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THE INFLUENCE OF A MINDLIN FORCE ON THE AXISYMMETRIC INTERACTION BETWEEN AN INFINITE PLATE AND AN ELASTIC HALFSPACE

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ABSTRACT The classical problem concerning the axisymmetric flexure of an infinite elastic plate resting on a homogeneous isotropic elastic halfspace is extended to include the effects of a Mindlin force, namely, a concentrated force which acts at an interior point of the halfspace along the axis of symmetry. Formal integral relationships and numerical results are presented for the deflection of the infinite plate and for contact stress at the interface for the particular case where the plate is subjected to a uniform circular load.

1. INTRODUCTION

The present paper is concerned with the flexural interaction of a thin elastic plate of infinite extent with a homogeneous isotropic elastic half-space under the combined action of axisymmetric loads applied at the surface of the plate and in the interior of the elastic halfspace. In particular, the internal loading of the halfspace is restricted to Mindlin's [1] problem of a concentrated force which acts at a point in the interior of an elastic halfspace with a traction free boundary.

As in previous investigations of the infinite plate problem, (see e.g. Hetenyi [2] Sneddon et al. [3]) the contact at the infinite plate-elastic medium boundary is assumed to be frictionless. Accordingly the plate is free to deform horizontally relative to the elastic halfspace. If bonded contact conditions are assumed at the interface (i.e. continuity of horizontal displacements) then the flexural behaviour of the plate should also take into consideration the effects of membrane stresses induced by frictional forces introduced at the interface. It is further assumed that there is no loss of contact at the plate-elastic medium interface in the vertical direction due to the action of the external and internal load systems. Such an assumption would require the interface to be capable of sustaining tensile normal tractions. From a physical point of view this assumption would seem to be inconsistent with conditions associated with a smooth interface. However, in relation to possible applications in the area of structural foundation problems it could be assumed that the compressive stresses induced by the self weight of the plate are sufficiently large to prevent any loss of contact at the interface.

Using Hankel transform techniques outlined by Sneddon [4], formal integral expressions are developed for the plate deflection, contact stress, etc., for the case where the plate is subjected to an arbitrary axisymmetric external load. Numerical results are given for the particular instance where the external load corresponds to a uniform circular load.

The basic problem discussed here is of interest in connection with the 'prestressing' of structural foundations against uplift loads; the interior force approximately represents the influence of the anchor region. These solutions can be further extended to investigate the influence of distributed axisymmetric internal loadings.

2. AXISYMMETRIC LOADING OF THE ELASTIC HALFSPACE

Firstly, the problem of a homogeneous isotropic elastic halfspace which is subjected to an axisymmetric normal traction f(r) on its plane boundary and a concentrated force P at a distance c from the origin of coordinates is investigated. It is assumed that the direction of the force P is in the negative z-direction (Fig. 1).

It can be shown that the transformed value of the surface displacement of the halfspace in the z-direction, $u_z^I(r, 0)$ (= $w_I(r)$) is related to the transformed value of the applied normal stress f(r) in the following manner

$$\bar{w}_I^0(\xi) = \frac{a(1-\nu_s)}{G_s\xi}\,\bar{f}^0(\xi) \tag{1}$$

where a is a typical length parameter of the problem and G_s and ν_s are respectively the linear elastic shear modulus and Poisson's ratio of the elastic material. In (1), $\bar{w}_I^0(\xi)$ denotes the zeroth-order Hankel transforms of the surface displacement defined as

$$\bar{w}_I^0(\xi) = H_0\{w_I(r) \ ; \ \xi\} \tag{2}$$

where the operator H_0 is given by the equation

$$H_0\{\Phi(r) \ ; \ \xi\} = \int_0^\infty r\Phi(r) J_0(\xi r/a) \, dr \ . \tag{3}$$

Similarly, $\bar{f}^0(\xi)$ denotes the zeroth-order Hankel transform of the normal stress $f(r)$. The corresponding Hankel inversion theorem is

$$\Phi(r) = \frac{1}{a^2} \int_0^\infty \xi \bar{\Phi}^0(\xi) J_0(\xi r/a) \, d\xi \ . \tag{4}$$

The problem of a concentrated force acting at a point in the interior of an isotropic elastic halfspace with a traction free boundary was first solved by Mindlin [1] and later by Dean et al. [5]. By using a combination of solutions corresponding to Kelvin's problem for a concentrated force acting at the interior of an infinite space and appropriate distribution of nuclei of strain, Mindlin was able to obtain an exact closed form solution to the problem of the halfspace loaded internally by a concentrated force which acts in a direction normal to the traction free plane boundary. The surface displacement $u_z^{II}(r, 0)$ $(=w_{II}(r))$ of the halfspace due to the action of the internal load P, is given by

$$w_{II}(r) = \frac{P(1-\nu_s)}{4\pi G_s} \left[\frac{2}{(r^2+c^2)^{1/2}} + \frac{c^2}{(1-\nu_s)\,(r^2+c^2)^{3/2}} \right] . \tag{5}$$

The zeroth order Hankel transform of the surface displacement (5) can be written as (see e.g. Erdelyi et al. [6])

$$\bar{w}_{II}^0(\xi) = \frac{a(1-\nu_s)}{G_s \xi} \bar{s}^0(\xi) \tag{6}$$

where

$$\bar{s}^0(\xi) = \frac{P}{4\pi} \left[2 + \frac{(\xi c/a)}{(1-\nu_s)} \right] e^{-\xi c/a} \ . \tag{7}$$

3. THE INFINITE PLATE PROBLEM

This section considers the axisymmetric flexure of the infinite elastic plate subjected to a surface load $p(r)$ and resting on a homogeneous isotropic elastic halfspace which is loaded internally by the concentrated force P (Fig. 1). Since there is no loss of contact at the interface, the surface displacements of the elastic halfspace correspond to the flexural deflections of the plate w(r). The contact stress at the interface is denoted by q(r). The transformed values of the plate deflection, the contact stress and the external load are defined by

$$[\bar{w}^0(\xi)\ ;\ \bar{q}^0(\xi)\ ;\ \bar{p}^0(\xi)] = H_0\{[w(r)\ ;\ q(r)\ ;\ p(r)]\ ;\ \xi\}. \tag{8}$$

For thin plates which satisfy the Poisson-Kirchoff plate theory, the differential equation governing axisymmetric flexure reduces to (see Timoshenko and Woinowsky-Krieger [7])

$$D\tilde{\nabla}^4 w(r) + q(r) = p(r) \tag{9}$$

where

$$\tilde{\nabla}^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \tag{10}$$

is the Laplace operator in plane polar coordinates; $D(= E_p h^3/12(1-\nu_p^2))$ is the flexural rigidity of the plate; h is the plate thickness and E_p and ν_p are the elastic constants of the plate material. Operating on (9) with the zeroth-order Hankel transform we obtain

$$D\,\frac{\xi^4}{a^4}\bar{w}^0(\xi) + \bar{q}^0(\xi) = \bar{p}^0(\xi)\ . \tag{11}$$

The surface displacement of the halfspace due to the combined action of the contact stress q(r) and the internal load P can be obtained by the combination of results (1) and (6); thus

$$\bar{w}^0(\xi) = \frac{a(1-\nu_s)}{G_s\xi}[\bar{q}^0(\xi) - \bar{s}^0(\xi)]\ . \tag{12}$$

The elimination of $\bar{q}^0(\xi)$ between (11) and (12), and the subsequent application of the inversion theorem (4) yields the following expression for the plate deflection:

$$w(r) = \frac{(1-\nu_s)}{G_s a}\int_0^\infty [\bar{p}^0(\xi) - \frac{P}{\pi}\{\tfrac{1}{2} + \frac{\xi c/a}{4(1-\nu_s)}\}e^{-\xi c/a}]\frac{J_0(\xi r/a)}{\{1+R\xi^3\}}d\xi \tag{13}$$

where

$$R = \frac{1}{6}\,\frac{E_p}{E_s}\,\left(\frac{1-\nu_s^2}{1-\nu_p^2}\right)\left(\frac{h}{a}\right)^3 . \tag{14}$$

Similarly, the contact stress is given by

$$q(r) = \frac{1}{a^2}\int_0^\infty [\xi\bar{p}^0(\xi) + \frac{P\xi^4 R}{\pi a^2}\{\tfrac{1}{2} + \frac{\xi c/a}{4(1-\nu_s^2)}\}e^{-\xi c/a}]\frac{J_0(\xi r/a)}{\{1+R\xi^3\}}\,d\xi. \tag{15}$$

The flexural moments (M_r, M_θ) and the shearing force (V_r) in the infinite plate take the forms

$$(M_r; M_\theta; V_r) = R\int_0^\infty [\bar{p}^0(\xi) - \frac{P}{\pi}\{\tfrac{1}{2} + \frac{\xi c/a}{4(1-\nu_s)}\}e^{-\xi c/a}]\frac{(m_r; m_\theta; v_r)}{\{1+R\xi^3\}}\,d\xi \quad (16a)$$

where

$$m_r = \xi^2 J_0(\xi r/a) - (1-\nu_p)\frac{\xi r}{a}J_1(\xi r/a)$$

$$m_\theta = \nu_p\xi^2 J_0(\xi r/a) + (1-\nu_p)\frac{\xi r}{a}J_1(\xi r/a) \quad (16b)$$

$$v_r = -\frac{\xi^3}{a}J_1(\xi r/a).$$

Of particular interest is the case where the plate is subjected to external and internal concentrated loads of equal magnitude. As a result of (3) and (8), $\bar{p}^0(\xi) = P/2\pi$; the equations (13) and (15) now reduce to the non-dimensional forms

$$\frac{w(r)}{[P(1-\nu_s)/\pi G_s a]} = \int_0^\infty [\tfrac{1}{2} - \{\tfrac{1}{2} + \frac{\xi c/a}{4(1-\nu_s)}\}e^{-\xi c/a}]\frac{J_0(\xi r/a)}{\{1+R\xi^3\}}\,d\xi$$

$$\frac{q(r)}{[P/\pi a^2]} = \int_0^\infty \xi[\tfrac{1}{2} + R\xi^3\{\tfrac{1}{2} + \frac{\xi c/a}{4(1-\nu_s)}\}e^{-\xi c/a}]\frac{J_0(\xi r/a)}{\{1+R\xi^3\}}\,d\xi \;. \quad (17)$$

As the parameter $c/a \to \infty$, the influence of the internal loading diminishes and the solutions (17) reduce to their corresponding expressions obtained independently by Holl [8] and Hogg [9] for the classical problem. As $c/a \to 0$ the plate deflection becomes zero for all values of r; in this case the thin plate is subjected to a doublet of concentrated forces at the origin. Similarly, since

$$\int_0^\infty \xi J_0(\xi r/a)\,d\xi = 0 \quad (18)$$

the contact stress at the plate-elastic halfspace interface also tends to zero for all values of r as $c/a \to 0$.

The special case of external loading $p(r)$ used to graphically illustrate the effects of the internal concentrated force on the plate deflection and the contact stress, corresponds to a circular load of radius a_0 and stress intensity p_0. In particular, we assume that $P = (p_0\pi a_0^2)$. The transformed value of the external load is

$$\bar{p}^0(\xi) = \frac{J_1(\xi)}{\xi} \quad (19)$$

where the length parameter a has been set equal to the radius of the external load a_0. The final solutions for the deflection of the infinite plate and the contact stress at the interface are

$$\frac{w(r)}{[P(1-\nu_s)/\pi G_s a_0]} = \int_0^\infty \left[\frac{J_1(\xi)}{\xi} - \left\{\tfrac{1}{2} + \frac{\xi c/a_0}{4(1-\nu_s)}\right\} e^{-\xi c/a_0}\right] \frac{J_0(\xi r/a_0)}{[1+R_0\xi^3]}\, d\xi \qquad (20a)$$

$$\frac{q(r)}{[P/\pi a_0^2]} = \int_0^\infty \left[J_1(\xi) + R_0\xi^4 \left\{\tfrac{1}{2} + \frac{\xi c/a_0}{4(1-\nu_s)}\right\} e^{-\xi c/a_0}\right] \frac{J_0(\xi r/a_0)}{[1+R_0\xi^3]}\, d\xi \qquad (20b)$$

where

$$R_0 = [R]_{a = a_0}.$$

4. NUMERICAL RESULTS

To illustrate the quantitative importance of the internal concentrated force, the integral expressions for the plate deflection (20a) and the contact stress (20b) are evaluated. The infinite integrals of the type (20) do not appear to reduce to any integrals known from the literature. A direct numerical integration technique is employed for the evaluation of the non-dimensional expressions for the plate deflection $w'(r)$ $(= \pi G_s a_0 w(r)/P(1-\nu_s))$ and the contact stress $q'(r)$ $(= \pi a_0^2 q(r)/P)$. The numerical integration is performed by representing the integral as an infinite series bounded by subsequent zeros of $J_1(\xi)J_0(\xi r/a_0)$ and $J_0(\xi r/a_0)$. Integration, which proceeds by one interval at a time, is carried out by using a 15 point Gauss-Legendre quadrature. The convergence of the series is slow and the procedure was terminated when the contribution from each partial integration was less than 0.01 percent. The accuracy of this numerical integration procedure was checked by comparing the numerical results with those of certain standard infinite integrals involving products of first- and zeroth-order Bessel functions.

$$\left[\text{e.g.} \int_0^\infty \frac{J_1(\xi)J_0(r\xi/a)}{[1+\xi^2]}\, d\xi = I_1(1)K_0(r/a) \text{ for } (r/a) > 1.\right]$$

Also, for the purpose of these numerical computations the following material parameters have been utilized: Poisson's ratio of the elastic halfspace $\nu_s = 0, 0.5$; relative rigidity parameter $R_0 = 0.1, 1.0$. It should also be noted that the internal force P acts in the negative z-direction (Fig. 1).

The Figs. 2-5 illustrate the variation of the dimensionless plate deflection w'(r) along a radial direction, computed for different locations (c/a_0) of the internal concentrated force, Poisson's ratio (ν_s) and relative rigidity (R_0). The magnitude and distribution of the plate deflection is considerably altered as the internal concentrated force migrates to the surface of the halfspace. As $(c/a_0) \to 0$ the plate deflections become negative and we may expect tensile tractions to develop at certain locations of the interface. Since the theoretical development of the interaction problem does not account for the development of such tensile stresses (large c.f. the applied external stress) the numerical results become inadmissible as $(c/a_0) \to 0$. As $(c/a_0) \to \infty$, the plate deflections converge to the appropriate result related to the unloaded elastic halfspace. Numerical computations indicate that when $(c/a_0) > 6.0$, the effects of the internal concentrated force on the plate deflection become insignificant. The Figs. 6-9, similarly, illustrate the variation of the contact stress at the plate-elastic halfspace interface along a radial direction. The contact stresses are generally compressive for $(c/a_0) > 0.4$ although tensile tractions of small magnitude $(< 0.01\ p_0)$ tend to develop in the regions $(r/a_0) > 2$. Numerical computations indicate that when $(c/a_0) > 5$ the contact stress distributions are unaffected by the internal concentrated force. The procedure outlined above can be further extended to numerically evaluate the flexural moments and shearing force in the infinite plate.

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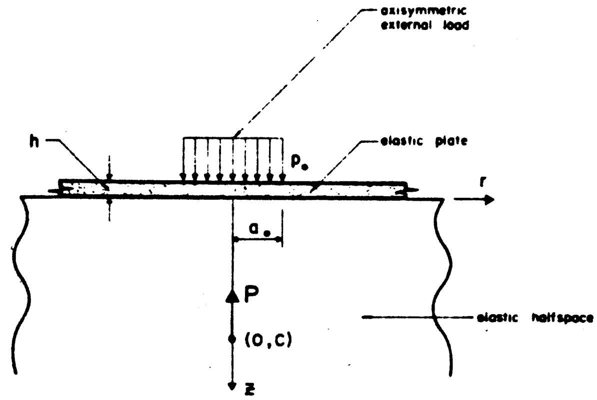

Fig. 1 The geometry of the infinite plate problem

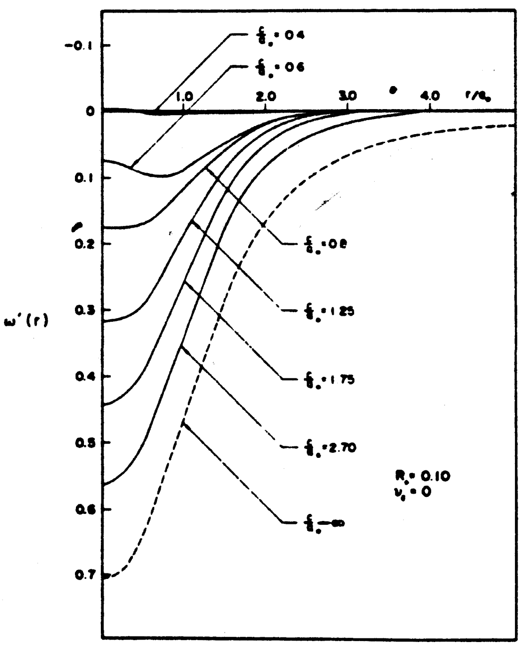

Fig. 2 Dimensionless deflection of the infinite plate

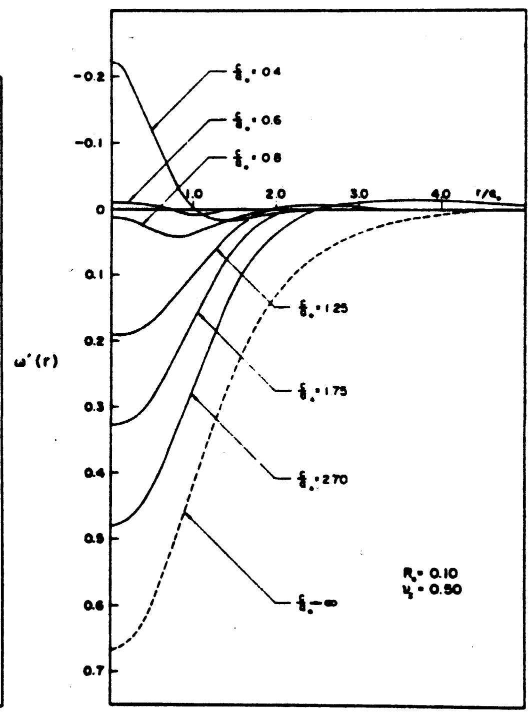

Fig. 3 Dimensionless deflection of the infinite plate

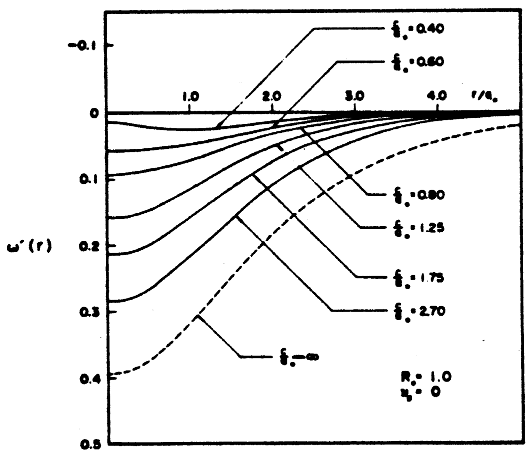

Fig. 4 Dimensionless deflection of the infinite plate

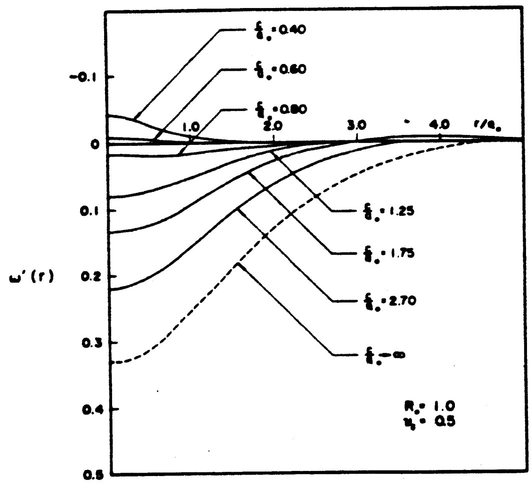

Fig. 5 Dimensionless deflection of the infinite plate

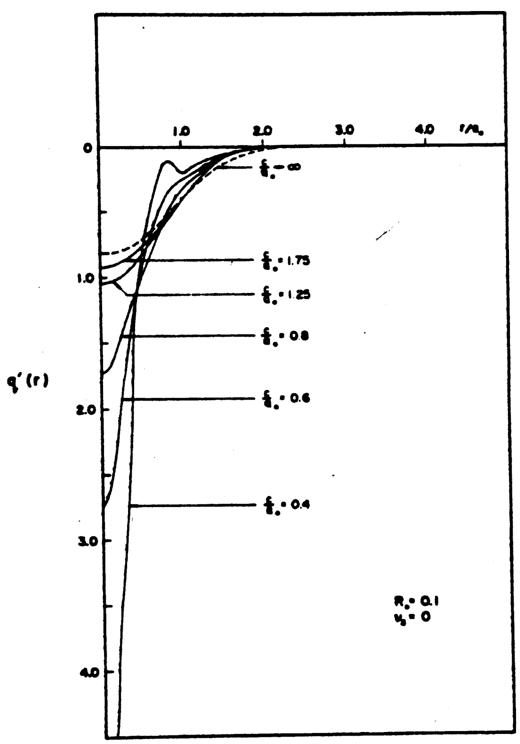

Fig. 6 Dimensionless contact stress distribution

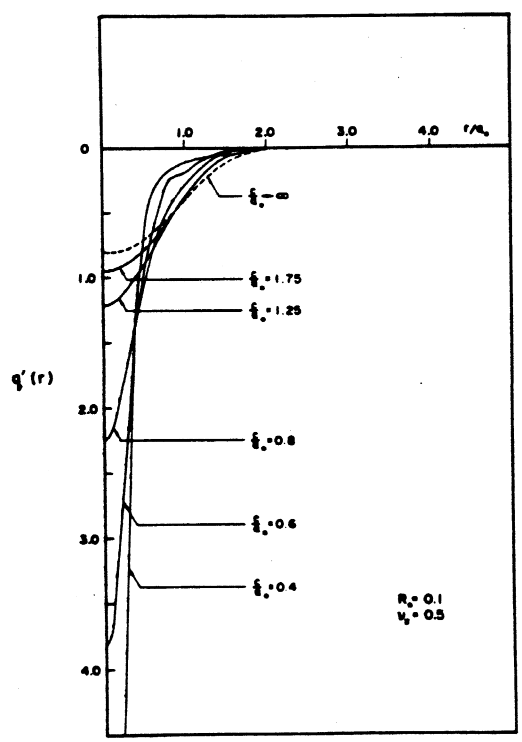

Fig. 7 Dimensionless contact stress distribution

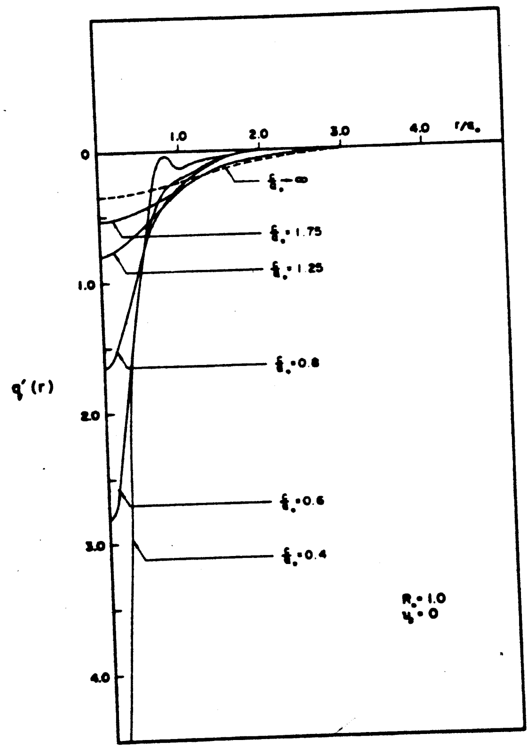

Fig. 8 Dimensionless contact stress distribution

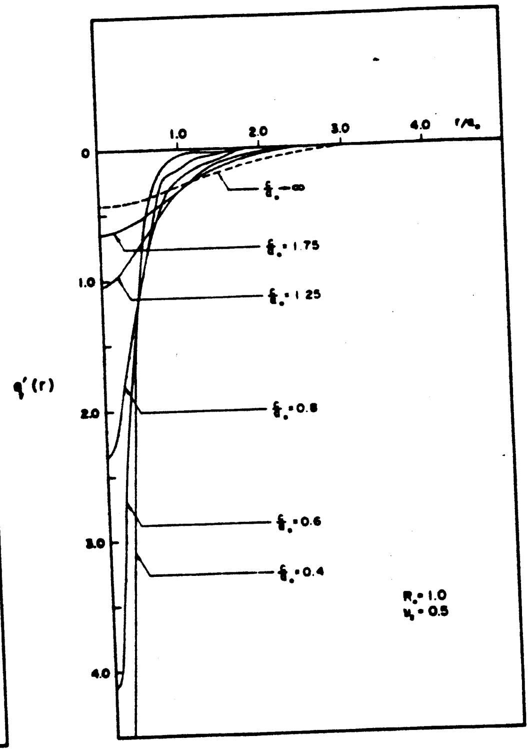

Fig. 9 Dimensionless contact stress distribution