# Contact Problem for Saturated Poroelastic Solid

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**Abstract:** The paper analytically examines the axisymmetric interaction between a rigid, circular, flat indentor and a poroelastic half-space that is saturated with a compressible fluid. The contact between the indentor and the poroelastic half-space is assumed to be smooth. The drainage conditions at the surface of the poroelastic half-space are considered as either completely drained, or partially drained, or completely undrained. By using the integral transform techniques, the paper develops the governing coupled integral equations. These governing integral equations are further reduced to systems of standard Fredholm integral equations of the second kind in the Laplace transform domain. Efficient computational algorithms are proposed to evaluate the time-dependent behavior of the rigid, circular indentor. The numerical results presented in the paper illustrate the manner in which the three variations in pore-pressure boundary conditions and the undrained compressibility of pore-fluid influence the consolidation response of the indentor.



## INTRODUCTION

The classical theory of soil consolidation under one-dimensional conditions was developed by Terzaghi (1925) and subsequently extended by Biot (1941, 1956) to include three-dimensional deformation of the porous medium and compressibility of the pore fluid. Due to the intrinsic interest of poroelasticity to the study of geomaterials, Biot's theory of soil consolidation has been applied to the study of a variety of boundary-value problems. The most notable contributions that deal with the surface loading of half-space regions are due to McNamee and Gibson (1960a,b), Schiffman and Fungaroli (1965), and Gibson et al. (1970).

This paper focuses on the axisymmetric indentation of a fluid saturated poroelastic half-space by a smooth rigid circular indentor. The indentation problem in contact mechanics has received considerable attention. The elastostatic problem of the indentation of an isotropic elastic half-space by a smooth circular indentor was first examined by Boussinesq (1885) who employed results of potential theory. The problem was reexamined by Harding and Sneddon (1945), who utilized the theory of dual integral equations. Since these studies, the axisymmetric circular indentation problem associated with an elastic half-space region was extensively investigated to include effects such as transverse isotropy and nonhomogeneity of the half-space, adhesion between the indentor and the elastic medium, nonuniform profile of the indentor, and flexibility of the indentor. Accounts of developments in this area are given by Galin (1961), Lur'e (1964), Selvadurai (1979), Gladwell (1980), and Johnson (1985).

In the context of geomechanics, the problem examined in this paper can be used to model the consolidation behavior of a circular foundation resting on a deep clay stratum. Analytical studies related to the indentation of fluid saturated media have been reported in the literature. Heinrich and Desoyer (1961) examined the indentation problem by assuming a priori the contact-stress distribution at the indentor-poroelastic half-space interface. In particular, the contact stress was prescribed by Boussinesq's result for the associated elasticity problem and this distribution was assumed to remain constant during the progress of consolidation. Agbezuge and Deresiewicz (1974, 1975) examined the axisymmetric indentation of a poroelastic half-space by rigid indentors with either a spherical or flat profile. In the study by Agbezuge and Deresiewicz (1974, 1975), an assumption that the Hankel transform of the unknown contact pressure varies slowly with time compared with the kernel functions was made in the mathematical formulation of the governing integral equations. Chiarella and Booker (1975) examined the indentation of a deep clay soil by a smooth, rigid indentor. The deep clay soil was modeled as a poroelastic half-space saturated with an incompressible fluid. The entire surface of the clay soil was permeable. Szefer and Gaszynski (1975) examined the problem of the axisymmetric indentation of a poroviscoelastic fluid-saturated half-space by a rigid, circular indentor. Gaszynski and Szefer (1978) extended the study to include formal results for the case of the indentation where an impermeable boundary condition is specified within the indentor and a permeable boundary

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condition is specified at the surface of the half-space exterior to the indentor. No numerical results were presented in the study by Gaszynski and Szefer (1978).

The objectives of this paper are the following: (1) To extend the analysis of the poroelastic contact problem to include compressibility of the pore fluid; (2) to present an alternative formulation to the development of the integral equations governing the smooth axisymmetric indentation of a poroelastic half-space; and (3) to take into account the influence of drainage conditions at the exterior surface of the poroelastic half-space. The role of compressibility of the pore fluid has been largely avoided in previous treatments of indentation problems. As has been observed by Bishop and Hight (1977) and Rice and Cleary (1976), many geomaterials including rocks and clays saturated with fluids containing low concentrations of released gases can exhibit undrained compressibility (e.g. Boise sandstone $\nu_u = 0.27$; Marine clay $\nu_u = 0.48$). The assessment of consolidation response of such soils is also important in marine environments where soil sediments can contain entrapped biogenic gases due to decaying organic matter. The overall effect of entrapped gases is to increase the compressibility of the pore fluid (Wheeler 1988). Such effect can disappear if there is movement of the gases into solution. With low concentrations of immiscible bubbles (e.g. air bubbles in water) the compressibility of the pore fluid can be maintained. In addition to extending the discussion to include undrained compressibility, the formulation of the title problem utilizes a simple solution representation to derive the governing integral equations. The solution of the resulting systems of coupled Fredholm integral equations of the second kind is achieved via efficient algorithms for the evaluation of the associated kernel functions and for the inversion of Laplace transforms. An important aspect of the work is also the presentation of numerical results in the range of small time factors.

## BASIC EQUATIONS

In the ensuing, we present a brief account of the basic equations referred to a Cartesian tensor notation. The constitutive equations governing the quasi-static response of a poroelastic medium, which consists of an isotropic poroelastic soil skeleton saturated with a compressible pore fluid takes the forms

$$\sigma_{ij} = \frac{2\mu\nu}{1 - 2\nu}\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - \frac{3(\nu_u - \nu)}{B(1 - 2\nu)(1 + \nu_u)}p\delta_{ij}; \tag{1a}$$

$$p = \frac{2\mu B^2(1 - 2\nu)(1 + \nu_u)^2}{9(\nu_u - \nu)(1 - 2\nu_u)}\zeta_v - \frac{2\mu B(1 + \nu_u)}{3(1 - 2\nu_u)}\varepsilon_{kk} \tag{1b}$$

where $p$ = pore-fluid pressure; $\zeta_v$ = volumetric strain in the pore fluid; $\sigma_{ij}$ = total stress tensor; and $\varepsilon_{ij}$ = soil skeleton strains defined by

$$\varepsilon_{ij} = (1/2)(u_{i,j} + u_{j,i}) \tag{2}$$

where $u_i$ = corresponding displacement components; and the comma denotes a partial derivative with respect to a spatial variable. In the absence of body forces, the quasi-static equations of equilibrium takes the forms

$$\sigma_{ij,j} = 0 \tag{3}$$

The equations governing quasi-static fluid flow are defined by Darcy's law, which takes the form

$$v_i = -\kappa p_{,i} \tag{4}$$

where $v_i$ = specific discharge vector in the pore fluid. The continuity equation associated with quasi-static fluid flow is

$$\frac{\partial\zeta_v}{\partial t} + \mathbf{v}_{i,i} = 0 \tag{5}$$

The foregoing basic equations are characterized by the five independent material parameters which are represented by the drained and undrained Poisson's ratios $\nu$ and $\nu_u$, respectively, the shear modulus $\mu$ (>0), Skempton's pore pressure coefficient $B$, and $\kappa$ ($= k/\gamma_w > 0$, where $k$ is the coefficient of permeability and $\gamma_w$ is the unit weight of pore fluid). Considering requirements for a positive definite strain energy potential, it can be shown that the material parameters have the following thermodynamic constraints: $\mu > 0$; $0 \leq B \leq 1$; $-1 < \nu < \nu_u \leq 0.5$ [see e.g. Rice and Clearly (1976)].

We take the Fourier integral transforms of (1)–(5) with respect to the horizontal coordinates ($x$, $y$, or $\gamma$, $\theta$) and the Laplace transform with respect to time (Fig. 1). In the following formulation, it is assumed that all the five material parameters in (1)–(5) do not vary in time during the consolidation process of the soil. Based on the theory of Fourier integral transforms [see e.g. Sneddon (1972)], it can be shown that the following sets of solution representations exist





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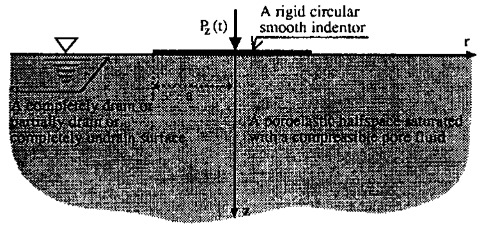


**FIG. 1. Axisymmetric Contact Problem in Poroelasticity**

for the field variables in a linear, isotropic, poroelastic medium of layer extent saturated with a compressible pore fluid. In either the temporal domain or the Laplace transform domain and in the cylindrical coordinate systems $(r, \theta, z)$ and $(\rho, \varphi, z)$, we have

$$\mathbf{u} = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\frac{1}{\rho}\,\mathbf{\Pi}_a\mathbf{w}K\rho\,d\varphi\,d\rho; \quad \mathbf{w} = \frac{\rho}{2\pi}\int_0^\infty\int_0^{2\pi}\mathbf{\Pi}_a^*\mathbf{u}K^*r\,d\theta\,dr;$$

$$\mathbf{T}_z = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\mathbf{\Pi}_a\boldsymbol{\tau}K\rho\,d\varphi\,d\rho; \quad \boldsymbol{\tau} = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\mathbf{\Pi}_a^*\mathbf{T}_zK^*r\,d\theta\,dr;$$

$$\mathbf{v} = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\rho\mathbf{\Pi}_a\boldsymbol{\vartheta}K\rho\,d\varphi\,d\rho; \quad \boldsymbol{\vartheta} = \frac{1}{2\pi\rho}\int_0^\infty\int_0^{2\pi}\mathbf{\Pi}_a^*\mathbf{v}K^*r\,d\theta\,dr;$$

$$p = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}p_wK\rho\,d\varphi\,d\rho; \quad p_w = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}pK^*r\,d\theta\,dr \tag{6a}$$

where $0 \le z < \infty$; $0 < t < \infty$; and the integrals are interpreted in the sense of a Cauchy principal value. The vector fields in (6*a*) are defined by

$$\mathbf{u} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}; \ \mathbf{T}_z = \begin{pmatrix} \sigma_{zr} \\ \sigma_{z\theta} \\ \sigma_{zz} \end{pmatrix}; \ \mathbf{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}; \ \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}; \ \boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}; \ \boldsymbol{\vartheta} = \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{pmatrix} \tag{6b}$$

and $K^*$ and $\mathbf{\Pi}_a^*$ = respectively, the complex conjugates of the Fourier matrix kernel functions $K$ and $\mathbf{\Pi}_a$ defined by

$$K = e^{+i\rho r\sin(\theta+\varphi)}; \ \mathbf{\Pi}_a = \begin{pmatrix} +i\sin(\theta+\varphi) & +i\cos(\theta+\varphi) & 0 \\ +i\cos(\theta+\varphi) & -i\sin(\theta+\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6c}$$

By assuming the initial condition $\zeta_v|_{t=0} = 0$, the basic (1)–(5) can be rewritten as two sets of first-order ordinary differential equations in the Fourier and Laplace transform domains; i.e.

$$(d/dz)\mathbf{V}_v(z) = \rho\mathbf{C}_v\mathbf{V}_v(z); \quad (d/dz)\mathbf{V}_u(z) = \rho\mathbf{C}_u\mathbf{V}_u(z) \tag{7a}$$

where

$$\mathbf{V}_v(z) = \begin{pmatrix} \hat{w}_2(z) \\ \dfrac{1}{\mu}\hat{\tau}_2(z) \end{pmatrix}; \ \mathbf{C}_v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \ \mathbf{V}_u(z) = \begin{bmatrix} \hat{w}_1(z) \\ \hat{w}_3(z) \\ \dfrac{1}{2\mu}\hat{\tau}_3(z) \\ \dfrac{1}{2\mu}\hat{\tau}_1(z) \\ \dfrac{1}{2\mu\alpha_d}\hat{p}_w(z) \\ \dfrac{1}{2\mu\kappa\alpha_d}\hat{\vartheta}_3(z) \end{bmatrix}; \ \mathbf{C}_u = \begin{bmatrix} 0 & -1 & 0 & 2 & 0 & 0 \\ \dfrac{\nu}{1-\nu} & 0 & \dfrac{1-2\nu}{1-\nu} & 0 & \nu_k & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \dfrac{1}{1-\nu} & 0 & \dfrac{-\nu}{1-\nu} & 0 & \nu_k & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \gamma^2-1 & 0 & 1-\gamma^2 & 0 & -\gamma^2 & 0 \end{bmatrix}$$

$$\gamma = \sqrt{\frac{1}{c}\frac{s}{\rho^2} + 1}; \quad c = \frac{2\mu B^2(1-\nu)(1+\nu_u)^2\kappa}{9(\nu_u-\nu)(1-\nu_u)}; \quad \nu_k = \frac{\nu_u-\nu}{(1-\nu)(1-\nu_u)}; \quad \alpha_d = \frac{B(1+\nu_u)}{3(1-\nu_u)} \tag{7b}$$

and the circumflex (ˆ) stands for the Laplace transform with respect to $t$, and $s$ is the Laplace transform parameter.

An algebraic governing equation of the field variables in the transform domains can be further





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obtained by solving the ordinary differential equations together with the four regularity conditions required as $z \to \infty$. Furthermore, substituting $z = 0$ in the algebraic governing equation we have the boundary algebraic equations for the eight variables at the surface of the poroelastic half-space. It can be shown that there are only four independent boundary algebraic equations that govern these eight variables; i.e.

$$\frac{(1 \quad 1)}{(1 \quad 1)}\mathbf{V}_v(0) = 0; \quad \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & \frac{1}{\gamma} & 1 & \frac{-1}{\gamma} & 1 & \frac{-1}{\gamma} \\ 1 & 1 & 3 - 4\nu_u & 3 - 4\nu_u & -\alpha_\gamma & \alpha_\gamma \end{bmatrix} \mathbf{V}_u(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

where

$$\alpha_\gamma = \frac{4(\nu_u - \nu)}{(1 - \nu)(\gamma^2 - 1)} \tag{9}$$

## CONTACT PROBLEM

We consider the problem of a rigid circular indentor which is in smooth contact with a fluid saturated poroelastic half-space and subjected to an axisymmetric load. Three types of drainage boundary conditions are considered in order to examine their influence on the consolidation behavior of the rigid circular indentor (see Fig. 1). For future reference, the mixed boundary conditions at the surface $z = 0$ of the poroelastic layer $0 \le z < \infty$ are represented in relation to a generalized asymmetric station of deformation; i.e.

$$\begin{aligned} \sigma_{zz}(r, \theta, 0, t) &= 0; && a < r < \infty \\ u_z(r, \theta, 0, t) &= D_z(r, \theta, t); && 0 \le r < a \\ \sigma_{rz}(r, \theta, 0, t) &= 0; && 0 \le r < \infty \\ \sigma_{\theta z}(r, \theta, 0, t) &= 0; && 0 \le r < \infty \end{aligned} \tag{10a}$$

The three types of drainage conditions at the surface of the poroelastic half-space are given by the following.

### Case I

The surface of the poroelastic half-space is assumed to be completely pervious both within and exterior to the indentor; i.e.

$$p(r, \theta, 0, t) = 0; \qquad 0 \le r < \infty \tag{10b}$$

### Case II

The interface between the indentor and the soil is assumed to be impervious and the exterior region is assumed to be pervious; i.e.

$$\begin{aligned} v_z(r, \theta, 0, t) &= 0; && 0 \le r < a \\ p(r, \theta, 0, t) &= 0; && a < r < \infty \end{aligned} \tag{10c}$$

### Case III

The surface of the poroelastic half-space is assumed to be completely impervious both within and exterior to the indentor; i.e.

$$v_z(r, \theta, 0, t) = 0; \qquad 0 \le r < \infty \tag{10d}$$

where $0 < t < \infty$; and $0 \le \theta < 2\pi$.

Among the three drainage boundary conditions, cases I and III are the limiting cases that can offer the plausible "extreme estimates" for all the "partial" drainage conditions that can occur at the surface of the consolidating half-space where the impermeability surface constraints can extend to a finite distance beyond the foundation. Case II is a particular example of such an intermediate drainage condition.

## GOVERNING INTEGRAL EQUATIONS

By using the solution representations (6) and the boundary algebraic (8), we can obtain the following relations among the relevant boundary variables in the transform domain: For case I, we have





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$$\hat{w}_3 = a_1(\gamma)\hat{\tau}_3; \quad \hat{p}_w = 0 \tag{11a}$$

For case II, we have

$$\hat{w}_3 = a_1(\gamma)\hat{\tau}_3 + a_2(\gamma)\hat{p}_w; \quad \hat{\vartheta}_3 = a_3(\gamma)\hat{\tau}_3 + a_4(\gamma)\hat{p}_w \tag{11b}$$

And for case III, we have

$$\hat{w}_3 = a_5(\gamma)\hat{\tau}_3; \quad \hat{\vartheta}_3 = 0 \tag{11c}$$

For all the cases, we have the zero shear tractions on the surface $z = 0$; i.e., $\hat{\tau}_1 = 0$, $\hat{\tau}_2 = 0$. The coefficients in (11$a$)–(11$c$) are defined by

$$a_1(\gamma) = \frac{1}{\alpha_a}\left[1 + \frac{\alpha_1}{2}k_1\right]; \quad a_2(\gamma) = \frac{1}{\alpha_a}\frac{3\alpha_1}{4B(1+\nu_u)}[1 + k_1]; \quad a_3(\gamma) = \frac{2}{3}\kappa B(1+\nu_u)k_2;$$

$$a_4(\gamma) = \kappa[1 + k_2]; \quad a_5(\gamma) = \frac{1}{\alpha_a}\left[1 + \frac{\alpha_1}{2}k_3\right]; \quad k_1 = \frac{1-\gamma}{\gamma + 1 - \alpha_1}; \quad k_2 = \frac{\gamma^2 - 1}{\gamma + 1 - \alpha_1};$$

$$k_3 = \frac{2 - \gamma(\gamma+1)}{\gamma(\gamma+1) - \alpha_1}; \quad \alpha_a = -\frac{\mu}{1-\nu}; \quad \alpha_1 = \frac{2(\nu_u - \nu)}{1-\nu} \tag{11d}$$

By using these expressions and the sets of solution representations (6), we can obtain the two-dimensional Fourier-transform-based integral equations, in the Laplace transform domain, for the rigid, circular indentor problems associated with the three types of drainage boundary conditions: For cases I and III, we have

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} a_k(\gamma)\hat{\tau}_3 K\, d\varphi\, d\rho = \hat{D}_z(r, \theta, s); \qquad 0 \le r \le a$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \hat{\tau}_3 K\rho\, d\varphi\, d\rho = 0; \qquad a < r < \infty \tag{12a}$$

For case II, we have

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} [a_1(\gamma)\hat{\tau}_3 + a_2(\gamma)\hat{p}_w]K\, d\varphi\, d\rho = \hat{D}_z(r, \theta, s); \qquad 0 \le r \le a$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} [a_3(\gamma)\hat{\tau}_3 + a_4(\gamma)\hat{p}_w]K\rho^2\, d\varphi\, d\rho = 0; \qquad 0 \le r < a$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \hat{\tau}_3 K\rho\, d\varphi\, d\rho = 0; \qquad a < r < \infty$$

$$\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} \hat{p}_w K\rho\, d\varphi\, d\rho = 0; \qquad a < r < \infty \tag{12b}$$

where $a_k(\gamma) = a_1(\gamma)$ for case I; $a_k(\gamma) = a_5(\gamma)$ for case III; and $0 \le \theta < 2\pi$.

We now reduce the preceding formulation to the case of an axisymmetric rigid indentor with a flat base, for which $\hat{D}_z(r, \theta, s) = \hat{D}_z(s)$. The two-dimensional (2D) integral equations (12) can be reduced to one-dimensional (1D) Hankel-transform-based integral equations by using Fourier-series expansions. The resulting equations are the following.

For cases I and III, we have

$$\int_0^\infty \left[1 + \frac{\alpha_1}{2}k_a\right]\hat{\tau}_{30}J_0(\rho r)\, d\rho = \alpha_a\hat{D}_z(s); \qquad 0 \le r \le a$$

$$\int_0^\infty \hat{\tau}_{30}J_0(\rho r)\rho\, d\rho = 0; \qquad a < r < \infty \tag{13a}$$

For case II, we have

$$\int_0^\infty \left\{\left[1 + \frac{\alpha_1}{2}k_1\right]\hat{\tau}_{30} + \frac{3\alpha_1}{4B(1+\nu_u)}[1 + k_1]\hat{p}_{w0}\right\}J_0(\rho r)\, d\rho = \alpha_a\hat{D}_z(s); \qquad 0 \le r \le a$$

$$\int_0^\infty \left\{k_2\hat{\tau}_{30} + \frac{3}{2B(1+\nu_u)}[1 + k_2]\hat{p}_{w0}\right\}J_0(\rho r)\rho^2\, d\rho = 0; \qquad 0 \le r < a$$

$$\int_0^\infty \hat{\tau}_{30}J_0(\rho r)\rho\, d\rho = 0; \qquad a < r < \infty$$

$$\int_0^\infty \hat{p}_{w0}J_0(\rho r)\rho\, d\rho = 0; \qquad a < r < \infty \tag{13b}$$



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where $k_a = k_1$ for case I; $k_a = k_3$ for case III; and $J_0$ = Bessel function of order zero.

For the purpose of evaluation of the total force on the indentor, we note that, for the three cases, the unknown contact stress beneath the rigid disk indentor can be expressed as follows.

$$\hat{\sigma}_{zz}(r, \theta, 0, s) = \int_0^\infty \hat{\tau}_{30} J_0(\rho r)\rho \, d\rho \tag{13c}$$

## FREDHOLM INTEGRAL EQUATIONS OF SECOND KIND

The sets of Hankel-transform-based integral equations (13*a*,*b*) are singular integral equations with unknown singularities. Further simplifications and reductions have to be made to account for the unknown singularities occurring in these integral equations. Isolating these singularities, it is found that the sets of integral equations can be reduced to the systems of Fredholm integral equations of the second kind in the Laplace-transform domain. Such systems of complex Fredholm integral equations of the second kind are standard and regular integral equations that can be numerically evaluated [see e.g. Kanwal (1971), Atkinson (1976), and Baker (1977)]. Based on the procedure suggested by Sneddon (1972), we define the following solution representation for $\hat{\tau}_{30}(\rho, s)$ and $\hat{p}_{w0}(\rho, s)$ in terms of the auxiliary functions $\hat{\phi}_0(r, s)$ and $\hat{\psi}_0(r, s)$ ($\hat{\psi}_0(0, s) = 0$): i.e.

$$\hat{\tau}_{30} = \int_0^a \hat{\phi}_0(x, s)\cos(\rho x) \, dx; \quad \hat{p}_{w0} = \frac{2B(1 + \nu_u)}{3\rho} \int_0^a \hat{\psi}_0(x, s)\sin(\rho x) \, dx \tag{14a,b}$$

The sets of integral equations (13*a*,*b*) can now be reduced to the integral equations of the Abel type. The Fredholm integral equations of the second kind in the Laplace transform domain for the contact problem can be obtained from these integral equations of the Abel type. For convenience, the following nondimensional variables $\hat{\phi}(x, s_1)$, $\hat{\psi}(x, s_1)$, $\hat{X}_z(s_1)$, and $\hat{Y}_z(s_1)$ are introduced.

$$\hat{\phi}_0(r, s) = \frac{-P_z}{2\pi a} \hat{\phi}(x, s_1); \quad \hat{D}_z(s) = \frac{1 - \nu}{4\mu a} P_z \hat{X}_z(s_1);$$

$$x = \frac{r}{a}; \quad \hat{\psi}_0(r, s) = \frac{-P_z}{2\pi a^2} \sqrt{s_1}\, \hat{\psi}(x, s_1); \quad \hat{P}_z(s) = P_z \hat{Y}_z(s_1); \quad s_1 = \frac{a^2 s}{c} \tag{15}$$

We then obtain the systems of complex Fredholm integral equations of the second kind in the following nondimensional forms.

For cases I and III, we have

$$\hat{\phi}(x, s_1) + \int_0^1 \hat{K}_a(x, y, s_1)\hat{\phi}(y, s_1) \, dy = \hat{X}_z(s_1) \tag{16a}$$

For case II, we have

$$\begin{pmatrix} \hat{\phi}(x, s_1) \\ \hat{\psi}(x, s_1) \end{pmatrix} + \int_0^1 \begin{pmatrix} \hat{K}_1(x, y, s_1)\hat{K}_2(x, y, s_1) \\ \hat{K}_3(x, y, s_1)\hat{K}_4(x, y, s_1) \end{pmatrix} \begin{pmatrix} \hat{\phi}(y, s_1) \\ \hat{\psi}(y, s_1) \end{pmatrix} dy = \begin{pmatrix} X_z(s_1) \\ 0 \end{pmatrix} \tag{16b}$$

where $0 \le x \le 1$; $\hat{K}_a = K_1$ for case I; and $\hat{K}_a = K_5$ for case III; and the kernel functions, $\hat{K}_1$ to $\hat{K}_5$, are given in the following nondimensional forms.

$$\hat{K}_1 = -\frac{\alpha_1}{\pi} \int_0^\infty \frac{s_1 \cos(\rho x)\cos(\rho y) \, d\rho}{(2 - \alpha_1)\rho(\rho + \sqrt{\rho^2 + s_1}) + s_1};$$

$$\hat{K}_2 = -\frac{\alpha_1}{2} \sqrt{s_1} \left[ \frac{2}{\pi} \int_0^\infty \frac{s_1 \cos(\rho x)\sin(\rho y) \, d\rho}{\rho[(2 - \alpha_1)\rho(\rho + \sqrt{\rho^2 + s_1}) + s_1]} - H(y - x) \right];$$

$$\hat{K}_3 = -\frac{\sqrt{s_1}}{2 - \alpha_1} \left[ \frac{2}{\pi} \int_0^\infty \frac{s_1 \sin(\rho x)\cos(\rho y) \, d\rho}{\rho[(2 - \alpha_1)\rho(\rho + \sqrt{\rho^2 + s_1}) + s_1]} - H(x - y) \right];$$

$$\hat{K}_4 = \frac{2}{\pi} \int_0^\infty \frac{s_1 \sin(\rho x)\sin(\rho y) \, d\rho}{\rho[\sqrt{\rho^2 + s_1} + (1 - \alpha_1)\rho]};$$

$$\hat{K}_5 = -\frac{\alpha_1}{\pi} \int_0^\infty \frac{s_1 + \rho\sqrt{\rho^2 + s_1} - \rho^2}{\rho[(1 - \alpha_1)\rho + \sqrt{\rho^2 + s_1}] + s_1} \cos(\rho x)\cos(\rho y) \, d\rho \tag{16c}$$

where $H(x)$ = Heaviside step function. It can be shown that the infinite integrals in (16*c*) with the independent variables $x$, $y$, and $s_1$ are absolutely convergent provided $\mathrm{Re}(s_1) > 0$. The transformed value of the total axial load $\hat{P}_z(s)$ can be evaluated in the following nondimensional form.

$$\int_0^1 \hat{\phi}(x, s_1)\, dx = \hat{Y}_z(s_1) \tag{16d}$$

The contact normal stress in the Laplace transform domain can be expressed in terms of the auxiliary function $\hat{\phi}(x, s)$ for the rigid, circular indentor problem associated with cases I, II, and III; i.e.

$$\hat{\sigma}_{zz}(r, \theta, 0, s_1) = \frac{-P_z}{\pi a^2}\left[\frac{\hat{\phi}(1, s_1)}{\sqrt{1 - r^2}} - \int_r^1 \frac{\partial\hat{\phi}(x, s_1)}{\partial x}\frac{1}{\sqrt{x^2 - r^2}}\, dx\right] \tag{17}$$

where $0 \leq r < 1$.

## NUMERICAL SOLUTIONS FOR INTEGRAL EQUATIONS

Considering the structure of the kernel functions (16$c$), it becomes evident that the systems of complex Fredholm integral equations of the second kind have no known exact solutions. In this paper, we adopt the following numerical scheme for the evaluation of time-dependent solutions of the complex integral equations in the Laplace transform domain. Details of this numerical scheme can be found in Yue (1992) and Yue and Selvadurai (1994). The complex Fredholm integral equations can be written as systems of real Fredholm integral equation of the second kind by separating the complex variables into their real and imaginary parts. For the numerical solution of the integral equations, the interval [0, 1] is divided into $N$ segments with ends defined by $r_k = (k - 1)/N$; $k = 1, 2, 3, \ldots, (N + 1)$, and the collocation points are $x_k = (r_k + r_{k+1})/2$, $k = 1, 2, 3, \ldots, N$. Consequently, we can convert the integral equations into the systems of linear algebraic simultaneous equations. These linear algebraic simultaneous equations can be written in the generalized matrix form

$$\sum_{j=1}^{L} [\mathbf{A}_{lj}]\{\hat{\mathbf{X}}_j\} = \{\mathbf{B}_l\} \tag{18}$$

where $l = 1, 2, 3, \ldots, L$; $L = 2N + 2$ for cases I and III; and $L = 4N + 2$ for case II.

The matrix equation (18) can be solved numerically to generate the unknown variables $\hat{\phi}(x_l, s_1)$, $\hat{\psi}(x_l, s_1)$, and $\hat{X}_z(s_1)$ [or $\hat{Y}_z(s_1)$] in the Laplace-transform domain. The time-dependent results of the variables can be evaluated by using a modified algorithm for inverse Laplace transforms. This modified algorithm is based on the work of Crump (1976) and the details are given by Yue (1992).

The numerical techniques adopted here involve three computational steps, which are essentially based on repeated numerical integrations. The first step involves the numerical integration of the infinite integrals, which occur in the kernel functions (16$c$). As noted previously, these infinite integrals are absolutely convergent. However, there are two unusual aspects in the numerical integrations. One refers to the infinite limit and the other is associated with the integrands. As the values of the independent variables $s$ and $\rho$ become large, the integrands become rapidly oscillatory functions. This property of the integrands slows down the convergence of the numerical integration and renders the numerical integration procedure unstable. Large values of the complex variable $s$ are associated with small values of $t$ in the numerical inversion of Laplace transforms. The technique adopted in this study overcomes these two numerical problems. This particular technique consists of an approximation technique and a proceeding limit technique. In the approximation technique, the integrands are separated into two parts. One is their asymptotic behavior as $s_1/\rho^2 \rightarrow 0$. The other is the difference between the integrands and their asymptotic functions. Closed-from results can be obtained for the infinite integrals associated with the asymptotic functions. The proceeding limit technique, based on adaptively iterative Simpson's quadrature, is used for the evaluation of the infinite integrals associated with the remaining terms. The second repeated numerical integration involves systems of Fredholm integral equations, the numerical solutions of which have been well investigated by many researchers [e.g. Baker (1977), Delves and Mohamed (1985)]. These studies show that with the increase in the segment number $N$ it is possible to obtain more accurate and readily convergent solutions for the integral equations. The last computational step involves the numerical inversion of Laplace transforms. Due to the accumulation of the errors in each repeated numerical integration, however, it is necessary to evaluate and test the convergence and accuracy of the time dependent results for the systems of Fredholm integral equations of the second kind in the Laplace transform domain. The details of these calculations are documented by Yue (1992). It is concluded from this verification that the numerical scheme and techniques adopted in this study provide highly stable and accurate solutions in the time domain for the systems of Fredholm integral equations of the second kind in the Laplace transform domain and that these procedures particularly overcome the numerical problems customarily associated with the initial stages ($10^{-4} \leq ct/a^2 \leq 10^{-2}$) of the consolidation of the poroelastic medium.

Table 1 illustrates a comparison between the degrees of consolidation indentation given by the current study and by Chiarella and Booker (1975), where the poroelastic half-space is fully





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**TABLE 1. Comparison between Degrees of Consolidation Settlement ($\nu_u$ = 0.5)**

| | $\nu$ = 0.0 | | $\nu$ = 0.1 | | $\nu$ = 0.2 | | $\nu$ = 0.3 | | $\nu$ = 0.4 | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\sqrt{ct/a^2}$ (1) | Current study (2) | Chiarella and Booker (1975) (3) | Current study (4) | Chiarella and Booker (1975) (5) | Current study (6) | Chiarella and Booker (1975) (7) | Current study (8) | Chiarella and Booker (1975) (9) | Current study (10) | Chiarella and Booker (1975) (11) |
| 0.2 | 0.250 | 0.230 | 0.267 | 0.257 | 0.287 | 0.285 | 0.310 | 0.314 | 0.339 | 0.359 |
| 0.4 | 0.412 | 0.384 | 0.433 | 0.414 | 0.456 | 0.448 | 0.483 | 0.483 | 0.513 | 0.517 |
| 0.6 | 0.529 | 0.509 | 0.550 | 0.543 | 0.572 | 0.569 | 0.597 | 0.603 | 0.625 | 0.647 |
| 0.8 | 0.614 | 0.603 | 0.633 | 0.630 | 0.653 | 0.655 | 0.675 | 0.670 | 0.700 | 0.716 |
| 1.0 | 0.676 | 0.655 | 0.692 | 0.690 | 0.710 | 0.709 | 0.730 | 0.743 | 0.751 | 0.770 |
| 1.2 | 0.722 | 0.707 | 0.737 | 0.733 | 0.752 | 0.750 | 0.770 | 0.776 | 0.788 | 0.836 |
| 1.4 | 0.757 | 0.741 | 0.770 | 0.762 | 0.784 | 0.779 | 0.800 | 0.810 | 0.816 | 0.845 |

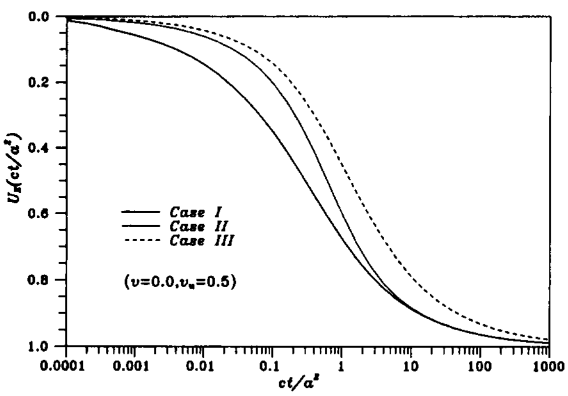

**FIG. 2. Effect of Drainage Conditions on Degree of Consolidation-Induced Indentation of Rigid Circular Indentor**

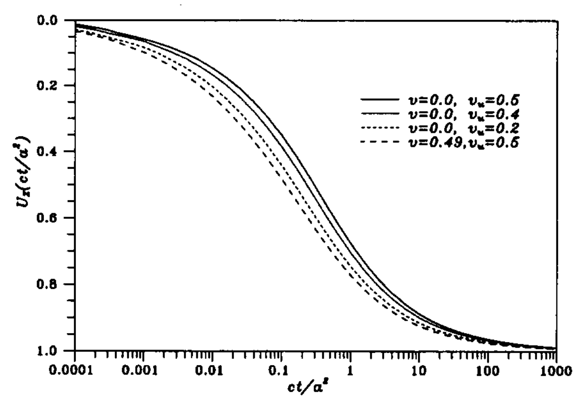

**FIG. 3. Effect of Poisson's Ratios on Degree of Consolidation-Induced Indentation of Rigid Circular Indentor: Case I (Completely Drained Boundary)**

saturated with an incompressible fluid (i.e. $B = 1$; $\nu_u = 0.5$) and its surface is completely permeable. As is evident the trends indicated by the results of the current study are consistent with the results obtained by Chiarella and Booker.

## NUMERICAL RESULTS

In the ensuing, we shall present numerical results for the axial indentation of the rigid circular indentor by assuming that it is subjected to a basic loading in the form of a Heaviside step function $P_z H(t)$. The axial displacement for the rigid circular indentor can then be expressed in the following nondimensional form.

$$\frac{a\mu D_z(T)}{P_z} = \frac{1-\nu}{4} X_z(T) \tag{19a}$$

and the degree of consolidation displacement of the rigid circular indentor can be defined as

$$U_z(T) = \frac{D_z(T) - D_z(0^+)}{D_z(\infty) - D_z(0^+)} = \frac{1 - \nu - (1 - \nu_u)X_z(T)}{\nu_u - \nu} \tag{19b}$$

In (19$b$), $D_z(0^+)$ and $D_z(\infty)$ are, respectively, the initial and final displacements of the rigid, circular indentor. They can be obtained analytically by taking, respectively, limits as $s \to \infty$ and $s \to 0$ in the integral equations (16); i.e.

$$\frac{a\mu D_z(0^+)}{P_z} = \frac{1-\nu_u}{4}; \qquad \frac{a\mu D_z(\infty)}{P_z} = \frac{1-\nu}{4} \tag{20}$$

In (19) $X_z(T)$ = a nondimensional function which can be numerically evaluated from (18). The function $X_z(T)$ includes only the constant $\alpha_1$, which is determined by the drained and undrained values of Poisson's ratios, and the conventional time factor $T(=ct/a^2)$.

The influence of drainage boundary conditions on the consolidation induced displacement of the rigid, circular indentor in smooth contact with a saturated poroelastic half-space can be observed in Fig. 2, where the degree of consolidation is plotted against the logarithm of the nondimensional time factor $ct/a^2$. As is evident, the drainage boundary conditions at the indenting surface have a significant effect on the consolidation induced displacement. The time for a certain degree of the consolidation induced displacement associated with the intermediate drainage boundary condition (case II) is bounded by the lower value that is associated with the completely permeable drainage boundary condition (case I) and the upper value that is associated with the completely impermeable drainage boundary condition (case III). The behavior of the rigid indentor corresponding to the drainage boundary condition defined in case II approaches that associated with case III at the initial stages ($T < 0.01$). For larger values of $T$ ($>1$), result for the case II approaches that of the case I.

In Figs. 3–5, the degree of consolidation induced indentation $U_z(T)$ is plotted against the time factor $T$ for four sets of Poisson's ratios. The curves in Figs. 3–5 for $(\nu, \nu_u) = (0.49, 0.5)$, $(0., 0.2)$, and $(0., 0.4)$ are almost indiscernible with those for $(\nu, \nu_u) = (0., 0.01)$, $(0.4, 0.5)$, and $(0.2, 0.5)$, respectively. From these results one could erroneously conclude that the time for a certain degree of consolidation induced indentation is not extremely sensitive to the values of the drained and undrained Poisson's ratios.

By introducing a modified time factor $T_m = c_m t/a^2 = 2\mu B^2 \kappa t/a^2$ that is independent of Poisson's ratios, we replot the degree of consolidation induced indentation associated with case I in Fig. 6. From Fig. 6 it is evident that the time for a particular degree of consolidation induced indentation is extremely sensitive to the values of Poisson's ratios; it demonstrates that the time for a particular level of consolidation is reduced as $\nu$ approaches $\nu_u$ and is almost directly proportional to $(\nu_u - \nu)$.

The nondimensional displacement of the rigid, circular indentor associated with case I is plotted in Fig. 7, where the seven sets of Poisson's ratios are employed. Fig. 7 indicates that the nondimensional magnitude of the consolidation induced displacement, i.e., $a\mu[D_z(T) - D_z(0)]/P_z$, is directly proportional to $(\nu_u - \nu)$. The rate of consolidation induced indentation (i.e. the slope of the curves) at any time increases with the increases in $(\nu_u - \nu)$ and reaches its maximum value when $\nu_u = 0.5$ and $\nu = 0$. Similar results are also observed for the consolidation induced displacement of the rigid, circular indentor, with drainage conditions prescribed by the cases II and III.

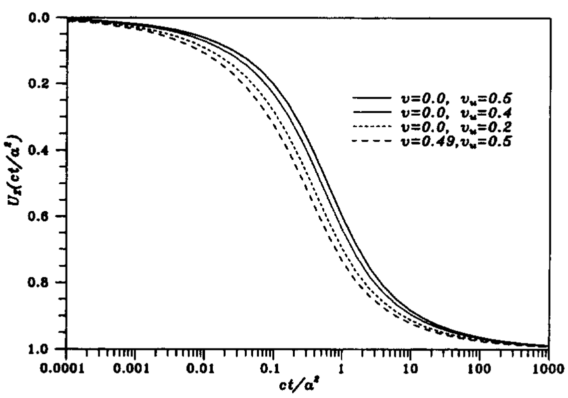


**FIG. 4. Effect of Poisson's Ratios on Degree of Consolidation-Induced Indentation of Rigid Circular Indentor: Case II (Partially Drained Boundary)**

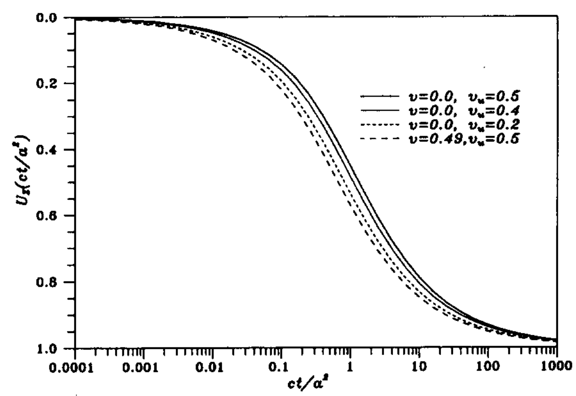


**FIG. 5. Effect of Poisson's Ratios on Degree of Conslidation Induced Indentation of Rigid Circular Indentor: Case III (Completely Undrained Boundary)**

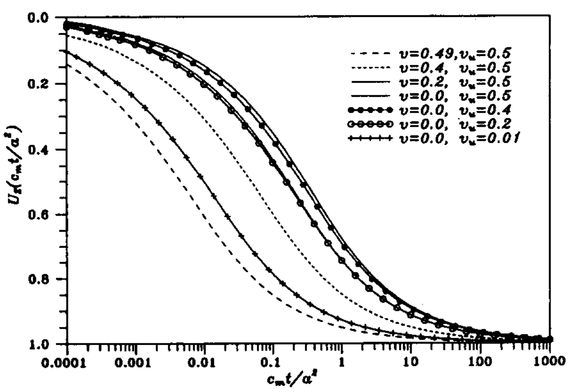


**FIG. 6. Degree of Consolidation Induced Indentation versus Modified Time Factor: Seven Poisson's Ratios and Completely Drained Boundary (Case I)**

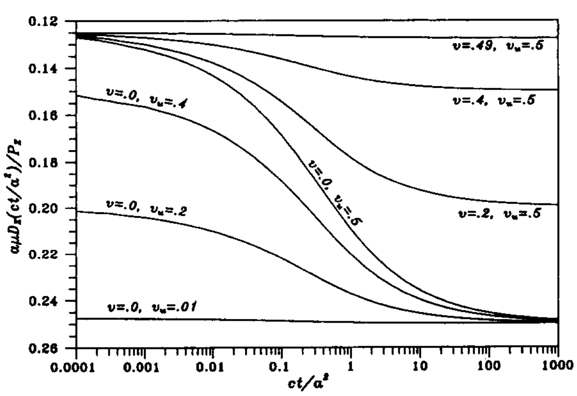


**FIG. 7. Axial Indentation of Rigid Indentor on Poroelastic Half-space: Seven Poisson's Ratios and Completely Drained Boundary (Case I)**





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## CONCLUSIONS

This paper examines the mathematical modeling of the quasi-static behavior of a rigid, circular indentor resting in smooth contact with a poroelastic half-space saturated with a compressible pore fluid. The rigid, circular indentor is subjected to an axial load and three types of drainage boundary conditions are considered. In the Laplace transform domain, these contact problems can be formulated as mixed-boundary-value problems associated with a half-space region. It is further shown that the associated integral equations can be reduced to systems of Fredholm integral equations of the second kind. The numerical solutions of these coupled Fredholm integral equations are used to generate the time-dependent behavior of the indentor, which is subjected to a step-function-type loading. Efficient computational algorithms have been used to develop the numerical solution of the sets of Fredholm integral equations in the Laplace transform domain.

The numerical results presented in the paper indicate that the drainage boundary conditions at the indenting surface of a saturated poroelastic half-space have an important influence on the consolidation rate of the rigid circular indentor. A further aspect that is considered in these studies relates to the influence of the undrained compressibility ($B$, $\nu_u$). The value of $B$ influences only the consolidation rate of the rigid indentor, while the value of $\nu_u$ has a strong effect on both the magnitude and the rate of the consolidation settlement of the rigid indentor. In the limit cases of either $\nu_u \rightarrow \nu$ or $B \rightarrow 0$, the poroelastic medium exhibits only an elastic deformation, no excess pore pressures are developed and consolidation effects are absent.

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