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A contact problem for a smooth rigid disc inclusion in a penny-shaped crack

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Introduction

Analysis of interaction between cracks and inclusions have important applications to the study of micro-mechanics of multiphase materials and to the examination of anchoring devices embedded in geological media. In multiphase composite materials with flat disc shaped inclusions, such interactions can occur during thermally induced fracture and damage evolution at the boundaries of the reinforcing particle. Examples of problems which investigate crack-inclusion interaction include the studies by Taya and Mura (1981), Tsai (1984), Selvadurai and Singh (1984, 1986), Tan and Selvadurai (1986), Selvadurai et al. (1989a,b; 1990; 1991) and Selvadurai (1985, 1989a,b). These investigations focus on the category of problems where the cracks either extend beyond the boundary of a disc shaped inclusion or delaminate at the plane interface. The modelling is applicable to situations where thermal loadings and the thermo-elastic mis-match between the matrix and the inclusion can induce either interfacial delamination or matrix cracking. Other situations can include debonding of the disc inclusion over an entire plane interface. Keer (1975) examined the problem of the axisymmetric loading of a penny-shaped crack by a rigid disc shaped inclusion which is bonded to one of its plane faces. The axial stiffness of the rigid inclusion is evaluated in exact closed form and the stresses at the region of contact are evaluated in integral form. The solution by Keer (1975) is an elegant exposition of the application of Hilbert transform techniques to the solution of this class of problem. Consequently the formulation accounts for the oscillatory form of the stress singularity at the tip of the crack which is located at the bonded rigid boundary.

Disc inclusion problems also have useful applications in the field of geomechanics where the loaded inclusion models the behaviour of the rigid anchoring device embedded in a geological medium. The solutions to the directly loaded rigid inclusion problem can be used to evaluate the axial stiffness of circular plate anchors predominantly in the elastic range (Selvadurai, 1976, 1989a,b; Rowe and Booker, 1979; Selvadurai et al., 1991).

This paper examines the axisymmetric problem of the complete indentation of the single face of a penny-shaped crack by a rigid disc inclusion. The contact between the inclusion and the crack is assumed to be smooth. Consequently, the analysis provides an estimate for the stiffness of the inclusion for the limiting case of frictionless contact (as opposed to full adhesive contact) between the inclusion and the elastic medium. The single integral equation governing the crack-inclusion interaction problem is numerically solved to develop results for the axial stiffness of the inclusion and the mixed mode stress intensity factors at the tip of the penny-shaped crack.

Fundamental equations

Owing to the axial symmetry of the crack-inclusion interaction problem it is convenient to employ the representation based on the strain potential approach of Love (1927). It can be shown that the solution to the displacement equations of equilibrium can be expressed in terms of a single function. In the application of the strain potential function to the crack-inclusion interaction problem (Fig. 1) it is convenient to adopt strain potential functions $\varphi^{(i)}(r, z)$ applicable to regions $z \in (0, \infty)$ and $z \in (0, -\infty)$ which are designated by $i = 1$ and $i = 2$ respectively. In the absence of body forces, functions $\varphi^{(i)}(r, z)$ satisfy the biharmonic equation

$$\nabla^2\nabla^2\varphi^{(i)}(r, z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is the axisymmetric form of Laplace's operator in cylindrical polar coordinates. The strain potential functions applicable to the regions 1 and 2 can be obtained by a Hankel transform development of (1). The relevant results can be expressed in the forms (Sneddon, 1977)

$$\varphi^{(1)}(r, z) = \int_0^\infty \xi[A(\xi) + zB(\xi)]\, e^{-\xi z} J_0(\xi r)\, d\xi \tag{3}$$

$$\varphi^{(2)}(r, z) = \int_0^\infty \xi[C(\xi) + \xi D(\xi)]\, e^{\xi z} J_0(\xi r)\, d\xi \tag{4}$$

where $A(\xi)$, $B(\xi)$, etc. are arbitrary functions. The relevant integral expressions for the displacements and stresses in the two regions ($z \geq 0$ and $z \leq 0$) can be obtained by making use of the results

$$2Gu_r^{(i)}(r, z) = -\frac{\partial^2\varphi^{(i)}}{\partial r\,\partial z} \tag{5}$$

$$2Gu_z^{(i)}(r, z) = 2(1-\nu)\nabla^2\varphi^{(i)} - \frac{\partial^2\varphi^{(i)}}{\partial z^2} \tag{6}$$

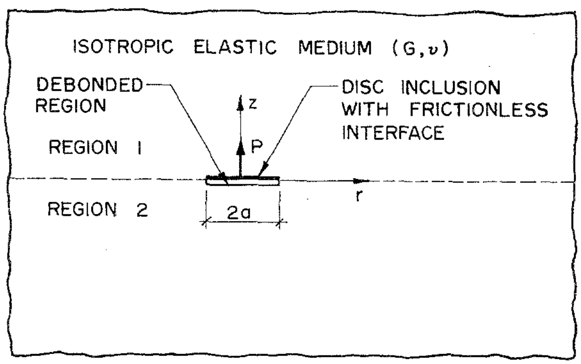

Figure 1
The indentation of the penny-shaped crack by a disc inclusion.

and

$$\sigma_{zz}^{(i)}(r, z) = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\varphi^{(i)} - \frac{\partial^2\varphi^{(i)}}{\partial z^2}\right] \tag{7}$$

$$\sigma_{rz}^{(i)}(r, z) = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\varphi^{(i)} - \frac{\partial^2\varphi^{(i)}}{\partial z^2}\right] \tag{8}$$

respectively.

The crack-inclusion interaction problem

We now focus attention on the problem of the complete indentation of a single face of a penny-shaped crack of radius 'a' by a rigid smooth disc inclusion also of radius 'a' (Fig. 1). It is assumed that axisymmetric force P acting on the inclusion induces a displacement Δ in the z-direction. The mixed boundary conditions associated with the problem are as follows:

$$u_z^{(1)}(r, 0) = \Delta; \qquad 0 \le r \le a \tag{9}$$

$$\sigma_{rz}^{(1)}(r, 0) = 0; \qquad 0 < r < a \tag{10}$$

$$\sigma_{rz}^{(2)}(r, 0) = 0; \qquad 0 < r < a \tag{11}$$

$$\sigma_{zz}^{(2)}(r, 0) = 0; \qquad 0 < r < a \tag{12}$$

$$u_z^{(1)}(r, 0) = u_z^{(2)}(r, 0); \qquad a \le r < \infty \tag{13}$$

$$u_r^{(1)}(r, 0) = u_r^{(2)}(r, 0); \qquad a \le r < \infty \tag{14}$$

$$\sigma_{zz}^{(1)}(r, 0) = \sigma_{zz}^{(2)}(r, 0); \qquad a \le r < \infty \tag{15}$$

$$\sigma_{rz}^{(1)}(r, 0) = \sigma_{zz}^{(2)}(r, 0); \qquad a \le r < \infty. \tag{16}$$

Considering the strain potentials (3) and (4) and the relationships (5) to (8), the mixed boundary conditions can be expressed in terms of the functions $A(\xi)$, $B(\xi)$, etc. in the following forms:

$$\int_0^\infty \xi[\xi A(\xi) + 2(1-2\nu)B(\xi)]J_0(\xi r)\, d\xi = -2G\Delta; \qquad 0 \le r \le a \tag{17}$$

$$\int_0^\infty \xi^2[\xi A(\xi) - 2\nu B(\xi)]J_1(\xi r)\, d\xi = 0; \qquad 0 < r < a \tag{18}$$

$$\int_0^\infty \xi[\xi\{A(\xi) - C(\xi)\} + 2(1-2\nu)\{B(\xi) + D(\xi)\}]J_0(\xi r)\, d\xi = 0; \qquad a \le r < \infty \tag{19}$$

$$\int_0^\infty \xi[-\xi\{A(\xi) + C(\xi)\} + \{B(\xi) - D(\xi)\}]J_1(\xi r)\, d\xi = 0; \qquad a \le r < \infty \tag{20}$$

$$\int_0^\infty \xi^2[\xi\{A(\xi) + C(\xi)\} + (1-2\nu)\{B(\xi) - D(\xi)\}]J_0(\xi r)\, d\xi = 0; \qquad a < r < \infty \tag{21}$$

$$\int_0^\infty \xi^2[\xi\{A(\xi) - C(\xi)\} - 2\nu\{B(\xi) + D(\xi)\}]J_1(\xi r)\, d\xi = 0; \qquad a < r < \infty. \tag{22}$$

Avoiding details of the procedures, which can be found in the references cited previously, it can be shown that the set of integral equations (17) to (22) can be reduced to the forms

$$f_1(s) + \frac{2}{\pi}\frac{d}{ds}\int_0^s \frac{r\,dr}{[s^2 - r^2]^{1/2}}\int_a^\infty \frac{f_2(u)}{[u^2 - r^2]^{1/2}}$$

$$= -\frac{\Delta G}{(1-\nu)}\frac{d}{ds}\int_0^s \frac{r\,dr}{[s^2 - r^2]^{1/2}}$$

$$+ \frac{(1-2\nu)}{2(1-\nu)}\frac{d}{ds}\int_0^s \frac{r\,dr}{[s^2 - r^2]^{1/2}}\int_0^\infty Q(\xi)J_0(\xi r)\,d\xi; \qquad 0 < s < a \tag{23}$$

$$f_2(u) - \frac{2}{\pi}\frac{d}{du}\int_u^\infty \frac{r\,dr}{[r^2 - u^2]^{1/2}}\int_0^a \frac{f_1(s)\,ds}{[r^2 - s^2]^{1/2}}$$

$$= \frac{d}{du}\int_u^\infty \frac{r\,dr}{[r^2 - u^2]^{1/2}}\int_0^\infty P(\xi)J_0(\xi r)\,d\xi; \qquad a < u < \infty \tag{24}$$

where $f_1(\xi)$ and $f_2(\xi)$ are defined by

$$\xi[\xi A(\xi) + (1-2\nu)B(\xi)] = \frac{2}{\pi}\left[\int_0^a f_1(\xi)\cos(\xi s)\,ds + \int_a^\infty f_2(\xi)\sin(\xi s)\,ds\right] \tag{25}$$

and

$$Q(\xi) = \xi[2\nu D(\xi) + \xi C(\xi)] \tag{26}$$

$$P(\xi) = \xi[(1-2\nu)D(\xi) - \xi C(\xi)]. \tag{27}$$

The integral equations (23) and (24) can be further reduced to a single integral equation of the form

$$T_1(r) + \frac{1}{\pi^2}\int_0^a \frac{T_1(s)K_1(r,s)\,ds}{(r^2 - s^2)} - \frac{(1-2\nu)^2}{8a(1-\nu)^2}\int_0^a T_1(v)\,dv$$

$$= \frac{-1}{2(1-\nu^2)}; \qquad 0 < r < a \tag{28}$$

where

$$K_1(r,s) = \left[r\log_e\left\{\frac{a-r}{a+r}\right\} - s\log_e\left\{\frac{a-s}{a+s}\right\}\right] \tag{29}$$

and

$$T_1(r) = \frac{f_1(r)}{E\Delta}. \tag{30}$$

The solution of the integral equation (28) formally completes the solution of the mixed boundary value problem related to the smooth complete indentation of the penny-shaped crack by a rigid disc inclusion, defined by (9)–(16).

Load-displacement relationship for the inclusion

The axial stress distribution at the inclusion-elastic medium interface can be used to evaluate the load-displacement relationship for the inclusion. From (3) and (7) we have

$$\sigma_{zz}^{(1)}(r, 0) = \int_0^\infty \xi^2[\xi A(\xi) + (1-2\nu)B(\xi)]J_0(\xi r)\,d\xi. \tag{31}$$

The load-displacement relationship is obtained by considering the equilibrium equation for the inclusion. The total force P in the inclusion is given by

$$P = 2\pi \int_0^a \sigma_{zz}^{(1)}(r, 0)\,dr \tag{32}$$

which can be expressed in the form

$$\frac{P}{E\Delta} = 4 \int_0^a T_1(r)\,dr. \tag{33}$$

The stress intensity factors

The indentation of the penny-shaped crack induces both Mode I and Mode II stress intensity factors at the crack tip. The stress intensity factors are defined by the relationships

$$K_I^a = \lim_{r \to a+} \{2(r-a)\}^{1/2}\sigma_{zz}(r, 0) \tag{34}$$

$$K_{II}^a = \lim_{r \to a+} \{2(r-a)\}^{1/2}\sigma_{rz}(r, 0). \tag{35}$$

Making use of the results developed in the previous sections it can be shown that

$$\frac{K_I^a}{E\Delta} = \frac{2}{\pi}\frac{T_1(a)}{\sqrt{a}} \tag{36}$$

$$\frac{K_{II}^a}{E\Delta} = \frac{(1-2\nu)}{2\pi(1-\nu)a^{3/2}} \int_0^a T_1(v)\,dv. \tag{37}$$

Numerical results

The integral equation (28) governing the indentation of the penny-shaped crack can be solved by employing a numerical procedure. The interval $(0, a)$ is divided into N segments with r_i $(i = 1$ to $N+1)$ such that $r_i = (i-1)h$ and $h = a/N$. The equivalent matrix representation of (28) can be written as

$$[A_{ij}]\{T_1(r_j)\} = \{B_i\} \tag{38}$$

with $i, j = 1, 2, \ldots, N$; $B_i = -1/2(1-\nu^2)$ and the coefficients A_{ij} are given by

$$A_{ij} = \begin{cases} \dfrac{h}{\pi^2}\dfrac{K(r_i, r_j)}{(r_i^2 - r_j^2)} - \dfrac{(1-2\nu)^2}{8(1-\nu)^2}\dfrac{h}{a}, & \text{if } i \neq j \\[2ex] \dfrac{1}{\pi^2}\left[\dfrac{h}{2r_i}\log_e\left|\dfrac{a-r_i}{a+r_i}\right| - \dfrac{r_i h}{(a^2 - r_i^2)}\right] - \dfrac{(1-2\nu)^2}{8(1-\nu)^2}\dfrac{h}{a}, & \text{if } i = j. \end{cases} \tag{39}$$

Upon solution of (38), the load-displacement relationship for the indenting inclusion and the stress intensity factors at the tip of the penny-shaped crack can be evaluated by

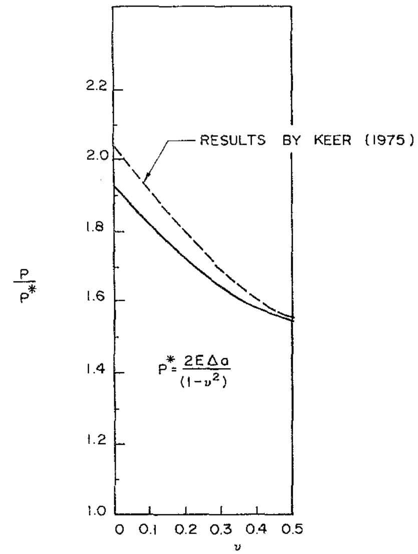

Figure 2
The stiffness of the indenting inclusion.

making use of the results (33), (36) and (37). The accuracy of the numerical scheme is verified by comparison with known exact solutions for contact problems.

The results for the axial stiffness of the indenting circular inclusion are presented in Fig. 2. The normalized results for P/P^* (where $P^* = 2E\Delta a/(1-v^2)$) are compared with equivalent results derived by Keer (1975) for the problem of the loading of a penny-shaped crack by an inclusion fully bonded to one of the crack surfaces. Keer's result is given by

$$P = \frac{4\pi G\Delta a}{(3-4v)}\left[1 + \frac{\{\log_e(3-4v)\}^2}{\pi^2}\right]. \tag{40}$$

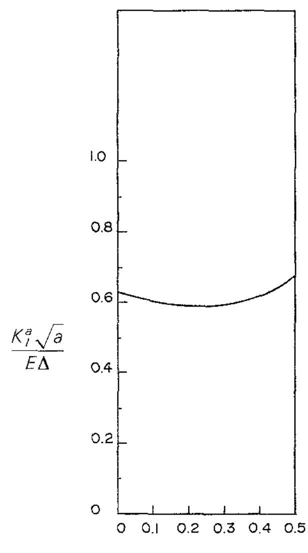

Figure 3
The mode I stress intensity factor at the crack tip.

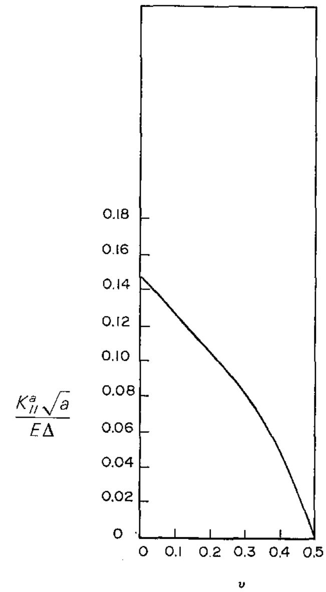

Figure 4
The mode II stress intensity factor at the crack tip.

The comparison of results indicates that adhesion at the inclusion-elastic medium has only a minor effect on the elastic stiffness. Also when $\nu = 1/2$, the solution for the adhesive contact problem converges to the result developed in this paper for the smooth contact. The accuracy of the numerical scheme is also verified for this particular limit. It is found that for $N = 10$, the error in the two sets of numerical results is approximately 1%.

The numerical results derived for the stress intensity factors K_I^a and K_{II}^a are shown in Figs. 3 and 4. The results indicate that the Mode I stress intensity factor is relatively insensitive to the Poisson's ratio of the elastic material. In contrast, the Mode II stress intensity factor is particularly sensitive to the Poisson's ratio. In the instance when $\nu = 1/2$, the shear stresses $\sigma_{rz}^{(1)}$; $\sigma_{rz}^{(2)}$ on $z = 0$ $(r > a)$ vanish with the result that $K_{II}^a = 0$.

Concluding remarks

The classical elasticity problem related to the complete indentation of a single face of a penny-shaped crack by a rigid smooth inclusion is examined. It is shown that the interaction problem can be reduced to the solution of a single Fredholm type integral equation of the second kind. This equation can be solved in a numerical fashion to generate results of engineering interest. These results are compared with equivalent results available in the literature for the adhesive indentation of a penny-shaped crack. It is shown that when $\nu = 1/2$, solutions for both the smooth contact and adhesive contact problems reduce to the same result. The dominant influences of interface conditions occur only in the estimate for the Mode II stress intensity factor evaluated for $\nu = 0$.

References

Keer, L. M., *Mixed boundary value problems for a penny-shaped cut.* J. Elasticity *5*, 89–98 (1975).

Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity* (4th ed.). Cambridge University Press, Cambridge 1927.

Rowe, R. K. and Booker, J. R., *The analysis of inclined anchor plates*. Proc. 3rd Int. Conf. Num. Meth. Geomech., Aachen *3*, 1227–1236 (1979).
Tan, C.-L. and Selvadurai, A. P. S., *Internally indented penny-shaped cracks: A comparison of analytical and boundary integral equation estimates*. Theor. & Appl. Fracture Mech. *5*, 181–188 (1986).
Taya, M. and Mura, T., *On the stiffness and strength of an aligned short-fibre reinforced composite containing fibre end cracks under uniaxial applied stress*. J. Appl. Mech. *48*, 361–367 (1981).
Tsai, Y. M., *Indentation of a penny-shaped crack by an oblate spheroidal rigid inclusion in a transversely isotropic elastic medium*. J. Appl. Mech. *51*, 811–815 (1984).
Selvadurai, A. P. S., *The load-deflexion characteristics of a deep rigid anchor in an elastic medium*. Geotechnique *26*, 603–612 (1976).
Selvadurai, A. P. S., *On an integral equation governing an internally indented penny-shaped crack*. Mech. Res. Comm. *12*, 347–351 (1985).
Selvadurai, A. P. S., *The influence of a boundary fracture on the elastic stiffness of a deeply embedded anchor plate*. Int. J. Numer. Analytical Meth. Geomech. *13*, 159–170 (1989a).
Selvadurai, A. P. S., *Elastic stiffness of a flat anchor region at a cracked geological interface*. J. Eng. Mech. Proc. ASCE *115*, 18–32 (1989b).
Selvadurai, A. P. S. and Singh, B. M., *On the expansion of a penny-shaped crack by a rigid circular disc inclusion*. Int. J. Fracture Mech. *25*, 69–77 (1984).
Selvadurai, A. P. S. and Singh, B. M., *Axisymmetric problems for an externally cracked elastic solid, II. Effects of a penny-shaped inclusion*. Int. J. Engng. Sci. *25*, 1477–1490 (1986).
Selvadurai, A. P. S., Singh, B. M. and Au, M. C., *Axial loading of a rigid disc inclusion with a debonded region*. Int. J. Solids Structures *25*, 783–795 (1989a).
Selvadurai, A. P. S., Singh, B. M. and Au, M. C., *Boussinesq's problem for a debonded boundary*. J. Appl. Math. Phys. (ZAMP) *42*, 1–14 (1989b).
Selvadurai, A. P. S., Au, M. C. and Singh, B. M., *A symmetric loading of an externally cracked elastic solid by an in-plane penny-shaped inclusion*. Theor. & Appl. Fracture Mech. *14*, 253–266 (1990).
Selvadurai, A. P. S., Singh, B. M. and Au, M. C., *The in-plane loading of a rigid disc inclusion embedded in an elastic halfspace*. J. Appl. Mech. Trans. ASME *58*, 362–369 (1991).
Sneddon, I. N., *Application of Integral Transforms in the Theory of Elasticity*. CISM Courses & Lect. No. 220, Springer-Verlag, Wien 1977.

Abstract

The present paper examines the problem of the complete indentation of the surface of a penny-shaped crack by a smooth rigid disc inclusion. The integral equation governing the problem is solved numerically to evaluate the axial stiffness of the rigid inclusion and the stress intensity factors at the tip of the penny-shaped crack.

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