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ON THE MECHANICS OF A RIGID DISC INCLUSION EMBEDDED IN A FLUID SATURATED POROELASTIC MEDIUM

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Abstract—The paper utilizes the classical integral transform techniques to develop the systems of Fredholm integral equations of the second kind, in the Laplace transform domain, governing the generalized displacement or loading of a rigid disc inclusion embedded in either permeable or impermeable bonded contact with a fluid saturated poroelastic infinite space. The generalized displacements correspond to an axial displacement, a rotation about the axial axis, a rotation about a diametral axis and an in-plane translation. The coupled integral equations in the Laplace transform domain are solved in a numerical fashion to generate results of technological interest. The numerical procedures focus on quadrature schemes for the solution of the integral equations and a procedure used for the inversion of Laplace transforms. The closed-form solutions of the integral equations as either $t \to 0^+$ or $t \to +\infty$ and as $\nu \to \nu_u$ are also obtained. The numerical results are presented for the time-dependent displacements and rotations of the inclusion subjected to Heaviside-step function type loads and moments, and for the time-dependent relaxation of the force and moment resultants in an inclusion subjected to Heaviside-step function type displacements and rotations. In particular, the influence of the compressibility of the pore fluid on the time dependent responses of the inclusion is documented.

INTRODUCTION

The mechanics of inclusions embedded in deformable media has always occupied an important position in applied mechanics and in the theory of materials. The notion of an inclusion, three-dimensional or otherwise can be used to describe an inhomogeneity or defect which is encountered in composite and multiphase materials. Inclusions are introduced as a strengthening mechanism for otherwise brittle materials which are susceptible to fracture and damage due to both mechanical and environmental loads. Classical studies in elasticity related to three-dimensional inclusions embedded in bonded contact with an elastic medium are given by Goodier [1], Dewey [2], Edwards [3], Robinson [4], Eshelby [5], and Lur'e [6]. Since these seminal works, studies related to inclusion problems have been extended to cover: (i) the influence of non-classical interface phenomena including friction separation and slip, (ii) the influence of anisotropic materials, (iii) dynamic and wave propagation effects, (iv) interactions with nuclei of strain and (v) the influence of boundaries and neighbouring inclusions. Extensive reviews of the subject of inclusions embedded in elastic media are given in the review articles by Sternberg [7], Willis [8], Walpole [9], Mura [10] and in the texts by Gladwell [11] and Mura [12].

The disc inclusion is a particular limiting case of the general class of three-dimensional inclusions with an ellipsoidal or spheroidal shape. Despite its highly idealized configuration, the mechanics of disc inclusions has been the subject of a sustained research effort. In particular, the idealization of the inclusion, as a defect or inhomogeneity of infinitesimal thickness enables the formulation of such inclusion problems as appropriate mixed boundary value problems associated with a halfspace region. This aspect has been successfully exploited in the analysis of disc inclusions embedded elastic media. Collins [13] examined the problem of axial loading of a

rigid circular disc inclusion embedded in an elastic medium. The in-plane translation of a rigid circular disc inclusion embedded in an elastic infinite space was examined by Keer [14]. Since these early studies, the disc inclusion problem associated with an elastic medium has been extensively studied to include: (i) flexibility of the inclusion, (ii) delamination at the inclusion–elastic medium interface, (iii) interaction of the inclusion with cracks, (iv) interaction with external forces, (v) interaction with neighbouring inclusions, (vi) influence of anisotropic materials, (vii) dynamic loads and wave propagation effects and (viii) elliptical configuration of the disc inclusion. Accounts of these developments are given by Mura [10], Yue [15] and in a forthcoming article by Selvadurai [16].

Many of the current studies related to disc inclusion problems have been motivated by their potential application to studies in mechanics of multiphase media reinforced by low concentrations of such inclusions. In the area of geomechanics, the embedded disc inclusion also serves as a model for the examination of the compliance of ground anchors [17, 18] and for the study of *in situ* testing devices used to determine the properties of geomaterials [19]. When dealing with geomaterials in particular, the assumptions of elastic behavior places a restriction on the applicability of such results to media which exhibit time-dependent phenomena. Geomaterials can exhibit time-dependent constitutive responses as a result of consolidation and/or creep effects. The objective of the current investigation is to extend the studies related to embedded disc inclusions to include effects of poroelasticity. In particular, the generalized theory of poroelasticity proposed by Biot [20, 21] is used to examine the time-dependent behavior of embedded rigid disc inclusions which are subjected to axial, rotational or in-plane translational displacements or the equivalent generalized forces and moments.

The classical theory of poroelasticity has provided the basis for the solutions of many problems of practical interest to mechanics of fluid saturated geomaterials. The consideration of an ideal fluid and elastic behavior of the deformable porous solid is regarded as a useful first approximation for the treatment of geomaterials in which the applied loads do not induce irreversible deformations in the porous solid. The extension of Biot's basic formulations to include such irreversible phenomena can be contemplated only as exercises in computational modelling [22, 23]. Even within the context of Biot's formulation, the majority of investigations have focused on the analysis of poroelastic media which are saturated with incompressible fluids. With poroelastic soils and rocks which are saturated with pore fluids which contain dissolved or distributed air voids or other gaseous phases, it becomes important to consider the influence of undrained compressibility of the medium. This aspect of undrained compressibility has been investigated by Rice and Cleary [24] who develop certain fundamental solutions for stress diffusion in isotropic porelastic media. Bishop and Hight [25] have also developed estimates for Skempton's pore pressure parameter B for instances where the poroelastic medium exhibits undrained compressibility effects.

The present study focuses on the application of Biot's theory for a poroelastic medium, saturated with a compressible pore fluid, to the examination of the mechanics of an embedded rigid circular disc inclusion. At first, the paper focuses for the development of the integral equations governing the inclusion problems. Fourier and Laplace transforms techniques are applied to generate the systems of Fredholm integral equations of the second-kind governing the basic displacment modes of the embedded inclusion which can exhibit *impermeable* or *permeable* interface conditions. Secondly, numerical solutions for the integral equations governing the three modes of displacement of the embedded rigid disc inclusion will be presented by adopting a numerical scheme based on the Laplace transform inversion. Finally, the time-dependent results for the translational and rotational stiffness and compliance of the embedded rigid inclusion are evaluated for various choices of the governing constitutive parameters of the porelastic medium.

GOVERNING EQUATIONS

In the ensuing we shall present a brief account of the governing equations referred to a Cartesian tensor notation. The constitutive equations governing the quasi-static response of a poroelastic medium, which consists of an isotropic poroelastic soil skeleton saturated with a compressible pore fluid take the forms

$$\sigma_{ij} = \frac{2\mu\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)}p\delta_{ij}$$

$$p = \frac{2\mu B^2(1-2\nu)(1+\nu_u)^2}{9(\nu_u - \nu)(1-2\nu_u)}\zeta_v - \frac{2\mu B(1+\nu_u)}{3(1-2\nu_u)}\varepsilon_{kk} \tag{1a}$$

where δ_{ij} is the Kronecker delta; p is the pore fluid pressure; ζ_v is the volumetric strain in the pore fluid; σ_{ij} is the total stress tensor; ε_{ij} are the soil skeleton strains defined by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1b}$$

u_i are the corresponding displacement components and the comma denotes a partial derivative with respect to a spatial variable ($i, j = x, y$ or z). In the absence of body forces, the quasi-static equations of equilibrium take the forms

$$\sigma_{ij,j} = 0. \tag{1c}$$

The equations governing quasi-static fluid flow are defined by Darcy's law which takes the form

$$v_i = -\kappa p_{,i} \tag{1d}$$

where v_i is the specific discharge vector in the pore fluid. The continuity equation associated with quasi-static fluid flow is

$$\frac{\partial \zeta_v}{\partial t} + v_{i,i} = 0. \tag{1e}$$

The governing equations are characterized by the five independent material parameters which are the following: the drained and undrained Poisson's ratios ν and ν_u respectively, the shear modulus μ, Skempton's pore pressure coefficient B, and the κ ($= k/\gamma_w$, where k is the coefficient of permeability and γ_w is the unit weight of pore fluid). Considering requirements for a positive definite strain energy potential, it can be shown that the material parameters have the following thermodynamic constraints: $\mu > 0$; $0 \le B \le 1$; $-1 < < \nu_u \le 0.5$; $\kappa > 0$ [24].

Applying the theory of Fourier integral transforms [26], it can be shown that the following sets of solution representations exist for the field variables in a linear, isotropic, poroelastic halfspace saturated with a compressible pore fluid. Considering the cylindrical polar coordinate systems (r, θ, z) and (ρ, φ, z), we have, in either the temporal domain or the Laplace transform domain,

$$\mathbf{u}(r,\theta,z,t) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \mathbf{\Pi}_a \mathbf{w} K \,\mathrm{d}\varphi\,\mathrm{d}\rho, \qquad \mathbf{w}(\rho,\varphi,z,t) = \frac{\rho}{2\pi}\int_0^\infty\int_0^{2\pi} \mathbf{\Pi}_a^* \mathbf{u} K^* r \,\mathrm{d}\theta\,\mathrm{d}r$$

$$\mathbf{T}_z(r,\theta,z,t) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \mathbf{\Pi}_a \mathbf{Y}_z K\rho \,\mathrm{d}\varphi\,\mathrm{d}\rho, \qquad \mathbf{Y}_z(\rho,\varphi,z,t) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \mathbf{\Pi}_a^* \mathbf{T}_z K^* r \,\mathrm{d}\theta\,\mathrm{d}r$$

$$\mathbf{v}(r,\theta,z,t) = \frac{2}{2\pi}\int_0^\infty\int_0^{2\pi} \mathbf{\Pi}_a \vartheta K\rho^2 \,\mathrm{d}\varphi\,\mathrm{d}\rho, \qquad \vartheta(\rho,\varphi,z,t) = \frac{1}{2\pi\rho}\int_0^\infty\int_0^{2\pi} \mathbf{\Pi}_a^* \mathbf{v} K^* r \,\mathrm{d}\theta\,\mathrm{d}r$$

$$p(r,\theta,z,t) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} p_w K\rho \,\mathrm{d}\varphi\,\mathrm{d}\rho, \qquad p_w(\rho,\varphi,z,t) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} p K^* r \,\mathrm{d}\theta\,\mathrm{d}r \tag{2a}$$

where $0^+ \le z < \infty$, or $-\infty < z \le 0^-$, $0 < t < \infty$, and the integrals are interpreted in the sense of a Cauchy principal value. The vectors in (2a) are defined by $\mathbf{u} = (u_r, u_\theta, u_z)^T$, $\mathbf{T}_z = (\sigma_{rz}, \sigma_{\theta z}, \sigma_{zz})^T$, $\mathbf{v} = (v_r, v_\theta, v_z)^T$, $\mathbf{w} = (w_1, w_2, w_3)^T$, $\mathbf{Y}_z = (\tau_1, \tau_2, \tau_3)^T$, and $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3)^T$. The superscript T stands for the transpose of matrix. K^* and $\mathbf{\Pi}_a^*$ are, respectively, the complex conjugates of the Fourier matrix kernel functions K and $\mathbf{\Pi}_a$ defined by

$$\mathbf{K} = e^{i\rho r \sin(\theta+\varphi)}, \qquad \mathbf{\Pi}_a = \begin{pmatrix} i\sin(\theta+\varphi) & i\cos(\theta+\varphi) & 0 \\ i\cos(\theta+\varphi) & -i\sin(\theta+\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2b}$$

By assuming the initial condition $\zeta_v|_{t=0} = 0$, the governing equations can be re-writen as two sets of first-order ordinary differential equations in the Fourier and Laplace transform domains; i.e.

$$\frac{d}{dz}\mathbf{V}_v(\rho, \varphi, z, s) = \rho \mathbf{C}_v \mathbf{V}_v(\rho, \varphi, z, s), \qquad \frac{d}{dz}\mathbf{V}_u(\rho, \varphi, z, s) = \rho \mathbf{C}_u \mathbf{V}_u(\rho, \varphi, z, s) \tag{3a}$$

where

$$\mathbf{V}_v = \left(\hat{w}_2, \frac{1}{\mu}\hat{\tau}\right)^T; \qquad \mathbf{V}_u = \left(\hat{w}_1, \hat{w}_3, \frac{1}{2\mu}\hat{\tau}_3, \frac{1}{2\mu}\hat{\tau}_1, \frac{1}{2\mu\alpha_d}\hat{p}_w, \frac{1}{2\mu\kappa\alpha_d}\hat{\vartheta}_3\right)^T;$$

and

$$\mathbf{C}_v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{C}_u = \begin{pmatrix} 0 & -1 & 0 & 2 & 0 & 0 \\ \frac{\nu}{1-\nu} & 0 & \frac{1-2\nu}{1-\nu} & 0 & \nu_k & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{1-\nu} & 0 & \frac{-\nu}{1-\nu} & 0 & \nu_k & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \gamma^2-1 & 0 & 1-\gamma^2 & 0 & -\gamma^2 & 0 \end{pmatrix}$$

$$c = \frac{2\mu B^2(1-\nu)(1+\nu_u)^2\kappa}{9(\nu_u-\nu)(1-\nu_u)}, \qquad \nu_k = \frac{\nu_u-\nu}{(1-\nu)(1-\nu_u)}, \qquad \alpha_d = \frac{B(1+\nu_u)}{3(1-\nu_u)}, \qquad \gamma = \sqrt{\frac{1}{c}\frac{s}{\rho^2}+1} \tag{3b}$$

and the superscript ^ stands for the Laplace transform with respect to t, and s is the Laplace transform parameter.

Solution representations governing the field variables in the halfspace regions ($0^+ \le z < \infty$) and ($-\infty < z \le 0^-$) can be further obtained by solving the ordinary differential equations together with the four regularity conditions required as $z \to +\infty$ and $z \to -\infty$, respectively. It can be shown that there are four independent boundary equations which govern the eight variables at the boundary surface of the halfspace region:

(i) For the halfspace region ($0^+ \le z < \infty$), we have

$$(1 \quad 1)\mathbf{V}_v(0^+) = 0; \qquad \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & \frac{-1}{\gamma} & -1 & \frac{1}{\gamma} & -1 & \frac{1}{\gamma} \\ 1 & 1 & 3-4\nu_u & 3-4\nu_u & -\alpha_\gamma & \alpha_\gamma \end{pmatrix} \mathbf{V}_u(0^+) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4a}$$

(ii) For the halfspace region ($-\infty < z \leq 0^-$), we have

$$(1 \quad -1)\mathbf{V}_v(0^-) = 0; \qquad \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & \dfrac{-1}{\gamma} & 1 & \dfrac{1}{\gamma} & 1 & \dfrac{1}{\gamma} \\ -1 & 1 & 4\nu_u - 3 & 3 - 4\nu_u & \alpha_\gamma & \alpha_\gamma \end{pmatrix} \quad \mathbf{V}_u(0^-) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4b}$$

where $\mathbf{V}_v(0^\pm) = \mathbf{V}_v(\rho, \varphi, 0^\pm, s)$; $\mathbf{V}_u(0^\pm) = \mathbf{V}_u(\rho, \varphi, 0^\pm, s)$; and

$$\alpha_\gamma = \frac{4(\nu_u - \nu)}{(1-\nu)(\gamma^2 - 1)}. \tag{4c}$$

THE DISC INCLUSION PROBLEMS

We consider the generalized displacements of a disc inclusion in a poroelastic medium saturated with a compressible pore fluid (Fig. 1). The generalized displacements induced at the inclusion–poroelastic medium interface in the r, θ and z directions are denoted by, $\mathbf{U}(r, \theta, t) = (U_r, U_\theta, U_z)^{\mathrm{T}}$. The associated force and moment resultants acting on the plane inclusion are denoted by $\mathbf{P}(t) = (P_x, P_y, P_z)^{\mathrm{T}}$ and $\mathbf{M}(t) = (M_x, M_y, M_z)^{\mathrm{T}}$, respectively. The interface between the saturated poroelastic medium and the inclusion is assumed to be either *impermeable* or *permeable*. By considering the symmetry or asymmetry of the embedded disc inclusion problems it can be shown that all of these generalized loading problems can be formulated in relation to mixed boundary value problems associated with the halfspace regions occupying $z \in [0^+, +\infty)$ and $z \in (-\infty, 0^-]$. The notation 0^+ and 0^- are intended to designate the plane surfaces of the halfspace regions corresponding to $z \geqslant 0$ and $z \leq 0$ respectively. For convenience we introduce two regions $\tilde{A}$ and $\bar{A}$ which correspond to the regions on $z = 0$ occupied respectively by the inclusion and its exterior. It may be noted that $\tilde{A} \cup \bar{A} = A$ where A is the entire plane $z = 0$ containing the plane inclusion and $\tilde{A} \cap \bar{A} = \{0\}$, implying that there is no separation, interpenetration or cracking induced by the loading of the embedded inclusion for any time $t \in (0, +\infty)$. Considering the generalized displacements we have,

$$\mathbf{u}(r, \theta, 0^+, t) = \mathbf{u}(r, \theta, 0^-, t) = \mathbf{U}(r, \theta, t), \quad \text{for} \quad r, \theta \in \tilde{A}. \tag{5a}$$

The pore pressure boundary conditions at the interface are either

$$p(r, \theta, 0^+, t) = p(r, \theta, 0^-, t) = 0, \quad \text{for} \quad r, \theta \in \tilde{A} \tag{5b}$$

or

$$v_z(r, \theta, 0^+, t) = v_z(r, \theta, 0^-, t) = 0, \quad \text{for} \quad r, \theta \in \tilde{A} \tag{5c}$$

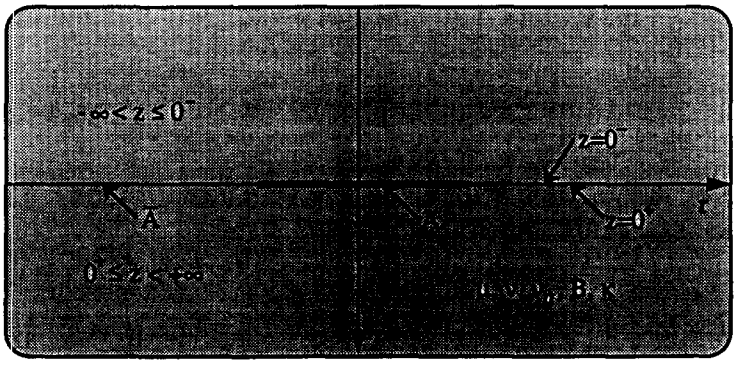

Fig. 1. A disc inclusion embedded in bonded contact with a poroelastic medium saturated with a compressible pore fluid.

which correspond to *permeable* and *impermeable* interfaces respectively. The definition of the boundary conditions can be made complete with the specification of continuity conditions associated with the region $\bar{A}$. We have, in general,

$$\mathbf{u}(r,\theta,0^+,t)=\mathbf{u}(r,\theta,0^-,t),\qquad \mathbf{T}_z(r,\theta,0^+,t)=\mathbf{T}_z(r,\theta,0^-,t),\quad \text{for}\quad r,\theta\in\bar{A}$$

$$p(r,\theta,0^+,t)=p(r,\theta,0^-,t),\qquad v_z(r,\theta,0^+,t)=v_z(r,\theta,0^-,t),\quad \text{for}\quad r,\theta\in\bar{A}. \tag{5d}$$

The force and moment resultants acting on the inclusion can be expressed in terms of the contact at the interfaces as follows:

$$\mathbf{P}(t)=-\iint_{\bar{A}}\mathbf{\Pi}_b[\mathbf{T}_z(r,\theta,0^+,t)-\mathbf{T}_z(r,\theta,0^-,t)]r\,\mathrm{d}\theta\,\mathrm{d}r$$

$$\mathbf{M}(t)=-\iint_{\bar{A}}\mathbf{\Pi}_c[\mathbf{T}_z(r,\theta,0^+,t)-\mathbf{T}_z(r,\theta,0^-,t)]r^2\,\mathrm{d}\theta\,\mathrm{d}r \tag{5e}$$

where

$$\mathbf{\Pi}_b=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix},\qquad \mathbf{\Pi}_c=\begin{pmatrix}0 & 0 & \sin\theta\\ 0 & 0 & \cos\theta\\ 0 & 1 & 0\end{pmatrix}. \tag{5f}$$

2-D INTEGRAL EQUATIONS GOVERNING A PLANE INCLUSION

Considering the continuity conditions for variables $\mathbf{u}$ and p at the plane containing the *permeable* inclusion ($z=0, 0\le r<+\infty, 0\le\theta<2\pi$), we have $\hat{\mathbf{u}}(r,\theta,0^\pm,s)=\hat{\mathbf{u}}(r,\theta,0,s)$ and $\hat{p}(r,\theta,0^\pm,s)=\hat{p}(r,\theta,0,s)$. The corresponding conditions in the transform domain are $\hat{\mathbf{w}}(\rho,\varphi,0^\pm,s)=\hat{\mathbf{w}}(\rho,\varphi,0,s)(=\hat{\mathbf{w}}(0))$ and $\hat{p}_w(\rho,\varphi,0^\pm,s)=\hat{p}_w(\rho,\varphi,0,s)(=\hat{p}_w(0))$. Using equations (2) and the boundary conditions (5a, 5b, 5d), we obtain the following system of 2-D Fourier transform based integral equations in the Laplace transform domain:

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\mathbf{\Pi}_a\hat{\mathbf{w}}(0)K\,\mathrm{d}\varphi\,\mathrm{d}\rho=\hat{\mathbf{U}}(r,\theta,s),\qquad r,\theta\in\tilde{A}$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\hat{p}_w(0)K\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho=0,\qquad r,\theta\in\tilde{A}$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\mathbf{\Pi}_a[\hat{\mathbf{Y}}_z(0^+)-\hat{\mathbf{Y}}_z(0^-)]K\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho=\mathbf{0},\qquad r,\theta\in\bar{A}$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}[\hat{\vartheta}_3(0^+)-\hat{\vartheta}_3(0^-)]K\rho^2\,\mathrm{d}\varphi\,\mathrm{d}\rho=0,\qquad r,\theta\in\bar{A}. \tag{6}$$

Using equations (4), the variables $\hat{\mathbf{Y}}_z(0^\pm)(=\hat{\mathbf{Y}}_z(\rho,\varphi,0^\pm,s))$ and $\hat{\vartheta}_3(0^\pm)(=\hat{\vartheta}_3(\rho,\varphi,0^\pm,s))$ can be expressed in terms of $\hat{\mathbf{w}}(0)$ and $\hat{p}_w(0)$: i.e.

$$\begin{pmatrix}\hat{t}_3(0^\pm)\\ \hat{t}_1(0^\pm)\\ \hat{\vartheta}_3(0^\pm)\\ \hat{t}_2(0^\pm)\end{pmatrix}=\begin{pmatrix}b_{11}(\gamma) & \mp b_{12}(\gamma) & b_{13}(\gamma) & 0\\ \mp b_{21}(\gamma) & b_{22}(\gamma) & \mp b_{23}(\gamma) & 0\\ \mp b_{31}(\gamma) & b_{32}(\gamma) & \mp b_{33}(\gamma) & 0\\ 0 & 0 & 0 & \mp\mu\end{pmatrix}\begin{pmatrix}\hat{w}_1(0)\\ \hat{w}_3(0)\\ \hat{p}_w(0)\\ \hat{w}_2(0)\end{pmatrix} \tag{7}$$

where the coefficients $b_{ij}(\gamma)$ $(i, j = 1, 2, 3)$ are given in Appendix A.

Substituting the expressions (7) for $\hat{\mathbf{Y}}_z(0^\pm)$ and $\hat{\vartheta}_3(0^\pm)$, into the equations (6), we decouple the system of 2-D integral equations (6) as two systems in terms of $\hat{w}_3(0)$ and $\hat{\mathbf{w}}_p(0)$ $(= [\hat{w}_1(0), \hat{w}_2(0), \hat{p}_w(0)]^{\mathrm{T}})$, respectively. The first system is

$$\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \hat{w}_3(0) K \, \mathrm{d}\varphi \, \mathrm{d}\rho = \hat{U}_z(r, \theta, s); \qquad r, \theta \in \tilde{A}$$

$$\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} b_{12}(\gamma)\hat{w}_3(0) K\rho \, \mathrm{d}\varphi \, \mathrm{d}\rho = 0; \qquad r, \theta \in \bar{A} \tag{8a}$$

and the second system is

$$\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \mathbf{\Pi}_d \hat{\mathbf{w}}_p(0) K \, \mathrm{d}\varphi \, \mathrm{d}\rho = \hat{\mathbf{U}}_h(r, \theta, s); \qquad r, \theta \in \tilde{A}$$

$$\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \mathbf{\Pi}_d \mathbf{B}_a \hat{\mathbf{w}}_p(0) K\rho \, \mathrm{d}\varphi \, \mathrm{d}\rho = \mathbf{0}; \qquad r, \theta \in \bar{A} \tag{8b}$$

where $\hat{\mathbf{U}}_h = (\hat{U}_r, \hat{U}_\theta, 0)$, and

$$\mathbf{\Pi}_d = \begin{pmatrix} i\sin(\varphi+\theta) & i\cos(\varphi+\theta) & 0 \\ i\cos(\varphi+\theta) & -i\sin(\varphi+\theta) & 0 \\ 0 & 0 & \rho \end{pmatrix}, \qquad \mathbf{B}_a = \begin{pmatrix} b_{21}(\gamma) & 0 & b_{23}(\gamma) \\ 0 & \mu & 0 \\ b_{31}(\gamma) & 0 & b_{33}(\gamma) \end{pmatrix}. \tag{8c}$$

Similarly, considering the continuity conditions for variables $\mathbf{u}$ and v_z at the plane containing the impermeable inclusion $(z = 0, 0 \le r < +\infty, 0 \le \theta < 2\pi)$, we have $\hat{\mathbf{w}}(\rho, \varphi, 0^\pm, s) = \hat{\mathbf{w}}(\rho, \varphi, 0, s)(= \hat{\mathbf{w}}(0))$ and $\hat{\vartheta}_3(\rho, \varphi, 0^\pm, s) = \hat{\vartheta}_3(\rho, \varphi, 0, s)(= \hat{\vartheta}_3(0))$. Using equations (2) and the boundary conditions (5a, 5c, 5d), we obtain the following system of 2-D integral equations in the Laplace transform domain:

$$\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \mathbf{\Pi}_a \hat{\mathbf{w}}(0) K \, \mathrm{d}\varphi \, \mathrm{d}\rho = \hat{\mathbf{U}}(r, \theta, s), \qquad r, \theta \in \tilde{A}$$

$$\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \hat{\vartheta}_3(0) K\rho^2 \, \mathrm{d}\varphi \, \mathrm{d}\rho = 0, \qquad r, \theta \in \tilde{A}$$

$$\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \mathbf{\Pi}_a [\hat{\mathbf{Y}}_z(0^+) - \hat{\mathbf{Y}}_z(0^-)] K\rho \, \mathrm{d}\varphi \, \mathrm{d}\rho = \mathbf{0}, \qquad r, \theta \in \bar{A}$$

$$\frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} [\hat{p}_w(0^+) - \hat{p}_w(0^-)] K\rho \, \mathrm{d}\varphi \, \mathrm{d}\rho = 0, \qquad r, \theta \in \bar{A}. \tag{9}$$

The variables $\hat{\mathbf{Y}}_z(0^\pm)(= \hat{\mathbf{Y}}_z(\rho, \varphi, 0^\pm, s))$ and $\hat{p}_w(0^\pm)(= \hat{p}_w(\rho, \varphi, 0^\pm, s))$ can be expressed in terms of $\hat{\mathbf{w}}(0)$ and $\hat{\vartheta}_3(0)$ by solving the equations (4), i.e.

$$\begin{pmatrix} \hat{\tau}_3(0^\pm) \\ \hat{\tau}_1(0^\pm) \\ \hat{p}_w(0^\pm) \\ \hat{\tau}_2(0^\pm) \end{pmatrix} = \begin{pmatrix} c_{11}(\gamma) & \mp c_{12}(\gamma) & \mp c_{13}(\gamma) & 0 \\ \mp c_{21}(\gamma) & c_{22}(\gamma) & c_{23}(\gamma) & 0 \\ c_{31}(\gamma) & \mp c_{32}(\gamma) & \mp c_{33}(\gamma) & 0 \\ 0 & 0 & 0 & \mp\mu \end{pmatrix} \begin{pmatrix} \hat{w}_1(0) \\ \hat{w}_3(0) \\ \hat{\vartheta}_3(0) \\ \hat{w}_2(0) \end{pmatrix}. \tag{10}$$

where the coefficients c_{ij} $(i, j = 1, 2, 3)$ are given in Appendix A.

Substituting the expressions (10) of $\hat{\mathbf{Y}}_z(0^\pm)$ and $\hat{p}_w(0^\pm)$ into equations (9), we decouple the

system of 2-D integral equations as two systems in terms of $\hat{\mathbf{w}}_v(0)(=[\hat{w}_3(0), \hat{\vartheta}_3(0)]^{\mathrm{T}})$ and $\hat{\mathbf{w}}_h(0)(=[\hat{w}_1(0), \hat{w}_2(0)]^{\mathrm{T}})$, respectively. The first system is

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \mathbf{\Lambda}_a\hat{\mathbf{w}}_v(0)K\,\mathrm{d}\varphi\,\mathrm{d}\rho = \hat{\mathbf{U}}_a(r,\theta,s); \qquad r,\theta\in\tilde{A}$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \mathbf{C}_a\hat{\mathbf{w}}_v(0)K\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho = \mathbf{0}; \qquad r,\theta\in\bar{A} \tag{11a}$$

and the second system is

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \mathbf{\Pi}_e\hat{\mathbf{w}}_h(0)K\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho = \hat{\mathbf{U}}_b(r,\theta,s); \qquad r,\theta\in\tilde{A}$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \mathbf{\Pi}_e\mathbf{C}_b\hat{\mathbf{w}}_h(0)K\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho = \mathbf{0}; \qquad r,\theta\in\bar{A} \tag{11b}$$

where $\hat{\mathbf{U}}_a = (\hat{U}_z, 0)^{\mathrm{T}}$, $\hat{\mathbf{U}}_b = (\hat{U}_r, \hat{U}_\theta)^{\mathrm{T}}$, and

$$\mathbf{\Pi}_e = \begin{pmatrix} i\sin(\varphi+\theta) & i\cos(\varphi+\theta) \\ i\cos(\varphi+\theta) & -i\sin(\varphi+\theta)\end{pmatrix}, \qquad \mathbf{\Lambda}_a = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2\end{pmatrix}$$

$$\mathbf{C}_a = \begin{pmatrix} c_{12}(\gamma) & c_{13}(\gamma) \\ \rho c_{32}(\gamma) & \rho c_{33}(\gamma)\end{pmatrix}, \qquad \mathbf{C}_b = \begin{pmatrix} c_{21}(\gamma) & 0 \\ 0 & \mu\end{pmatrix}. \tag{11c}$$

The 2-D integral equations governing the plane inclusion problems are now decoupled into two systems associated with the vertical and horizontal displacements of the inclusions, respectively. In the Laplace transform domain, the non-zero force and moments acting on the permeable or impermeable plane inclusion associated with the vertical displacements can be expressed as follows:

$$\begin{pmatrix} \hat{P}_z(s) \\ \hat{M}_x(s) \\ \hat{M}_y(s)\end{pmatrix} = -2\iint_{\tilde{A}} \begin{pmatrix} 1 \\ r\sin\theta \\ r\cos\theta\end{pmatrix}\hat{\sigma}_{zz}(r,\theta,0,s)r\,\mathrm{d}\theta\,\mathrm{d}r \tag{12a}$$

where the normal stress is given by

$$\hat{\sigma}_{zz}(r,\theta,0,s) = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \hat{\tau}_3(\rho,\varphi,0,s)K\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho. \tag{12b}$$

The unknown variable $\hat{\tau}_3(\rho,\varphi,0,s)$ is obtained by the solution of the following systems of 2-D integral equations, i.e.

(i) For a *permeable* plane inclusion, we have

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} a_1(\gamma)\hat{\tau}_3(\rho,\varphi,0,s)K\,\mathrm{d}\varphi\,\mathrm{d}\rho = \hat{U}_z(r,\theta,s), \qquad r,\theta\in\tilde{A}$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \hat{\tau}_3(\rho,\varphi,0,s)K\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho = 0, \qquad r,\theta\in\bar{A}. \tag{13a}$$

(ii) For an *impermeable* plane inclusion, have

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \mathbf{A}_a\hat{\mathbf{Y}}_a(\rho,\varphi,0,s)K\,\mathrm{d}\varphi\,\mathrm{d}\rho = \hat{\mathbf{U}}_a(r,\theta,s), \qquad r,\theta\in\tilde{A}$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \hat{\mathbf{Y}}_a(\rho,\varphi,0,s)K\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho = \mathbf{0}, \qquad r,\theta\in\bar{A} \tag{13b}$$

where $\hat{\mathbf{Y}}_a(\rho,\varphi,0,s) = [\hat{\tau}_3(0), \hat{p}_w(0)]^{\mathrm{T}}$, and

$$\mathbf{A}_a = \begin{pmatrix} a_1(\gamma) & a_2(\gamma) \\ \rho^2 a_3(\gamma) & \rho^2 a_4(\gamma)\end{pmatrix}. \tag{13c}$$

The coefficients $a_j(\gamma)$ $(j = 1, 2, 3, 4)$ are defined in Appendix A.

In the Laplace transform domain, the non-zero forces and moment acting on either the permeable or the impermeable plane inclusion associated with the horizontal displacement can be expressed as follows:

$$\begin{pmatrix} \hat{P}_x(s) \\ \hat{P}_y(s) \\ \hat{M}_z(s) \end{pmatrix} = -2 \iint_{\tilde{A}} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \\ 0 & r \end{pmatrix} \begin{pmatrix} \hat{\sigma}_{rz}(r, \theta, 0, s) \\ \hat{\sigma}_{\theta z}(r, \theta, 0, s) \end{pmatrix} r \, d\theta \, dr \tag{14a}$$

where the contact shear stresses are governed by,

$$\begin{pmatrix} \hat{\sigma}_{rz}(r, \theta, 0, s) \\ \hat{\sigma}_{\theta z}(r, \theta, 0, s) \end{pmatrix} = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \mathbf{\Pi}_e \hat{\mathbf{Y}}_b(\rho, \varphi, 0, s) K\rho \, d\varphi \, d\rho. \tag{14b}$$

The unknown variables $\hat{\mathbf{Y}}_b(\rho, \varphi, 0, s)(= [\hat{t}_1(0), \hat{t}_2(0)]^{\mathrm{T}})$ are governed by the following systems of 2-D integral equations, i.e.

(i) For a *permeable* plane inclusion, we have

$$\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \mathbf{\Pi}_d \mathbf{B}_h \hat{\mathbf{Y}}_h(\rho, \varphi, 0, s) \, d\varphi \, d\rho = \hat{\mathbf{U}}_h(r, \theta, s), \qquad r, \theta \in \tilde{A}$$

$$\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \mathbf{\Pi}_d \hat{\mathbf{Y}}_h(\rho, \varphi, 0, s) \rho \, d\varphi \, d\rho = \mathbf{0}, \qquad r, \theta \in \bar{A}. \tag{15a}$$

(ii) For an *impermeable* plane inclusion,

$$\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \mathbf{\Pi}_e \mathbf{B}_b \hat{\mathbf{Y}}_b(\rho, \varphi, 0, s) K \, d\varphi \, d\rho = \hat{\mathbf{U}}_b(r, \theta, s), \qquad r, \theta \in \tilde{A}$$

$$\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \mathbf{\Pi}_e \hat{\mathbf{Y}}_b(\rho, \varphi, 0, s) K\rho \, d\varphi \, d\rho = \mathbf{0}, \qquad r, \theta \in \bar{A} \tag{15b}$$

where $\hat{\mathbf{Y}}_h(\rho, \varphi, 0, s) = (\hat{t}_1(0), \hat{t}_2(0), \hat{\vartheta}_3(0))^{\mathrm{T}}$, and

$$\mathbf{B}_h = \begin{pmatrix} b_1(\gamma) & 0 & b_2(\gamma) \\ 0 & \dfrac{-1}{\mu} & 0 \\ b_3(\gamma) & 0 & b_4(\gamma) \end{pmatrix}, \qquad \mathbf{B}_b = \begin{pmatrix} b_1(\gamma) & 0 \\ 0 & \dfrac{-1}{\mu} \end{pmatrix}. \tag{15c}$$

The coefficients $b_j(\gamma)$ $(j = 1, \sim, 6)$ are defined in Appendix A.

INTEGRAL EQUATIONS GOVERNING A RIGID DISC INCLUSION

We consider the particular case of the displacements of a rigid disc inclusion which is embedded in bonded contact with a poroelastic medium of infinite extent (Fig. 2). The

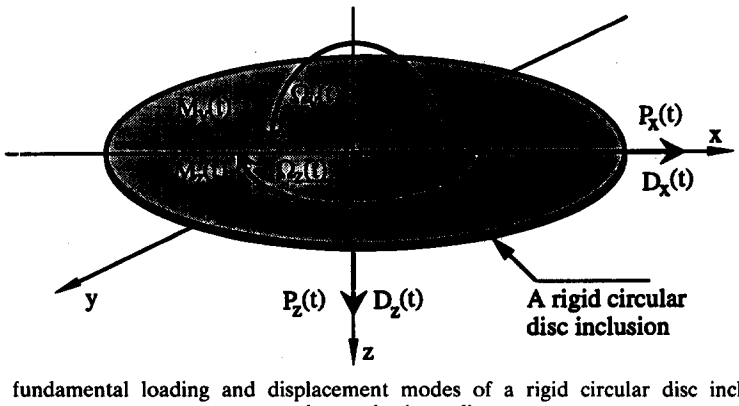

Fig. 2. The fundamental loading and displacement modes of a rigid circular disc inclusion in a saturated poroelastic medium.

generalized displacement vector for the inclusion, in the Laplace transform domain, has components

$$\hat{U}_z(r,\theta,s)=\hat{D}_z(s)+\hat{\Omega}_y(s)r\cos\theta,\quad \hat{U}_r(r,\theta,s)=\hat{D}_x(s)\cos\theta,\quad \hat{U}_\theta(r,\theta,s)=-\hat{D}_x(s)\sin\theta+r\hat{\Omega}_z(s) \tag{16}$$

where $0\le r\le a$, $0\le\theta<2\pi$, a is the radius of rigid disc, $\hat{D}_x(s)$ and $\hat{D}_z(s)$ are, respectively, the horizontal and vertical translations at the centre of rigid disc inclusion, and $\hat{\Omega}_y(s)$ and $\hat{\Omega}_z(s)$ are, respectively, the central and axial rotations of the rigid disc inclusion about the y and z axes.

Substituting the vertical displacements into the systems of 2-D integral equations (13), we can find, with the help of Fourier series expansions, the following solutions for the unknown variables.

$$\begin{aligned}\hat{\tau}_3(\rho,\varphi,0,s)&=\hat{\tau}_{30}(\rho,s)-2i\sin\varphi\hat{\tau}_{31}(\rho,s)\\ \hat{p}_w(\rho,\varphi,0,s)&=\hat{p}_{w0}(\rho,s)-2i\sin\varphi\hat{p}_{w1}(\rho,s).\end{aligned} \tag{17a}$$

The force and the moments acting on the inclusion can be expressed in the following simplified form:

$$\hat{P}_z(s)=-4\pi a\int_0^\infty\hat{\tau}_{30}(\rho,s)J_1(\rho a)\,d\rho;\qquad \hat{M}_x(s)=0;\qquad \hat{M}_y(s)=-4\pi a^2\int_0^\infty\hat{\tau}_{31}(\rho,s)J_2(\rho a)\,d\rho \tag{17b}$$

and the normal stress at the interface plane of the inclusion are,

$$\hat{\sigma}_{zz}(r,\theta,0,s)=\int_0^\infty\hat{\tau}_{30}(\rho,s)J_0(\rho r)\rho\,d\rho+2\cos\theta\int_0^\infty\hat{\tau}_{31}(\rho,s)J_1(\rho r)\rho\,d\rho. \tag{17c}$$

The unknown variables $\hat{\tau}_{3j}(\rho,s)$ and $\hat{p}_{wj}(\rho,s)$ $(j=0,1)$ in equations (17) are governed by the following sets of integral equations.

(i) For the *permeable* inclusion problem, we have

$$\begin{aligned}&\int_0^\infty a_1(\gamma)\hat{\tau}_{3j}(\rho,s)J_j(\rho r)\,d\rho=\hat{D}_j(r,s), && 0\le r\le a\\ &\int_0^\infty\hat{\tau}_{3j}(\rho,s)J_j(\rho r)\rho\,d\rho=0, && a<r<\infty\end{aligned} \tag{18a}$$

where $j=0,1$; $\hat{D}_0(r,s)=\hat{D}_z(s)$; $\hat{D}_1(r,s)=\frac{r}{2}\hat{\Omega}_y(s)$; and $J_j(\rho r)$ is the Bessel function of order j.

(ii) For the *impermeable* inclusion problem, we have

$$\begin{aligned}&\int_0^\infty\mathbf{A}_b\hat{\mathbf{Y}}_j(\rho,s)J_j(\rho r)\,d\rho=\hat{\mathbf{D}}_j(r,s), && 0\le r\le a\\ &\int_0^\infty\hat{\mathbf{Y}}_j(\rho,s)J_j(\rho r)\rho\,d\rho=\mathbf{0}, && a<r<\infty\end{aligned} \tag{18b}$$

where $j=0,1$; $\hat{\mathbf{Y}}_j(\rho,s)=[\hat{\tau}_{3j}(\rho,s),\hat{p}_{wj}(\rho,s)]^{\mathrm{T}}$; $\hat{\mathbf{D}}_0(r,s)=[\hat{D}_z(s),0]^{\mathrm{T}}$; and $\hat{\mathbf{D}}_1(r,s)=\left[\frac{r}{2}\hat{\Omega}_y(s),0\right]^{\mathrm{T}}$.

Substituting the boundary condition for the horizontal displacements given in equations (16) into the systems of 2-D integral equations (15), we can use the Fourier series expansion technique to find the following solutions for the unknown variables.

$$\begin{aligned}\hat{\tau}_1(\rho,\varphi,0,s)&=-i[\hat{N}(\rho,s)+\hat{M}(\rho,s)]\sin\varphi\\ \hat{\tau}_2(\rho,\varphi,0,s)&=\hat{\tau}_{20}(\rho,s)+[\hat{N}(\rho,s)-\hat{M}(\rho,s)]\cos\varphi\\ \hat{\vartheta}_3(\rho,\varphi,0,s)&=-2i\hat{\vartheta}_{31}(\rho,s)\sin\varphi.\end{aligned} \tag{19a}$$

The forces and moment acting on the rigid inclusion can be expressed in the following simplified forms:

$$\hat{P}_x(s) = -4\pi a \int_0^\infty \hat{N}(\rho, s) J_1(\rho a)\, \mathrm{d}\rho, \qquad \hat{P}_y(s) = 0, \qquad \hat{M}_z(s) = -4\pi a^2 \int_0^\infty \hat{\tau}_{20}(\rho, s) J_1(\rho a)\, \mathrm{d}\rho. \tag{19b}$$

Similarly, the associated shear stresses at the interface plane of the inclusion are given by

$$\hat{\sigma}_{rz}(r, \theta, 0, s) = \cos\theta \left[\int_0^\infty \hat{N}(\rho, s) J_0(\rho r)\rho\, \mathrm{d}\rho - \int_0^\infty \hat{M}(\rho, s) J_2(\rho r)\rho\, \mathrm{d}\rho \right]$$

$$\hat{\sigma}_{\theta z}(r, \theta, 0, s) = -\sin\theta \left[\int_0^\infty \hat{N}(\rho, s) J_0(\rho r)\rho\, \mathrm{d}\rho + \int_0^\infty \hat{M}(\rho, s) J_2(\rho r)\rho\, \mathrm{d}\rho \right] + \int_0^\infty \hat{\tau}_{20}(\rho, s) J_1(\rho r)\rho\, \mathrm{d}\rho. \tag{19c}$$

The unknown variables $\hat{N}(\rho, s)$ and $\hat{M}(\rho, s)$ associated with the in-plane translation of the rigid inclusion are governed by the following sets of integral equations:

$$\int_0^\infty \mathbf{\Pi}_0 \mathbf{B}_0 \hat{\mathbf{Y}}_d(\rho, s)\, \mathrm{d}\rho = \hat{\mathbf{H}}_0(s), \qquad 0 \le r \le a$$

$$\int_0^\infty \mathbf{\Pi}_0 \hat{\mathbf{Y}}_d(\rho, s)\rho\, \mathrm{d}\rho = \mathbf{0}, \qquad a < r < \infty \tag{20a}$$

where (i) for the *permeable* inclusion, $\hat{\mathbf{Y}}_d(\rho, s) = [\hat{N}(\rho, s), \hat{M}(\rho, s), \hat{\vartheta}_{31}(\rho, s)]^{\mathrm{T}}$, $\mathbf{H}_0 = [\hat{D}_x(s), 0, 0]^{\mathrm{T}}$, and

$$\mathbf{B}_0 = \begin{pmatrix} b_5(\gamma) & b_6(\gamma) & b_2(\gamma) \\ b_6(\gamma) & b_5(\gamma) & b_2(\gamma) \\ \frac{1}{2} b_3(\gamma) & \frac{1}{2} b_3(\gamma) & b_4(\gamma) \end{pmatrix}, \qquad \mathbf{\Pi}_0 = \begin{pmatrix} J_0(\rho r) & 0 & 0 \\ 0 & J_2(\rho r) & 0 \\ 0 & & \rho J_1(\rho r) \end{pmatrix} \tag{20b}$$

and (ii) for the *impermeable* inclusion, $\hat{\mathbf{Y}}_d(\rho, s) = [\hat{N}(\rho, s), \hat{M}(\rho, s)]^{\mathrm{T}}$, $\mathbf{H}_0 = [\hat{D}_x(s), 0]^{\mathrm{T}}$, and

$$\mathbf{B}_0 = \begin{pmatrix} b_5(\gamma) & b_6(\gamma) \\ b_6(\gamma) & b_5(\gamma) \end{pmatrix}, \qquad \mathbf{\Pi}_0 = \begin{pmatrix} J_0(\rho r) & 0 \\ 0 & J_2(\rho r) \end{pmatrix}. \tag{20c}$$

The unknown variable $\hat{\tau}_{20}(\rho, s)$ associated with the axial rotation of a rigid disc inclusion with both *permeable* and *impermeable* interfaces is governed by the following dual integral equations:

$$\int_0^\infty \hat{\tau}_{20}(\rho, s) J_1(\rho r)\, \mathrm{d}\rho = -\mu r \hat{\Omega}_z(s), \qquad 0 \le r \le a$$

$$\int_0^\infty \hat{\tau}_{20}(\rho, s) J_1(\rho r)\rho\, \mathrm{d}\rho = 0, \qquad a < r < \infty. \tag{21}$$

From the above results, it is evident that the integral equations governing the four basic displacement (loading) modes of the rigid disc inclusion are decoupled. Consequently, there is only one loading mode $P_z(t)$ ($M_y(t)$, $P_x(t)$, or $M_z(t)$) corresponding to the basic displacement mode $D_z(t)$ ($\Omega_y(t)$, $D_x(t)$ or $\Omega_z(t)$). In particular, it is found that the dual integral equations (21) for the axial rotation of the rigid inclusion can be solved in exact closed-from. The

relationships among the axial moment, the associated rotation and the contact shear stress are given by

$$M_z(t) = \frac{32\mu a^3}{3}\Omega_z(t), \qquad \sigma_{\theta z}(r, t) = \frac{-4\mu r}{\pi\sqrt{a^2 - r^2}}\Omega_z(t). \tag{22}$$

This solution is identical to that obtained for the analogous problems in classical elasticity [27].

FREDHOLM INTEGRAL EQUATIONS GOVERNING THE RIGID DISC INCLUSION PROBLEMS

The sets of integral equations (18) and (20) are singular integral equations with unknown singularities. Such singularities require detailed consideration in the numerical evaluation of the sets of integral equations. For example, Small and Booker [28] have numerically evaluated this type of singular integral equations for the axisymmetric translation of a rigid disc inclusion by assuming the unknown variables have the same order singularity as that in the elastostatic problem. Further simplifications and reductions have to be made to eliminate and separate the unknown singularities from the integral equations. It is found that all the sets of integral equations can be reduced into the systems of Fredholm integral equations of the second kind. Based on the procedure given by Sneddon [26], we define the following solution representations for the unknown variables

$$\hat{\tau}_{30}(\rho, s) = \int_0^a \hat{\phi}_0(x, s)\cos(\rho x)\,\mathrm{d}x, \qquad \hat{p}_{w0}(\rho, s) = \frac{B}{\rho}\int_0^a \hat{\psi}_0(x, 0)\sin(\rho x)\,\mathrm{d}x$$

$$\hat{\tau}_{31}(\rho, s) = \int_0^a \hat{\phi}_1(x, s)\sin(\rho x)\,\mathrm{d}x, \qquad \hat{p}_{w1}(\rho, s) = \sqrt{\frac{\pi}{2}}\frac{B}{\sqrt{\rho}}\int_0^a \hat{\psi}_1(x, s)J_{3/2}(\rho x)\,\mathrm{d}x$$

$$\hat{N}(\rho, s) = \int_0^a \hat{\phi}(x, s)\cos(\rho x)\,\mathrm{d}x, \qquad \hat{M}(\rho, s) = \sqrt{\frac{\pi\rho}{2}}\int_0^a \hat{\psi}(x, s)J_{3/2}(\rho x)\,\mathrm{d}x$$

$$\hat{\vartheta}_{31}(\rho, s) = \frac{\kappa B}{\rho}\sqrt{\frac{a^2 s}{c}}\int_0^a \hat{\varphi}(x, s)\sin(\rho x)\,\mathrm{d}x \tag{23}$$

where $\hat{\psi}_0(0, s) = 0$ and $\lim_{x\to 0}\sqrt{x}\,\hat{\psi}_1(x, s) = 0$.

It is worth noting the following integral relationships play a key role in the derivations [26, 29–31]:

$$J_a(\rho r) = \frac{2^{1-b}}{\Gamma(b)}\frac{\rho^b}{r^a}\int_0^r \frac{x^{a-b+1}}{(r^2 - x^2)^{1-b}}J_{a-b}(\rho x)\,\mathrm{d}x \tag{24a}$$

where $\mathrm{Re}(b) > 0$, $\mathrm{Re}(a - b) > -1$.

$$\int_0^\infty \rho^{b-a+1}J_a(\rho r)J_b(\rho x)\,\mathrm{d}\rho = \frac{2^{b+1-a}}{\Gamma(a-b)}\frac{x^b r^{-a}}{(r^2 - x^2)^{b+1-a}}H(r - x) \tag{24b}$$

where $\mathrm{Re}(a) > \mathrm{Re}(b) > -1$ with $r \neq x$ or $\mathrm{Re}(a) > \mathrm{Re}(b + 1) > 0$ with $r = x$, and $H(x)$ is a Heaviside step function.

$$g(y) = \sqrt{\frac{2}{\pi}}\int_0^y \frac{f(x)}{\sqrt{y^2 - x^2}}\,\mathrm{d}x, \qquad f(x) = \sqrt{\frac{2}{\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{yg(y)}{\sqrt{x^2 - y^2}}\,\mathrm{d}y. \tag{24c}$$

This is a pair of integral equations of the Abel type.

The sets of integral equations (18) and (20) can now be reduced to the integral equations of the Abel type. The Fredholm integral equations of the second kind in the Laplace transform domain for the inclusion problems can be obtained from the integral equations of the Abel type.

(a) Fredholm integral equations governing the axial translation of the rigid disc inclusion

Introducing the non-dimensional variables

$$x=\frac{r}{a},\qquad \hat{D}_z(s)=\frac{P_z(3-4\nu)}{32a\mu(1-\nu)}\hat{X}_z(s_1),\qquad \hat{P}_z(s)=P_z\hat{Y}_z(s_1)$$

$$s_1=\frac{a^2s}{c},\qquad \hat{\phi}_0(r,s)=\frac{-P_z}{4\pi a}\hat{\phi}_a(x,s_1),\qquad \hat{\psi}_0(r,s)=\frac{-P_z\sqrt{s_1}}{4\pi a^2}\hat{\psi}_a(x,s_1) \tag{25}$$

we can then obtain the non-dimensional forms of the integral equations as follows:
(i) for a *permeable* inclusion, we have

$$\hat{\phi}_a(x,s_1)+\int_0^1\hat{K}_1(x,y,s_1)\hat{\phi}_a(y,s_1)\,\mathrm{d}y=\hat{X}_z(s_1) \tag{26a}$$

and (ii) for an *impermeable* inclusion, we have

$$\begin{pmatrix}\hat{\phi}_a(x,s_1)\\ \hat{\psi}_a(x,s_1)\end{pmatrix}+\int_0^1\begin{pmatrix}\hat{K}_1(x,y,s_1) & \hat{K}_2(x,y,s_1)\\ \hat{K}_3(x,y,s_1) & \hat{K}_4(x,y,s_1)\end{pmatrix}\begin{pmatrix}\hat{\phi}_a(y,s_1)\\ \hat{\psi}_a(y,s_1)\end{pmatrix}\mathrm{d}y=\begin{pmatrix}\hat{X}_z(s_1)\\ 0\end{pmatrix} \tag{26b}$$

where $0\le x\le 1$; and the non-dimensional kernel functions are

$$\hat{K}_1(x,y,s_1)=-\alpha_2I_1(x,y,s_1),\qquad \hat{K}_2(x,y,s_1)=-\alpha_3\sqrt{s_1}\,[I_2(x,y,s_1)-H(y-x)]$$

$$\hat{K}_4(x,y,s_1)=I_3(x,y,s_1),\qquad \hat{K}_3(x,y,s_1)=-\alpha_4\sqrt{s_1}\,[I_2(y,x,s_1)-H(x-y)] \tag{26c}$$

where

$$I_1(x,y,s_1)=\frac{2s_1}{\pi}\int_0^\infty\frac{\cos(\rho x)\cos(\rho y)}{(\rho+\sqrt{\rho^2+s_1})^2}\mathrm{d}\rho,\qquad I_2(x,y,s_1)=\frac{2s_1}{\pi}\int_0^\infty\frac{\cos(\rho x)\sin(\rho y)}{\rho(\rho+\sqrt{\rho^2+s_1})^2}\mathrm{d}\rho$$

$$I_3(x,y,s_1)=\frac{2s_1}{\pi}\int_0^\infty\frac{\sin(\rho x)\sin(\rho y)}{\rho(\rho+\sqrt{\rho^2+s_1})}\mathrm{d}\rho. \tag{26d}$$

For both permeable and impermeable cases, we have the following non-dimensional expression for the applied axial load:

$$\int_0^1\hat{\phi}_a(y,s_1)\,\mathrm{d}y=\hat{Y}_z(s_1). \tag{26e}$$

(b) Fredholm integral equations governing the central rotation of the rigid disc inclusion

Introducing the following non-dimensional variables

$$\hat{\Omega}_y(s)=\frac{M_y(3-4\nu)}{64a^3\mu(1-\nu)}\hat{X}_y(x,s_1),\qquad \hat{M}_y(s)=M_y\hat{Y}_y(s_1)$$

$$\hat{\phi}_1(r,s)=\frac{-M_y}{8\pi a^2}\hat{\phi}_b(x,s_1),\qquad \hat{\psi}_1(r,s)=\frac{-M_y\sqrt{xs_1}}{8\pi a^{5/2}}\hat{\psi}_b(x,s_1) \tag{27}$$

we can then obtain the non-dimensional forms of the integral equations as follows:
(i) for a *permeable* inclusion, we have

$$\hat{\phi}_b(x,s_1)+\int_0^1\hat{K}_1(x,y,s_1)\hat{\phi}_b(y,s_1)\,\mathrm{d}y=x\hat{X}_y(s_1) \tag{28a}$$

and (ii) for an *impermeable* inclusion, we have

$$\begin{pmatrix}\hat{\phi}_b(x,s_1)\\ \hat{\psi}_b(x,s_1)\end{pmatrix}+\int_0^1\begin{pmatrix}\hat{K}_1(x,y,s_1) & \hat{K}_2(x,y,s_1)\\ \hat{K}_3(x,y,s_1) & \hat{K}_4(x,y,s_1)\end{pmatrix}\begin{pmatrix}\hat{\phi}_b(y,s_1)\\ \hat{\psi}_b(y,s_1)\end{pmatrix}\mathrm{d}y=\begin{pmatrix}x\hat{X}_y(s_1)\\ 0\end{pmatrix} \tag{28b}$$

where $0 \le x \le 1$; and the non-dimensional kernel functions are

$$\hat{K}_1(x, y, s_1) = -\alpha_2 I_1(x, y, s_1), \qquad \hat{K}_2(x, y, s_1) = -\alpha_3\sqrt{s_1}\left[I_2(x, y, s_1) - \frac{x}{y}H(y-x)\right]$$

$$\hat{K}_4(x, y, s_1) = I_3(x, y, s_1), \qquad \hat{K}_3(x, y, s_1) = -\alpha_4\sqrt{s_1}\left[I_2(y, x, s_1) - \frac{y}{x}H(x-y)\right] \tag{28c}$$

where

$$I_1(x, y, s_1) = \frac{2s_1}{\pi}\int_0^\infty \frac{\sin(\rho x)\sin(\rho y)}{[\rho + \sqrt{\rho^2 + s_1}]^2}\,\mathrm{d}\rho,$$

$$I_2(x, y, s_1) = \frac{2s_1}{\pi}\int_0^\infty \frac{\sin(\rho x)\left[\dfrac{\sin(\rho y)}{\rho y} - \cos(\rho y)\right]}{\rho[\rho + \sqrt{\rho^2 + s_1}]^2}\,\mathrm{d}\rho$$

$$I_3(x, y, s_1) = \frac{2s_1}{\pi}\int_0^\infty \frac{\left[\dfrac{\sin(\rho x)}{\rho x} - \cos(\rho x)\right]\left[\dfrac{\sin(\rho y)}{\rho y} - \cos(\rho y)\right]}{\rho[\rho + \sqrt{\rho^2 + s_1}]}\,\mathrm{d}\rho. \tag{28d}$$

For both permeable and impermeable cases, we have the following non-dimensional expression for the applied central moment:

$$\int_0^1 \hat{\phi}_b(y, s_1)y\,\mathrm{d}y = \hat{Y}_y(s_1). \tag{28e}$$

(c) Fredholm integral equations governing the in-plane translation of the rigid disc inclusion

Introducing the following non-dimensional variables:

$$\hat{D}_x(s) = \frac{P_x(7-8\nu)}{64a\mu(1-\nu)}\hat{X}_x(x, s_1), \qquad \hat{P}_x(s) = P_x\hat{Y}_x(s_1)$$

$$\hat{\phi}(r, s) = \frac{-P_x}{4\pi a}\hat{\phi}_c(x, s_1), \qquad \hat{\psi}(r, s) = \frac{-P_x}{4\pi}\sqrt{\frac{x}{a}}\,\hat{\psi}_c(x, s_1), \qquad \hat{\varphi}(r, s) = \frac{-P_x}{4\pi}\hat{\varphi}_c(x, s_1) \tag{29}$$

we can then obtain the non-dimensional forms of the integral equations as follows:
(i) for a *permeable* inclusion, we have

$$\begin{pmatrix} 1 & \alpha_1 & 0 \\ \alpha_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{\phi}_c(x, s_1) \\ \hat{\psi}_c(x, s_1) \\ \hat{\varphi}_c(x, s_1) \end{pmatrix} + \int_0^1 \begin{pmatrix} K_1(x, y, s_1) & K_2(x, y, s_1) & K_3(x, y, s_1) \\ K_4(x, y, s_1) & K_5(x, y, s_1) & K_6(x, y, s_1) \\ K_7(x, y, s_1) & K_8(x, y, s_1) & K_9(x, y, s_1) \end{pmatrix} \times \begin{pmatrix} \hat{\phi}_c(y, s_1) \\ \hat{\psi}_c(y, s_1) \\ \hat{\varphi}_c(y, s_1) \end{pmatrix}\mathrm{d}y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\hat{X}_x(s_1) \tag{30a}$$

and (ii) for an *impermeable* inclusion, we have

$$\begin{pmatrix} 1 & \alpha_1 \\ \alpha_1 & 1 \end{pmatrix}\begin{pmatrix} \hat{\phi}_c(x, s_1) \\ \hat{\psi}_c(x, s_1) \end{pmatrix} + \int_0^1 \begin{pmatrix} K_1(x, y, s_1) & K_2(x, y, s_1) \\ K_4(x, y, s_1) & K_5(x, y, s_1) \end{pmatrix}\begin{pmatrix} \hat{\phi}_c(y, s_1) \\ \hat{\psi}_c(y, s_1) \end{pmatrix}\mathrm{d}y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\hat{X}_x(s_1) \tag{30b}$$

where $0 \le x < 1$; and the non-dimensional kernel functions are

$$K_1(x, y, s_1) = -\alpha_6 I_1(x, y, s_1), \qquad K_9(x, y, s_1) = I_5(x, y, s_1)$$

$$K_2(x, y, s_1) = \alpha_6[I_1(x, y, s_1) - I_2(x, y, s_1)] - \frac{\alpha_1}{y}H(y-x)$$

$$K_3(x, y, s_1) = \alpha_5[2I_4(x, y, s_1) + \sqrt{s_1}\,H(y-x)]$$

$$K_4(x, y, s_1) = \alpha_6[I_1(x, y, s_1) - I_2(y, x, s_1)] - \frac{\alpha_1}{x}H(x-y)$$

$$K_5(x, y, s_1) = -\alpha_6[I_1(x, y, s_1) - I_2(x, y, s_1) - I_2(y, x, s_1) + I_3(x, y, s_1)]$$

$$K_6(x, y, s_1) = \alpha_5\left[\frac{2}{x\sqrt{s_1}} I_5(x, y, s_1) - 2I_4(x, y, s_1) - \sqrt{s_1}\, H(y - x)\right]$$

$$K_7(x, y, s_1) = -\alpha_4\left[I_4(y, x, s_1) + \frac{\sqrt{s_1}}{2} H(x - y)\right]$$

$$K_8(x, y, s_1) = -\alpha_4\left[\frac{1}{y\sqrt{s_1}} I_5(x, y, s_1) - I_4(y, x, s_1) - \frac{\sqrt{s_1}}{2} H(x - y)\right] \tag{30c}$$

where

$$I_1(x, y, s_1) = \frac{2s_1}{\pi}\int_0^\infty [f_1(\rho, s_1) + f_2(\rho, s_1)]\cos(\rho x)\cos(\rho y)\,\mathrm{d}\rho$$

$$I_2(x, y, s_1) = \frac{2s_1}{\pi}\int_0^\infty \frac{1}{\rho y}[f_1(\rho, s_1) + f_2(\rho, s_1)]\cos(\rho x)\sin(\rho y)\,\mathrm{d}\rho$$

$$I_3(x, y, s_1) = \frac{2s_1}{\pi}\int_0^\infty \frac{1}{\rho^2 xy}[f_1(\rho, s_1) + f_2(\rho, s_1)]\sin(\rho x)\sin(\rho y)\,\mathrm{d}\rho$$

$$I_4(x, y, s_1) = \frac{2\sqrt{s_1}}{\pi}\int_0^\infty f_3(\rho, s_1)\cos(\rho x)\sin(\rho y)\,\mathrm{d}\rho,$$

$$I_5(x, y, s_1) = \frac{2s_1}{\pi}\int_0^\infty f_1(\rho, s_1)\sin(\rho x)\sin(\rho y)\,\mathrm{d}\rho \tag{30d}$$

$$f_1(\rho, s_1) = \frac{1}{\rho^2 + s_1 + \rho\sqrt{\rho^2 + s_1}}, \qquad f_3(\rho, s_1) = \frac{\rho^2 - s_1 - \rho\sqrt{\rho^2 + s_1}}{\rho^2 + s_1 + \rho\sqrt{\rho^2 + s_1}}$$

$$f_2(\rho, s_1) = \frac{\rho}{2\rho(\rho^2 + s_1) + (2\rho^2 + s_1)\sqrt{\rho^2 + s_1}}.$$

Again for both permeable and impermeable cases, we have the following non-dimensional expression for the applied in-plane load:

$$\int_0^1 \hat{\phi}_c(y, s_1)\,\mathrm{d}y = \hat{Y}_x(s_1). \tag{30e}$$

The material constants α_j ($j = 1 \sim 8$) in the above equations are defined in Appendix A. For completeness, we shall record here the stresses acting on a single face of the inclusion ($0 \le x < 1, 0 \le \theta < 2\pi$); i.e.

$$\sigma_{zz}(x, \theta, 0, t) = \frac{-P_z}{4\pi a^2}\left[\frac{\phi_a(1, t)}{\sqrt{1 - x^2}} - \int_x^1 \frac{\partial\phi_a(y, t)}{\partial y}\frac{1}{\sqrt{y^2 - x^2}}\,\mathrm{d}y\right] - \frac{M_y}{4\pi a^3}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 \frac{\phi_b(y, t)}{\sqrt{y^2 - x^2}}\,\mathrm{d}y\cos\theta$$

$$\sigma_{xz}(x, \theta, 0, t) = \frac{-P_x}{4\pi a^2}\left[\frac{\phi_c(1, t)}{\sqrt{1 - x^2}} - \int_x^1 \frac{\partial\phi_c(y, t)}{\partial y}\frac{1}{\sqrt{y^2 - x^2}}\,\mathrm{d}y + x\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 \frac{\psi_c(y, t)}{y\sqrt{y^2 - x^2}}\,\mathrm{d}y\cos 2\theta\right]$$

$$\sigma_{yx}(x, \theta, 0, t) = \frac{-P_x}{4\pi a^2}\left[x\frac{\mathrm{d}}{\mathrm{d}x}\int_x^1 \frac{\psi_c(y, t)}{y\sqrt{y^2 - x^2}}\,\mathrm{d}y\right]\sin 2\theta. \tag{31}$$

LIMITING RESPONSES

It is instructive to record certain limiting results obtained via the generalized Fredholm integral equations which govern the mechanics of a bonded rigid circular disc inclusion in a poroelastic medium, with either a *permeable* or an *impermeable* surface. The limiting cases pertain to the initial and final responses of the poroelastic medium as $t \to 0^+$ and $t \to +\infty$

respectively, as well as the response of the poroelastic medium as $v \to v_u$. The limiting response of the rigid inclusion as $t \to 0^+$ and $t \to +\infty$ can be obtained in exact closed-form from the Fredholm integral equations by using the Tauberian theorems for Laplace transforms. For both *permeable* and *impermeable* rigid inclusions, we have

$$\phi_0(x, T_a) = -\frac{8\mu(1-\chi)}{\pi(3-4\chi)} D_z(T_a), \qquad \phi_1(x, T_a) = -\frac{8\mu(1-\chi)}{\pi(3-4\chi)} \Omega_y(T_a)x,$$

$$\phi(x, T_a) = -\frac{16\mu(1-\chi)}{\pi(7-8\chi)} D_x(T_a), \qquad \psi(x, T_a) = 0 \tag{32a}$$

and the corresponding stiffnesses of the rigid disc inclusions are given by

$$P_z(T_a) = \frac{32\mu(1-\chi)a}{3-4\chi} D_z(T_a), \qquad M_y(T_a) = \frac{64\mu a^3(1-\chi)}{3(3-4\chi)} \Omega_y(T_a), \qquad P_x(T_a) = \frac{64\mu(1-\chi)a}{7-8\chi} D_x(T_a) \tag{32b}$$

where (i) for $t \to 0^+$, we have $T_a = 0^+$ and $\chi = v_u$; (ii) for $t \to +\infty$, we have $T_a = +\infty$ and $\chi = v$; (iii) for $v \to v_u$, we have $T_a = t$ and $\chi = v = v_u$.

These closed-form results are in agreement with elastostatic solutions for a rigid disc inclusion embedded in an elastic medium [13, 14, 18, 32–34].

NUMERICAL SOLUTION OF THE INTEGRAL EQUATIONS

An examination of the structure of the kernel functions occurring in these systems of Fredholm integral equations indicates that they are not amenable to exact solutions [35–38]. Consequently, it becomes necessary to adopt accurate and efficient techniques for the numerical solution of these integral equations. In this paper, we adopt the following numerical scheme for the evaluation of time-dependent solutions of the integral equations in the Laplace transform domain. First, we separate the complex variables in the integral equations into their real and imaginary parts. We then obtain new systems of Fredholm integral equations of the second kind associated with real variables. Secondly, we divide the integration interval [0, 1] into N segments with ends defined by $r_k = (k-1)/N$; $k = 1, 2, 3, \ldots, (N+1)$. The collocation points are $x_k = (r_k + r_{k+1})/2$, $k = 1, 2, 3, \ldots, N$. Consequently we can convert the integral equations into the systems of linear algebraic equations. These linear algebraic equations can be written in the generalized matrix form

$$\sum_{j=1}^{L} [A_{lj}]\{\hat{X}_j\} = \{B_l\} \tag{33}$$

where $l = 1, 2, 3, \ldots, L$, $L = 2N + 2$ for equations (26a) and (27a); $L = 4N + 2$ for equations (26b), (27b), and (30b); and $L = 6N + 2$ for equations (30a).

Equation (33) can be solved numerically to generate the unknown variables $\hat{X}_j(s_1)$ in the Laplace transform domain. The time-dependent results of the variables $X_j(ct/a^2)$ can be evaluated by using numerical inversion of Laplace transforms.

The numerical techniques adopted here involve three computational steps which are essentially based on repeated numerical integrations. The first step involves the numerical integration of the infinite integrals, which occur in the kernel functions. It can be shown that these infinite integrals are absolutely convergent. However, there are two unusual aspects in the numerical integrations. One refers to the infinite limit and the other is associated with the integrands. As the values of the complex variable s_1 become large, the integrands will become rapidly oscillatory functions. This property of the integrands will show the convergence of the numerical integration and render the numerical integration procedure unstable. In the

numerical inversion of Laplace transforms, large values of the complex variable s_1 are associated with small values of the time factor ct/a^2. The technique adopted in this study overcomes these two numerical problems. The details of this technique are documented in Appendix B. This particular technique consists of an approximation technique and a proceeding limit technique. In the approximation technique, the integrands are separated into two parts. The first relates to their asymptotic behavior as $s_1/\rho^2 \to 0$. The second relates to the difference between the integrands and their asymptotic functions. Closed-form results can be obtained for the infinite integrals associated with the asymptotic functions. The proceeding limit technique, which is based on an adaptively iterative Simpson's quadrature, is used for the evaluation of the infinite integrals associated with the remaining terms.

The second repeated numerical integration involves systems of Fredholm integral equations, the numerical solution of which have been well investigated by many researchers [37, 38]. These studies show that with the increase in the segment number N it is possible to obtain more accurate and readily convergent solutions for the integral equations.

The last computational step involves the numerical inversion of Laplace transforms. A modified algorithm is proposed and verified for the numerical inversion of Laplace transforms (see Appendix C). Due to the accumulation of errors in each repeated numerical integration, however, it is necessasry to evaluate and test the convergence and accuracy of the time-dependent results for the systems of Fredholm integral equations of the second kind in the Laplace transform domain. The details of these calculations are documented by Yue [15]. It is concluded from this verification that the numerical scheme and techniques adopted in this study provide highly stable and accurate solutions in the time domain for the systems of Fredholm integral equations of the second kind in the Laplace transform domain and in particular, these procedures overcome the numerical problems customarily associated with the initial stages ($10^{-4} \le ct/a^2 \le 10^{-2}$) of the consolidation of the poroelastic medium.

GENERALIZED RESPONSES OF A RIGID DISC INCLUSION

The inclusion can be subjected to at least two types of loading conditions which occur quite frequently in problems in geomechanics. The first category of loading is one where the time-dependent variation of either the axial force $P_z(t)(=P_zY_z(t))$ or the central moment $M_y(t)(=M_yY_y(t))$ or the in-plane force $P_x(t)(=P_xY_x(t))$ applied on the inclusion are known *a priori*. In the second category the time-dependent behaviour of either the axial translation $D_z(t)(=D_zX_z(t))$ or the central rotation $\Omega_y(t)(=\Omega_yX_y(t))$ or the in-plane translation $D_x(t)(=D_xX_x(t))$ of the rigid disc inclusion is prescribed. (It may be noted here that owing to the linearity of the differential equations governing a problem in poroelasticity, it is also possible to impose combinations of these generalized loadings.)

The most basic case in the first category of loading is a constant imposed generalized loading of the type

$$P_z(t) = P_zH(t), \qquad M_y(t) = M_yH(t), \qquad P_x(t) = P_xH(t) \tag{34}$$

where $H(\ )$ is a Heaviside step function. For this type of loading, the resulting generalized displacements and rotation take the forms

$$\frac{a\mu D_z(ct/a^2)}{P_z} = \frac{3-4\nu}{32(1-\nu)}X_z(ct/a^2)$$

$$\frac{a^3\mu\Omega_y(ct/a^2)}{M_y} = \frac{3-4\nu}{64(1-\nu)}X_y(ct/a^2)$$

$$\frac{a\mu D_x(ct/a^2)}{P_x} = \frac{7-8\nu}{64(1-\nu)}X_x(ct/a^2). \tag{35a}$$

We define the "Degree of Consolidation", associated with these three states of loading, as follows:

$$U_z(ct/a^2) = \frac{D_z(ct/a^2) - D_z(0^+)}{D_z(\infty) - D_z(0^+)} = \frac{X_z(ct/a^2) - \alpha_8}{1 - \alpha_8}$$

$$U_y(ct/a^2) = \frac{\Omega_y(ct/a^2) - \Omega_y(0^+)}{\Omega_y(\infty) - \Omega_y(0^+)} = \frac{X_y(ct/a^2) - 3\alpha_8}{3 - 3\alpha_8}$$

$$U_x(ct/a^2) = \frac{D_x(ct/a^2) - D_x(0^+)}{D_x(\infty) - D_x(0^+)} = \frac{X_x(ct/a^2) - \alpha_9}{1 - \alpha_9} \tag{35b}$$

where $X_\alpha(ct/a^2)\,(\alpha = x, y, z)$ are non-dimensional functions numerically evaluated from the equations (33) together with the applied constant forces and moment defined by equations (34). The functions $X_\alpha(ct/a^2)$ include only the material constants of the drained and undrained values of Poisson's ratios. The constants α_8 and α_9 are dependent on ν and ν_u (see Appendix A).

The second basic category of loading is a constant imposed generalized displacement of the type

$$D_z(t) = D_z H(t), \qquad \Omega_y(t) = \Omega_y H(t), \qquad D_x(t) = D_x H(t). \tag{36}$$

For this type of displacements and rotation, the induced forces and moment take the forms

$$\frac{P_z(ct/a^2)}{a\mu D_z} = \frac{32(1-\nu)}{3-4\nu} Y_z(ct/a^2)$$

$$\frac{M_y(ct/a^2)}{a^3\mu\Omega_y} = \frac{64(1-\nu)}{3-4\nu} Y_y(ct/a^2)$$

$$\frac{P_x(ct/a^2)}{a\mu D_x} = \frac{64(1-\mu)}{7-8\nu} Y_x(ct/a^2). \tag{37a}$$

The corresponding expressions for the "Degree of Relaxation" associated with the loadings are given by

$$V_z(ct/a^2) = \frac{P_z(ct/a^2) - P_z(0^+)}{P_z(\infty) - P_z(0^+)} = \frac{\alpha_8 Y_z(ct/a^2) - 1}{\alpha_8 - 1}$$

$$V_y(ct/a^2) = \frac{M_y(ct/a^2) - M_y(0^+)}{M_y(\infty) - M_y(0^+)} = \frac{3\alpha_8 Y_y(ct/a^2) - 1}{\alpha_8 - 1}$$

$$V_x(ct/a^2) = \frac{P_x(ct/a^2) - P_x(0^+)}{P_x(\infty) - P_x(0^+)} = \frac{\alpha_9 Y_x(ct/a^2) - 1}{\alpha_9 - 1} \tag{37b}$$

where $Y_\alpha(ct/a^2)\,(\alpha = x, y, z)$ are non-dimensional functions numerically evaluated from the equations (33) together with the applied load of either constant displacements or constant rotation of the rigid inclusion (36). The functions $X_\alpha(ct/a^2)$ include only the material constants of the drained and undrained values of Poisson's ratios.

NUMERICAL RESULTS

The time-dependent behavior of an embedded rigid disc inclusion is represented by the relationship between the fundamental loading mode $P_z(t)$ ($M_y(t)$, $P_x(t)$, or $M_z(t)$) and the corresponding fundamental displacement mode $D_z(t)$ ($\Omega_y(t)$, $D_x(t)$, or $\Omega_z(t)$). The time-dependent relationship between the axial moment $M_z(t)$ and the axial rotation $\Omega_z(t)$ of the embedded inclusion can be simply expressed in exact closed-form, i.e. $M_z(t) = \frac{32\mu a^3}{3}\Omega_z(t)$. The time dependent response of the associated fundamental mode, i.e. $D_z(t)$ ($\Omega_y(t)$, $D_x(t)$, $P_z(t)$

$M_y(t)$, or $P_x(t)$) can be examined by considering separately the influence of the interface drainage conditions and the five material parameters (μ, ν, ν_u, B, and κ). The influence of the three material parameters (μ, B, and κ) on the behavior of the rigid inclusion can be easily examined by using the equations (35) and (37). This is due to the fact that the non-dimensional functions $X_\alpha(ct/a^2)$ or $Y_\alpha(ct/a^2)$ ($\alpha = x, y, z$) depend only the drained and undrained Poisson ratios (ν, ν_u) and the interface drainage conditions. In the ensuing, we shall present the numerical results which will clearly illustrate the time dependent behavior of the rigid disc inclusion by taking account the influences of interface drainage conditions and the drained and undrained Poisson ratios ν and ν_u respectively.

The conventional time factor ct/a^2 is utilized in the presentation of the numerical results. It is noted that this conventional time factor depends on the drained and undrained Poisson's ratios [see equations (3b)]. Therefore, the influence of the drained and undrained Poisson ratios on the behavior of the rigid disc inclusion can be further understood by using a modified time factor $c_m t/a^2$ ($= 2\mu B^2 \kappa t/a^2$). This modified time factor $c_m t/a^2$ eliminates the dependency on Poisson's ratios ν and ν_u.

For the first category of loading, where the loads are prescribed in equation (34), the results corresponding to equations (35) are plotted against the conventional time factor ct/a^2 (Figs 3–5). For the second category of loading [equation (36)], the results corresponding to equations (37) are plotted against the conventional time factor ct/a^2 (Figs 6–8). Some general qualitative and specific results associated with the time-dependent behavior of the rigid disc inclusion are further summarized in Tables 1 and 2. These results are applicable to the four basic displacement modes and the four basic loading modes. It is noted that (i) in Table 1, $\Delta U(ct/a^2)$ refers to the difference in the degree of consolidation induced displacements between the permeable and the impermeable inclusions and (ii) in Table 2, $\Delta V(ct/a^2)$ refers to the difference in the degree of consolidation induced load relaxations between the permeable and the impermeable inclusions.

CONCLUSIONS

The paper develops the systems of coupled integral equations governing quasi-static behavior of a rigid circular disc inclusion embedded in bonded contact with a poroelastic infinite medium saturated with a compressible pore fluid. These coupled integral equations are developed for disc inclusions which display either *permeable* or *impermeable* interface conditions. The generalized displacements of the rigid disc inclusion include an axial displacement, rotation about two axes and an in-plane translation. Due to their symmetry or asymmetry, these inclusion problems can be formulated as mixed boundary value problems related to a halfspace region and referred to variables in the Laplace transform domain. Through the generalized treatment presented in the paper, it is established that the modes of deformation associated with the translation and rotation of the inclusion give rise to independent sets of transient problems. It is shown that all sets of integral equations governing the quasi-static response of the rigid disc inclusion can be reduced to the standard systems of Fredholm integral equations of the second kind in the Laplace transform domain.

The numerical evaluations of the systems of Fredholm integral equations of the second kind are used to demonstrate the time-dependent behavior of the inclusion which is subjected to either a step function-type loading or a step function-type displacement. It is shown that the kernel functions in the systems of Fredholm integral equations contain infinite integrals with rapidly oscillatory integrands. The evaluation of these infinite integrals is based on the proceeding limit technique and an approximation scheme. The computation schemes also involve Laplace transform inversion which is achieved via a modified algorithm. These numerical procedures offer accurate and efficient techniques for solution of this class of coupled systems of integral equations of the Fredholm type in the Laplace transform domain.

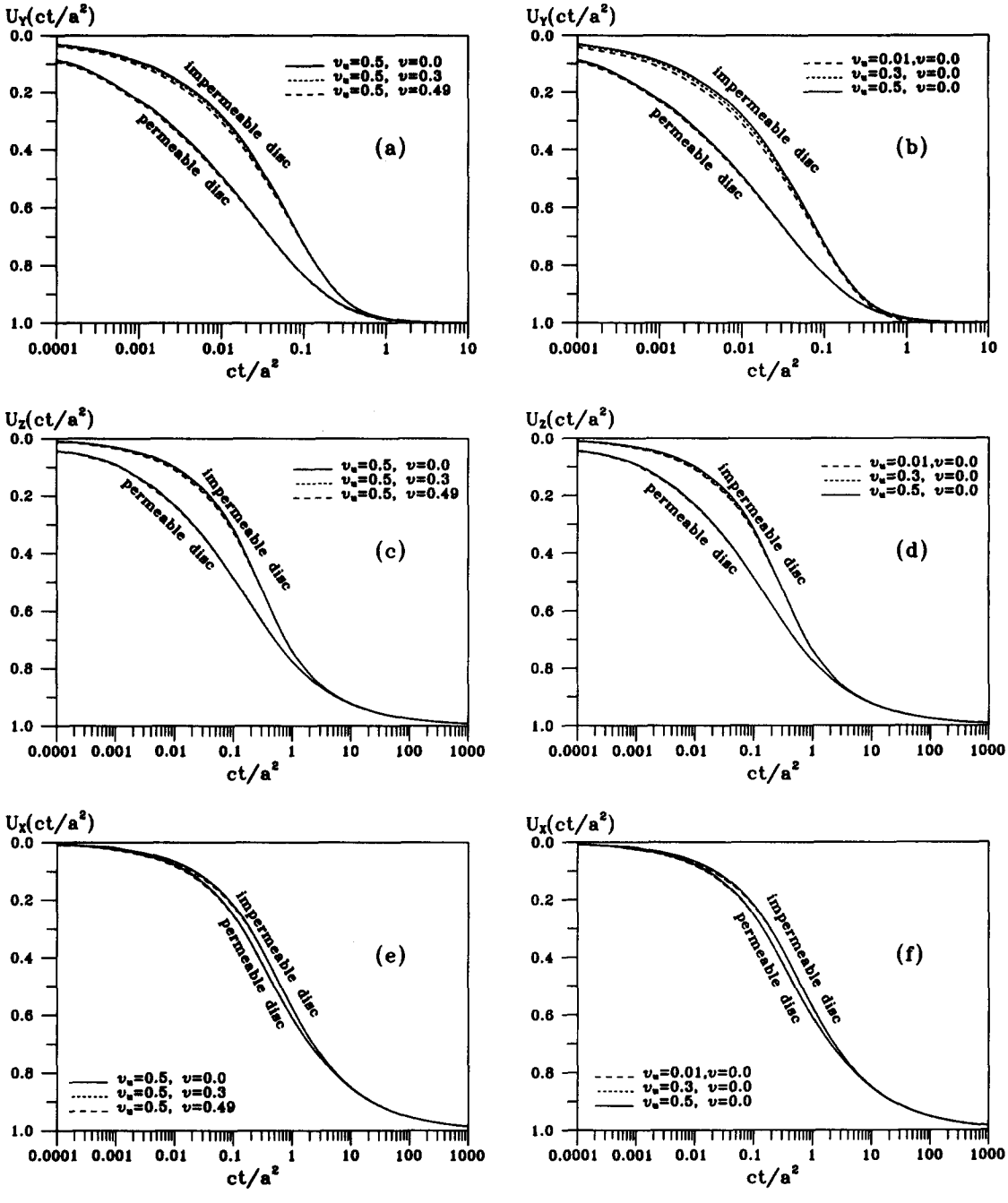

Fig. 3. Degree of consolidation induced displacements of a rigid inclusion vs the time factor ct/a^2: effect of interface drainages and Poisson's ratios.

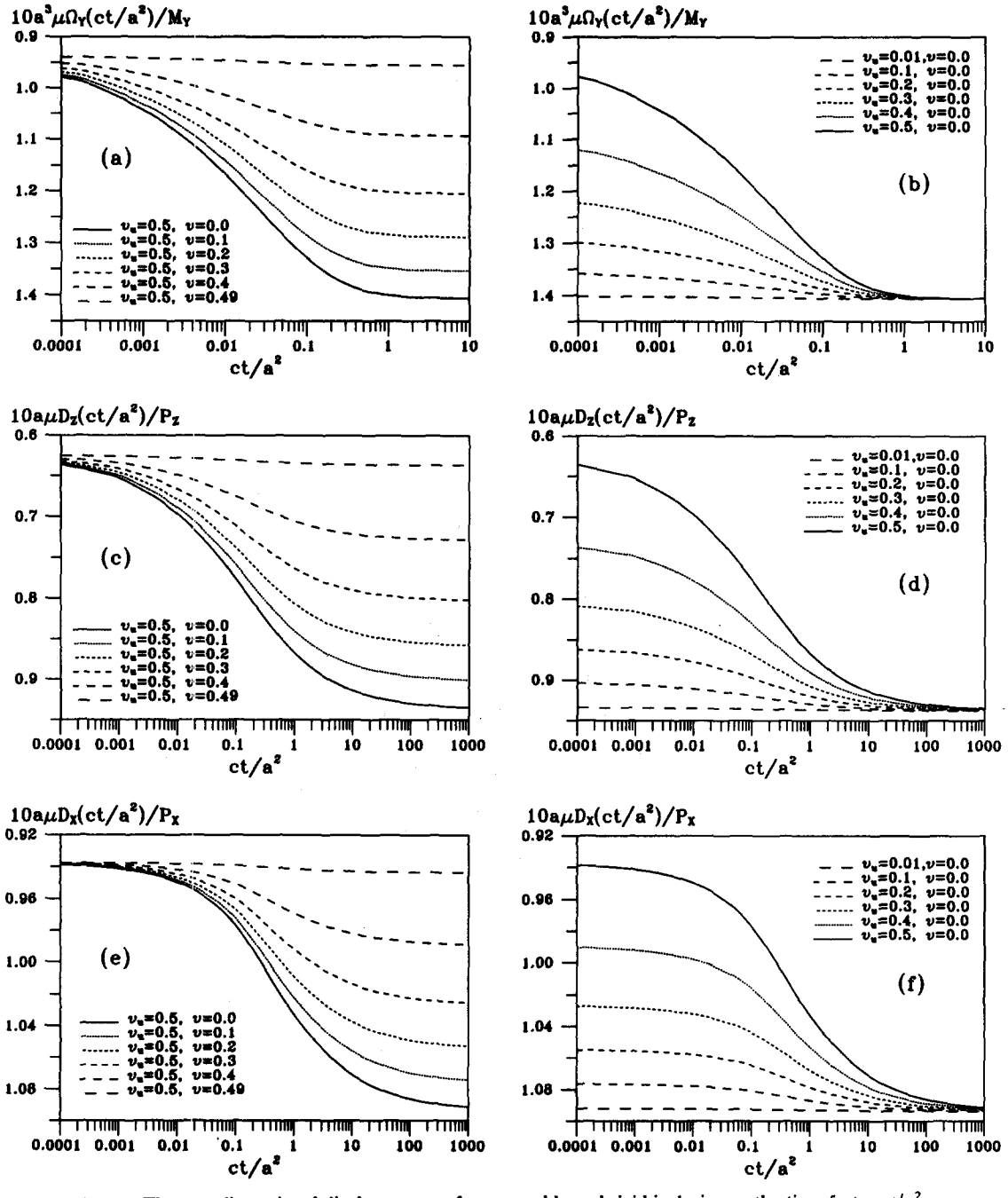

Fig. 4. The non-dimensional displacements of a permeable and rigid inclusion vs the time factor ct/a^2: effect of Poisson's ratios.

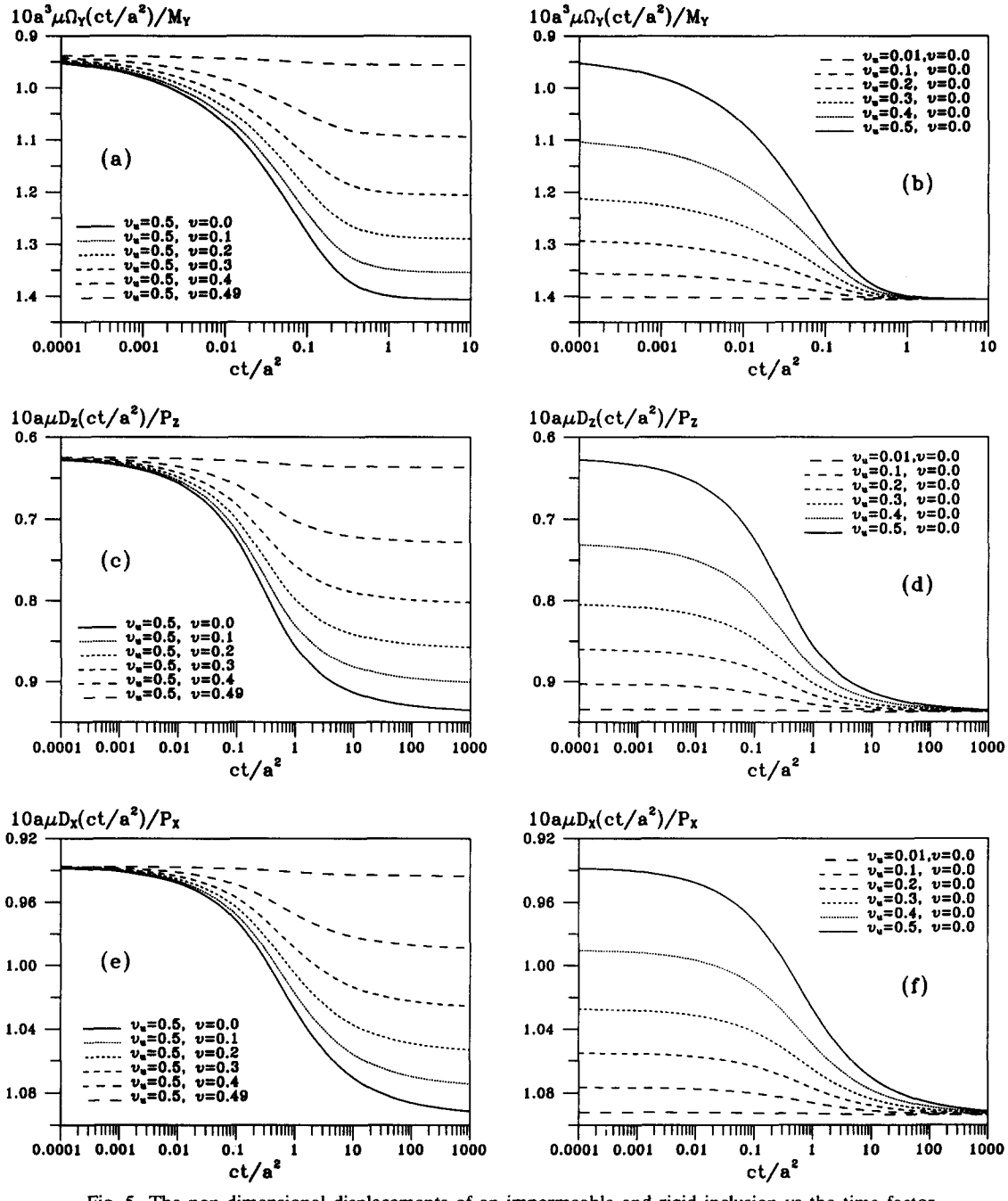

Fig. 5. The non-dimensional displacements of an impermeable and rigid inclusion vs the time factor ct/a^2: effect of Poisson's ratios.

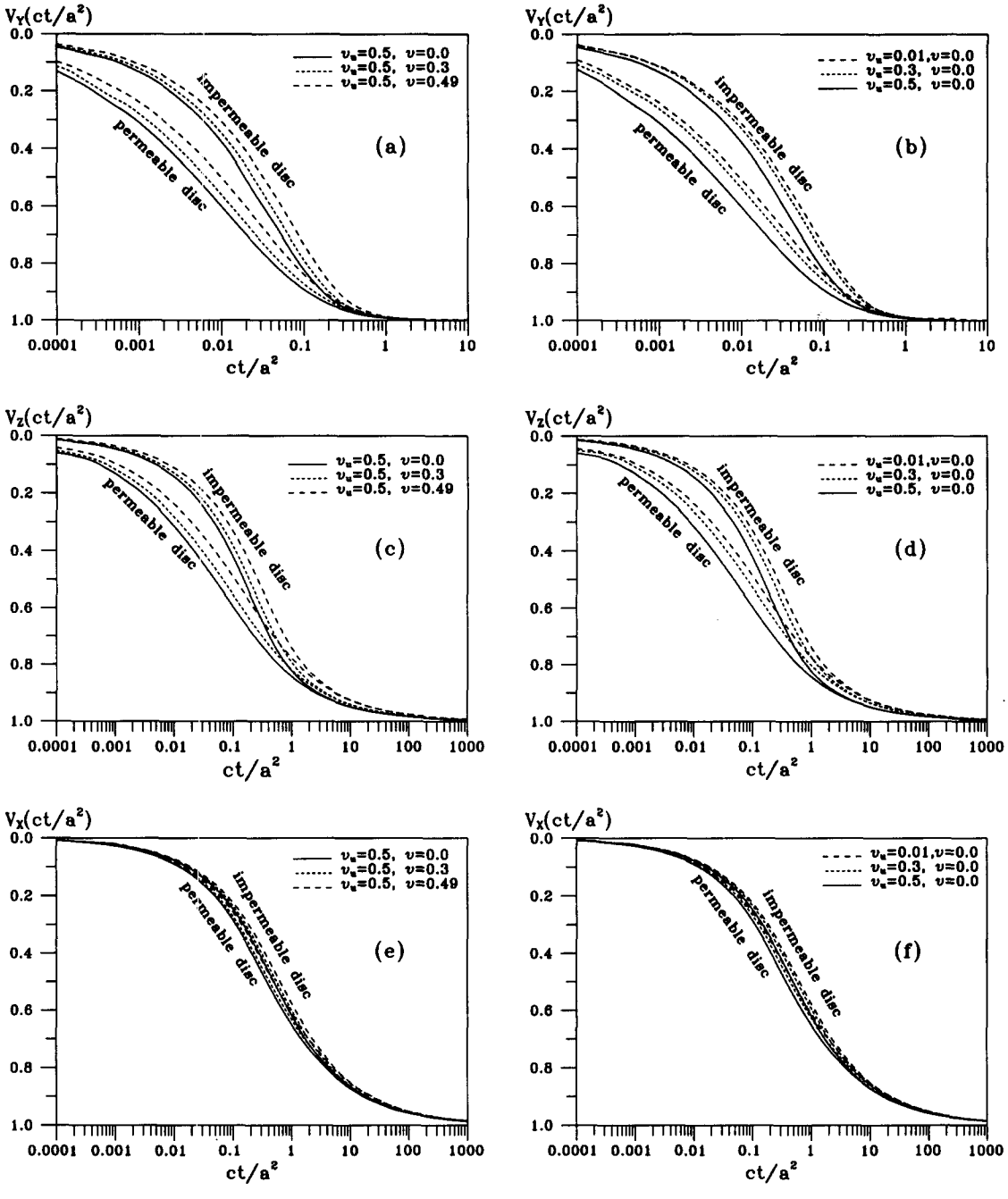

Fig. 6. Degree of consolidation induced load relaxations in a rigid inclusion vs the time factor ct/a^2: effect of interface drainage and Poisson's ratios.

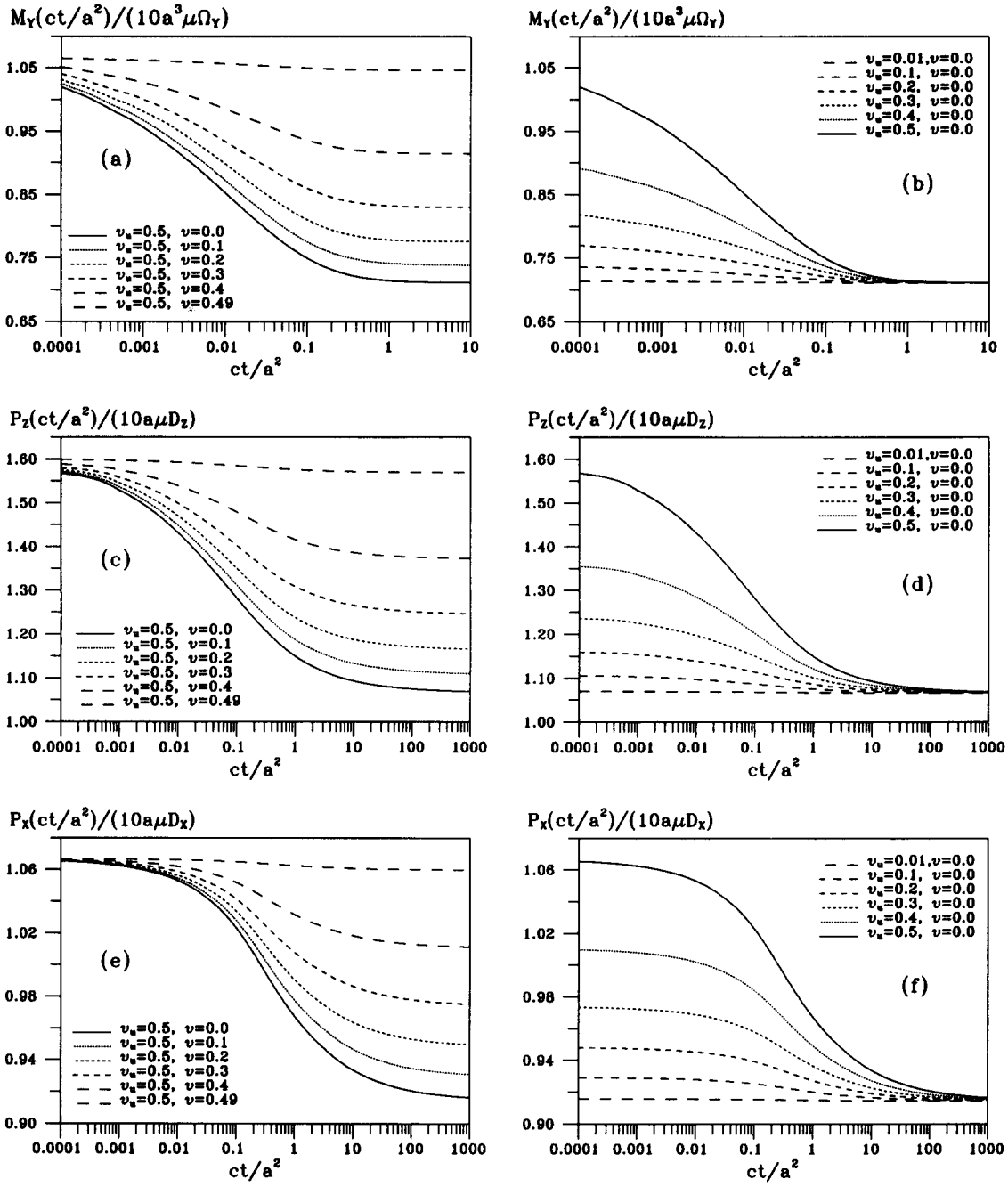

Fig. 7. The non-dimensional force and moment resultants in a permeable and rigid inclusion vs the time factor ct/a^2: effect of Poisson's ratios.

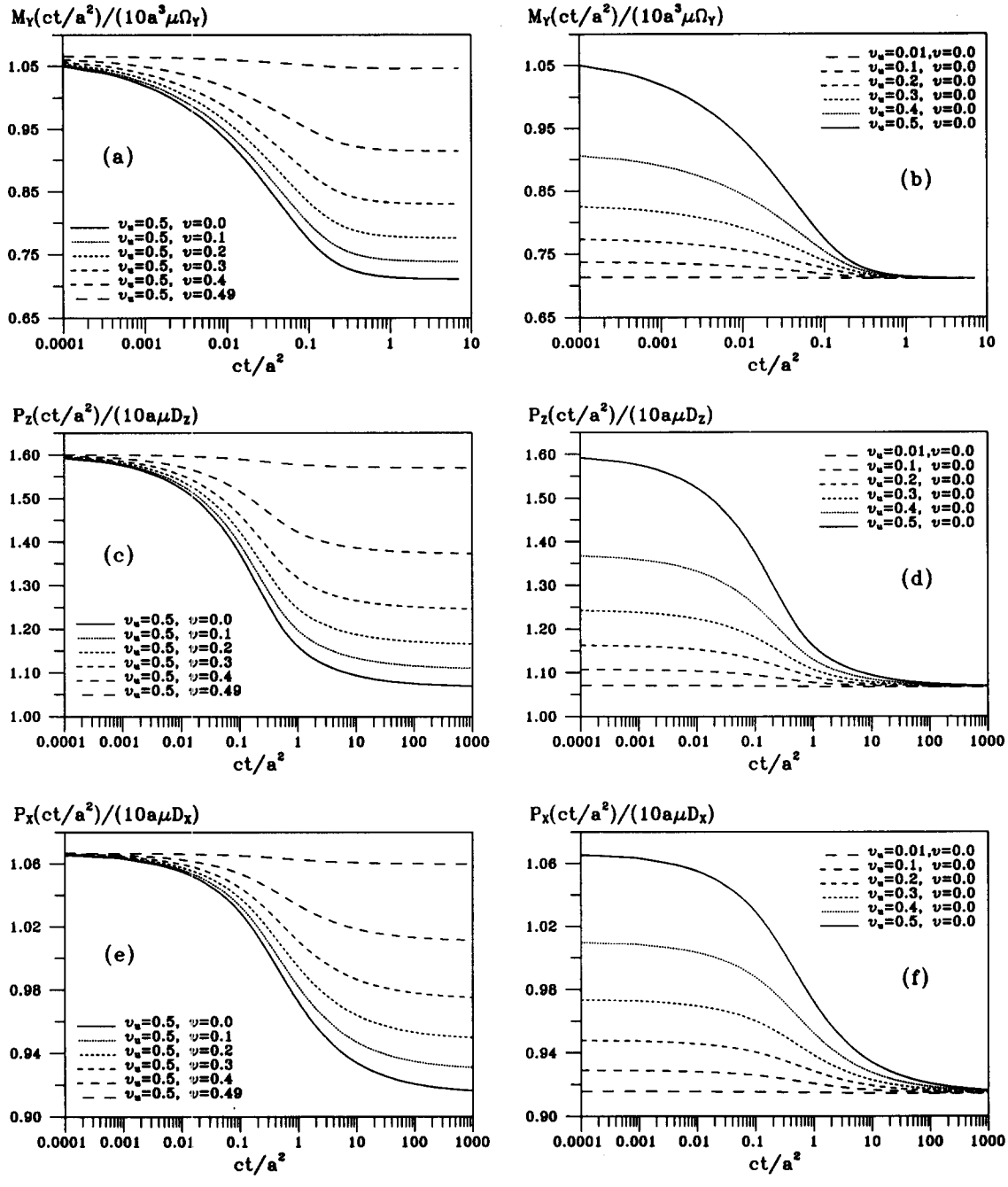

Fig. 8. The non-dimensional force and moment resultants in an impermeable and rigid inclusion vs the time factor ct/a^2: effect of Poisson's ratios.

Table 1. The time-dependent behavior of a rigid disc inclusion in a saturated poroelastic medium. Case (A): consolidation displacement due to an applied constant load

Displacement modes		Vertical displacement modes		Horizontal displacement modes	
		axial translation	central rotation	in-plane translation	axial rotation
Conditions and phenomena		$a\mu D_z(t)/P_z$	$a^3\mu\Omega_y(t)/M_y$	$a\mu D_x(t)/P_x$	$a^3\mu\Omega_z(t)/M_z$
Solution		numerical results from integral equations, $N > 1/\sqrt{ct/a^2}$			exact
Time for consolidation process		longer	shorter	much longer	shortest 0
Drainage	consolidation rate	significant	more significant	limited	no effect
	magnitude	no effect on the initial and final values of displacement			
	maximum $\Delta U(ct/a^2)$	17.3%	21.2%	4.8%	no effect
The four parameters μ, κ, B, a		consolidation time is inversely proportional to $2\mu\kappa B^2/a^2$			no effect
Poisson's ratios ν, ν_u	initial displacement	$\frac{3-4\nu_u}{32(1-\nu_u)}$	$\frac{3(3-4\nu_u)}{64(1-\nu_u)}$	$\frac{7-8\nu_u}{64(1-\nu_u)}$	$\frac{3}{64}$
	final displacement	$\frac{3-4\nu}{32(1-\nu)}$	$\frac{3(3-4\nu)}{64(1-\nu)}$	$\frac{7-8\nu}{64(1-\nu)}$	$\frac{3}{64}$
	consolidation displacement	$\frac{(\nu_u-\nu)U_z(ct/a^2)}{32(1-\nu)(1-\nu_u)}$	$\frac{3(\nu_u-\nu)U_y(ct/a^2)}{64(1-\nu)(1-\nu_u)}$	$\frac{(\nu_u-\nu)U_x(ct/a^2)}{64(1-\nu)(1-\nu_u)}$	0
	consolidation time	very limited effect by using the time factor ct/a^2			
		significant effect due to $\nu_b = \frac{(1-\nu)(1+\nu_u)^2}{9(\nu_u-\nu)(1-\nu_u)}$ in $c = \nu_b c_m$ by using the modified time factor $c_m t/a^2$			no effect

Table 2. The time-dependent behavior of a rigid disc inclusion in a saturated poroelastic medium. Case (B): force relaxation due to an applied constant displacement

Displacement modes		Vertical displacement modes		Horizontal displacement modes	
		axial force	central moment	in-plane force	axial moment
Conditions and phenomena		$P_z(t)/(a\mu D_z)$	$M_y(t)/(a^3\mu\Omega_y)$	$P_x(t)/(a\mu D_x)$	$M_z(t)/(a^3\mu\Omega_z)$
Solution		numerical results from integral equations, $N > 1/\sqrt{ct/a^2}$			exact
Time for relaxation process		longer	shorter	much longer	shortest 0
Drainage	relaxation rate	significant	more significant	limited	no effect
	magnitude	no effect on the initial and final values of forces			
	maximum $\Delta V(ct/a^2)$	20.0%	22.6%	5.0%	no effect
The four parameters μ, κ, B, a		relaxation time is inversely proportional to $2\mu\kappa B^2/a^2$			no effect
Poisson's ratios ν, ν_u	initial force	$\frac{32(1-\nu_u)}{(3-4\nu_u)}$	$\frac{64(1-\nu_u)}{3(3-4\nu_u)}$	$\frac{64(1-\nu_u)}{7-8\nu_u}$	$\frac{64}{3}$
	final force	$\frac{32(1-\nu)}{3-4\nu}$	$\frac{64(1-\nu)}{3(3-4\nu)}$	$\frac{64(1-\nu)}{3-4\nu}$	$\frac{64}{3}$
	relaxation force	$\frac{32(\nu-\nu_u)V_z(ct/a^2)}{(3-4\nu)(3-4\nu_u)}$	$\frac{64(\nu-\nu_u)V_y(ct/a^2)}{3(3-4\nu)(3-4\nu_u)}$	$\frac{64(\nu-\nu_u)V_x(ct/a^2)}{(7-8\nu)(7-8\nu_u)}$	0
	degree by ct/a^2	significant effect		limited effect	
	degree by $c_m t/a^2$	significant due to the counteracting effects of ν, ν_u in ν_b and in the integral equations			no effect

The time-dependent behavior of the rigid disc inclusion has been examined in detail by taking into account the influences of the interface drainage conditions and the five material parameters governing classical poroelasticity.

For an axisymmetric rotation about the z-axis, in particular, the state of stress induced in the poroelastic medium corresponds purely to shear stresses and the solution for the poroelasticity problem reduces to that for the analogous problem in classical elasticity. It is of interest to note that in the case where the embedded inclusion is subjected to an in-plane translation, the traction orthogonal to the direction of translation and within the inclusion region, are non-zero for any arbitrary time t. These tractions however reduce to zero as $t \to 0$ and $t \to +\infty$. The limiting responses as $t \to 0$, $t \to +\infty$, and $\nu \to \nu_u$ can also be recovered as special cases of the generalized systems of Fredholm integral equations of second kind. These closed-form results are in agreement with the appropriate solutions for disc inclusions embedded in elastic media.

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APPENDIX A

The coefficient matrices $\{b_{ij}(\gamma)\}$ and $\{c_{ij}(\gamma)\}$ $(i, j = 1, 2, 3)$ are defined by

$$b_{11} = b_{22} = \frac{2\mu}{D_b}[4\nu_u - 2 + \alpha_\gamma(1-\gamma)], \qquad c_{11} = c_{22} = \frac{2\mu}{D_c}[\alpha_\gamma(1-\gamma) - 2(1-2\nu_u)\gamma]$$

$$b_{12} = b_{21} = \frac{8\mu(\nu_u - 1)}{D_b}, \qquad c_{12} = c_{21} = \frac{8\mu(\nu_u - 1)\gamma}{D_c}$$

$$b_{13} = -b_{23} = \frac{\alpha_\gamma(\gamma - 1)}{\alpha_d D_b}, \qquad c_{13} = -c_{23} = \frac{\alpha_\gamma(1-\gamma)}{\kappa\alpha_d D_c}$$

$$b_{31} = -b_{32} = \frac{8\mu\kappa\alpha_d(1-\nu_u)(1-\gamma)}{D_b}, \qquad c_{31} = -c_{32} = \frac{8\mu\alpha_d(1-\nu_u)(1-\gamma)}{D_c}$$

$$b_{33} = \frac{\kappa}{D_b}[(8\nu_u - 6)\gamma + \alpha_\gamma(1-\gamma)], \qquad c_{33} = \frac{1}{\kappa D_c}[6 - 8\nu_u + \alpha_\gamma(\gamma - 1)]$$

$$D_b = 8\nu_u - 6 + \alpha_\gamma(1-\gamma), \qquad D_c = (8\nu_u - 6)\gamma + \alpha_\gamma(1-\gamma). \tag{A1}$$

The coefficients $a_j(\gamma)$ ($j = 1, 2, 3, 4$) and $b_j(\gamma)$ ($j = 1, \ldots, 6$) are defined by

$$a_1(\gamma) = -\frac{3-4\nu}{4\mu(1-\nu)}[1+\alpha_2 k_1(\gamma)], \qquad a_3(\gamma) = 2\kappa B\alpha_4 k_2(\gamma), \qquad a_4(\gamma) = \kappa[1+k_2(\gamma)]$$

$$a_2(\gamma) = -\frac{\alpha_3(3-4\nu)}{4\mu B(1-\nu)}[1+k_1(\gamma)], \qquad b_3(\gamma) = 2B\alpha_4 k_4(\gamma), \qquad b_4(\gamma) = \frac{1}{\kappa}[1+k_4(\gamma)]$$

$$b_1(\gamma) = -\frac{7-8\nu}{4\mu(1-\nu)}[1+\alpha_6 k_3(\gamma)] + \frac{1}{\mu}, \qquad b_2(\gamma) = -\frac{\alpha_5(7-8\nu)}{8\mu\kappa B(1-\nu)}[1+k_3(\gamma)]$$

$$b_5(\gamma) = \frac{1}{2}\left[b_1(\gamma) - \frac{1}{\mu}\right], \qquad b_6(\gamma) = \frac{(7-8\nu)}{8\mu(1-\nu)}[\alpha_1 - \alpha_6 k_3(\gamma)]$$

$$k_1(\gamma) = \frac{1-\gamma}{1+\gamma}, \qquad k_2(\gamma) = \gamma - 1, \qquad k_3(\gamma) = \frac{2}{(1+\gamma)\gamma} - 1, \qquad k_4(\gamma) = \frac{1-\gamma}{\gamma}. \tag{A2}$$

The constants α_j ($j = 1 \sim 6, 8, 9$) are defined by

$$\alpha_1 = \frac{1}{7-8\nu}, \qquad \alpha_2 = \frac{\nu_u - \nu}{(1-\nu_u)(3-4\nu)}, \qquad \alpha_3 = \frac{3(\nu_u - \nu)}{(1+\nu_u)(3-4\nu)}, \qquad \alpha_8 = \frac{(3-4\nu_u)(1-\nu)}{(3-4\nu)(1-\nu_u)}$$

$$\alpha_4 = \frac{1+\nu_u}{6(1-\nu_u)}, \qquad \alpha_5 = \frac{6(\nu_u - \nu)}{(1+\nu_u)(7-8\nu)}, \qquad \alpha_6 = \frac{\nu_u - \nu}{(1-\nu_u)(7-8\nu)}, \qquad \alpha_9 = \frac{(7-8\nu_u)(1-\nu)}{(7-8\nu)(1-\nu_u)}. \tag{A3}$$

APPENDIX B

Evaluation of the Infinite Integrals in the Kernel Functions

In general, the infinite integrals in the kernel functions can be written in terms of the three independent variables x, y, and s; i.e.

$$I_{1j}(x, y, s) = \frac{2}{\pi}\int_0^\infty f_j(\rho, s)\cos(\rho x)\cos(\rho y)\,\mathrm{d}\rho$$

$$I_{2j}(x, y, s) = \frac{2}{\pi}\int_0^\infty \frac{1}{\rho} f_j(\rho, s)\sin(\rho x)\cos(\rho y)\,\mathrm{d}\rho$$

$$I_{3j}(x, y, s) = \frac{2}{\pi}\int_0^\infty f_j(\rho, s)\sin(\rho x)\sin(\rho y)\,\mathrm{d}\rho \tag{B1}$$

where $0 \le x,\ y \le 1,\ s = s_r + is_i,\ s_r > 0,\ -\infty < s_i < +\infty,\ i = \sqrt{-1}$, and the functions $f_j(\rho, s)$ ($j = 1, 2, 3, 4, 5$) ($j \neq 2$ for $I_{2j}(x, y, s)$ and $I_{3j}(x, y, s)$) are

$$f_1(\rho, s) = \frac{s}{(\rho + \sqrt{\rho^2 + s})^2}, \qquad f_2(\rho, s) = \frac{s}{\rho(\rho + \sqrt{\rho^2 + s})}, \qquad f_3(\rho, s) = \frac{s}{\rho^2 + s + \rho\sqrt{\rho^2 + s}}$$

$$f_4(\rho, s) = \frac{\rho s}{2\rho(\rho^2 + s) + (2\rho^2 + s)\sqrt{\rho^2 + s}}, \qquad f_5(\rho, s) = \frac{\rho^2 - s - \rho\sqrt{\rho^2 + s}}{\rho^2 + s + \rho\sqrt{\rho^2 + s}}. \tag{B2}$$

The functions $f_j(\rho, s)$ ($j = 1, 2, 3, 4, 5$) have the approximations $a_j(\rho, s)$ as $|s|/\rho^2 \to 0$, where

$$a_1(\rho, s) = \frac{s}{4\rho^2 + 2s}, \qquad a_2(\rho, s) = \frac{2s}{4\rho^2 + s}, \qquad a_3(\rho, s) = \frac{2s}{4\rho^2 + 3s}$$

$$a_4(\rho, s) = \frac{s}{4\rho^2 + 4s}, \qquad a_5(\rho, s) = \frac{-9s}{12\rho^2 + 10s}. \tag{B3}$$

We can then re-write the infinite integrals (B1) in the following forms:

$$I_{1j}(x, y, s) = \frac{2}{\pi}\int_0^\infty [f_j(\rho, s) - a_j(\rho, s)]\cos(\rho x)\cos(\rho y)\,\mathrm{d}\rho + I_{1j}^o(x, y, s)$$

$$I_{2j}(x, y, s) = \frac{2}{\pi}\int_0^\infty \frac{1}{\rho}[f_j(\rho, s) - a_j(\rho, s)]\sin(\rho x)\cos(\rho y)\,\mathrm{d}\rho + I_{2j}^o(x, y, s)$$

$$I_{3j}(x, y, s) = \frac{2}{\pi}\int_0^\infty [f_j(\rho, s) - a_j(\rho, s)]\sin(\rho x)\sin(\rho y)\,\mathrm{d}\rho + I_{3j}^o(x, y, s) \tag{B4}$$

where $0 \le x,\ y \le 1$, $\mathrm{Re}(s) > 0$, and

$$I_{1j}^{o}(x, y, s) = \frac{2}{\pi}\int_0^\infty a_j(\rho, s)\cos(\rho x)\cos(\rho y)\,\mathrm{d}\rho$$

$$I_{2j}^{o}(x, y, s) = \frac{2}{\pi}\int_0^\infty \frac{1}{\rho} a_j(\rho, s)\sin(\rho x)\cos(\rho y)\,\mathrm{d}\rho$$

$$I_{3j}^{o}(x, y, s) = \frac{2}{\pi}\int_0^\infty a_j(\rho, s)\sin(\rho x)\sin(\rho y)\,\mathrm{d}\rho. \tag{B5}$$

The infinite integrals in (B5) are integrated in exact closed-form with the help of the following results:

$$\int_0^\infty \frac{s}{\rho^2+s}\cos(\rho\chi)\,\mathrm{d}\rho = \frac{\pi}{2}\sqrt{s}\,\mathrm{e}^{-\chi\sqrt{s}}, \qquad \int_0^\infty \frac{s}{\rho(\rho^2+s)}\sin(\rho\chi)\,\mathrm{d}\rho = \frac{\pi}{2}(1-\mathrm{e}^{-\chi\sqrt{s}}) \tag{B6}$$

where $\chi \ge 0$, $\mathrm{Re}(s) > 0$.

The adjusted infinite integrals in (B4) can be evaluated by using the proceeding limit technique; i.e.

$$\int_0^\infty F(\rho, x, y, s)\,\mathrm{d}\rho \approx \int_0^{A_0} F(\rho, x, y, s)\,\mathrm{d}\rho + \int_{A_0}^{A_1} F(\rho, x, y, s)\,\mathrm{d}\rho + \cdots + \int_{A_n}^{A_{n+1}} F(\rho, x, y, s)\,\mathrm{d}\rho \tag{B7}$$

where $0 < A_0 < A_1 \cdots < A_{n+1}$ is a sequence of numbers that approaches infinity. Each finite integral on the right-hand side is proper and can be calculated by using the Simpson's quadrature based adaptively iterative integration. The limits of $A_0, A_1, \ldots, A_{n+1}$ are chosen as the zeros of the oscillatory part of the integrands, i.e., either $\sin\rho(x \pm y)$ or $\cos\rho(x \pm y)$. The evaluation of these proceeding finite integrals is automatically terminated provided the following condition is satisfied

$$\left|\int_{A_n}^{A_{n+1}} F(\rho)\,\mathrm{d}\rho\right| \le \epsilon_c \tag{B8}$$

where ϵ_c is an assigned absolute error.

As a result, the adjusted infinite integrals in equations (B4) can be evaluated with greater accuracy and faster convergence, using the Simpson's quadrature based proceeding limit technique. With the above procedure, we can accommodate the problems associated with infinite integrals which contain rapidly oscillatory integrands.

APPENDIX C

Algorithm for the Numerical Inversion of Laplace Transforms

Let $f(t)$ be a real function of t, with $f(t) = 0$ for $t < 0$; the Laplace transform and its inversion formula are defined as follows:

$$\hat{f}(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t, \qquad f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \hat{f}(s)\mathrm{e}^{st}\,\mathrm{d}s \tag{C1}$$

where $s = a + i\omega$, $a > a_0 > 0$, $f(t) = O(\mathrm{e}^{at})$ as $t \to \infty$, a_0 is a constant that exceeds the real part of all singularities of $\hat{f}(s)$, and the integrals are in the sense of the Cauchy principal value. In the ensuing discussion, it is therefore assumed that equations (C1) are defined for $\mathrm{Re}(s) \ge a > 0$.

The following algorithm for numerical inversion of Laplace transforms was proposed by Crump [39]. The basic formula for approximating the inverse Laplace transform was derived by expanding the integral in the form of a Fourier series; i.e.

$$f(t) \approx \frac{\mathrm{e}^{at}}{T}\left\{\frac{1}{2}\hat{f}(a) + \mathrm{Re}\left[\sum_{n=1}^{\infty} \hat{f}\left(a + i\frac{n\pi}{T}\right)\mathrm{e}^{in\phi}\right]\right\} \tag{C2}$$

where $0 < t \le 2T$, $\phi = \pi t/T$.

The epsilon technique suggested by MacDonald [40] was incorporated in the algorithm for a faster convergent approximation of the infinite summation in (C2), i.e.

$$\epsilon_{2M+1}^{(1)} \approx \mathrm{Re}\left[\sum_{n=1}^{\infty} \hat{f}\left(a + i\frac{n\pi}{T}\right)\mathrm{e}^{in\phi}\right] \tag{C3}$$

where

$$\epsilon_0^{(m)} = 0, \qquad \epsilon_1^{(m)} = S_m = \mathrm{Re}\left[\sum_{n=1}^{m} \hat{f}\left(a + i\frac{n\pi}{T}\right)\mathrm{e}^{in\phi}\right]$$

$$\epsilon_{p+1}^{(m)} = \epsilon_{p-1}^{(m+1)} + \frac{1}{\epsilon_p^{(m+1)} - \epsilon_p^{(m)}}, \qquad m = 1, 2, 3, \ldots, 2M+1. \tag{C4}$$

The constant parameter a in equation (C2) was obtained from error analysis as follows:

$$a = \alpha_t - \frac{\ln(\epsilon_a)}{2T} \tag{C5}$$

where ϵ_a is a relative error and α_t is a number larger than $\max\{\mathrm{Re}(P)$; where P is a pole of $\hat{f}(s)\}$.

In order to apply the above algorithm to the study of poroelastic media, we made two modifications as follows. The first modification is implemented to reduce the round off error induced in the direct calculation of the partial sums of the Fourier series in Crump's algorithm. By defining,

$$b_{m+2} = b_{m+1} = 0, \qquad b_n = \hat{f}\left(a + i\frac{n\pi}{T}\right) + 2\cos\phi b_{n+1} - b_{n+2}, \qquad n = m, m-1, \ldots, 1 \tag{C6}$$

the partial sums can be evaluated more accurately by the following equation

$$S_m = \mathrm{Re}[b_1(\cos\phi + i\sin\phi) - b_2]. \tag{C7}$$

The second modification is suggested for overcoming Gibbs phenomena in Crump's algorithm. Instead of increasing the total number $2M + 1$ of the partial sums for convergence, we fix the total number $2M + 1$ and use a time dependent T as $T = \alpha_s t$, where α_s is a constant. Many functions such as the Heaviside step function, the exponential function and the error function have been employed to verify the accuracy and efficiency of the algorithm for inverting Laplace transforms. In this study, it was observed that by choosing the constants as $\epsilon_a = 10^{-6} \sim 10^{-8}$; $M = 5 \sim 15$; $\alpha_s = 0{\cdot}9 \sim 1.95$; $\alpha_t = 0$, the adopted algorithm gives optimal results in the time interval of $10^{-5} \leq t \leq 10^3$. This range of non-dimensional time factor meets the requirements applicable to the study of a variety of fluid saturated poroelastic media.