{4}\left(\frac{a}{b}\right)^3 c^\alpha\left[\int_0^{\sec^{-1}(b/a)}[1+2\tan^2\theta]H_1(\theta)\,d\theta\right]. \tag{27}$$

Solving numerically eqn (24) and using eqn (27), we get the numerical results for T/T_∞.

The values of $(T/T_\infty)c^{-\alpha}$ are plotted for different values of c and α for particular values of $a = 0.3$ and $b = 1$. The results are shown in Fig. 1.

REFERENCES

Chuaprasert, M. F. and Kassir, M. (1973). Torsion of non-homogeneous solid. *J. Engng Mech. Div., Proc. Am. Soc. Civil Engng* **99**, 703–714.

Cooke, J. C. (1963). Triple integral equations. *Quart. J. Mech. Appl. Math.* **16**, 193–201.

Dhaliwal, R. S. and Singh, B. M. (1978). Torsion by a circular die of a non-homogeneous elastic layer bonded to a non-homogeneous half-space. *Int. J. Engng Sci.* **14**, 649–658.

Freeman, N. J. and Keer, L. M. (1967). Torsion of a cylindrical rod welded to an elastic half-space. *J. Appl. Mech.* **34**, 687–692.

Gladwell, G. M. L. (1969). The forced torsional vibration of an elastic stratum. *Int. J. Engng Sci.* **7**, 1011–1024.

Kassir, M. K. (1970). The Reissner–Sagoci problem for a non-homogeneous solid. *Int. J. Engng Sci.* **8**, 875–885.

Reissner, E. and Sagoci, H. F. (1944). Forced torsional oscillations of an elastic half-space I. *J. Appl. Phys.* **15**, 652–654.

Sagoci, H. F. (1944). Forced torsional oscillations of an elastic half-space II. *J. Appl. Phys.* **15**, 655–662.

Sneddon, I. N. (1947). A note on a boundary value problem of Reissner and Sagoci. *J. Appl. Phys.* **18**, 130–132.

Sneddon, I. N. (1966). The Reissner–Sagoci problem. *Proc. Glas. Math. Assoc.* **7**, 136–144.

Uflyand, Ia. S. (1959). Torsion of an elastic layer. *Dokl. Akad. Nauk., S.S.S.R.* **129**, 997–999.