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SEPARATION AT A PRE-FRACTURED BI-MATERIAL GEOLOGICAL INTERFACE

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INTRODUCTION

The paper examines the problem of the separation at a pre-compressed, pre-fractured, bi-material elastic geological interface. The problem is of interest in the study of separation zones which can be created by proppant solids that are used in hydraulic fracturing techniques associated with energy and geothermal resources recovery endeavours. The paper examines the specific situation where the separation is induced by a rigid disc shaped proppant region of finite thickness. The integral equations associated with the mathematical modelling of the problem are simplified by introducing additional constraints on the axial displacements beyond the separation zone.

PROBLEM FORMULATION

We consider the problem where two isotropic elastic geological media are subjected to a pre-compression σ_v normal to a pre-fractured interface. The stress σ_v can be attributed to overburden stresses which maintain a smooth unilateral contact. The region can be subjected to horizontal stresses σ_h as shown in Figure 1 but these stresses will not participate in the ensuing treatments. The assumption of smooth contact at the pre-fractured interface is somewhat artificial. However in view of the fact that fracture fluids may be present at the interface, such an assumption is justifiable. We assume that the pre-fractured, pre-compressed bi-material region is wedged open by a rigid smooth circular disc of radius a and thickness Δ. Referring to Figure 1, the region $a \leq r \leq b$ beyond the disc is assumed to undergo separation and smooth contact is re-established beyond the separation zone.

The disc induced separation problem at the bi-material interface can be posed as a mixed boundary value problem in elasticity, where mixed boundary conditions are prescribed at the interface. We refer the analysis to a system of cylindrical polar coordinates (r, θ, z) and denote by $u_r^{(\alpha)}$, $u_z^{(\alpha)}$, $\sigma_{zz}^{(\alpha)}$ and $\sigma_{rz}^{(\alpha)}$ the relevant displacement and stress components applicable to the two halfspace regions $(\alpha = 1, 2)$.

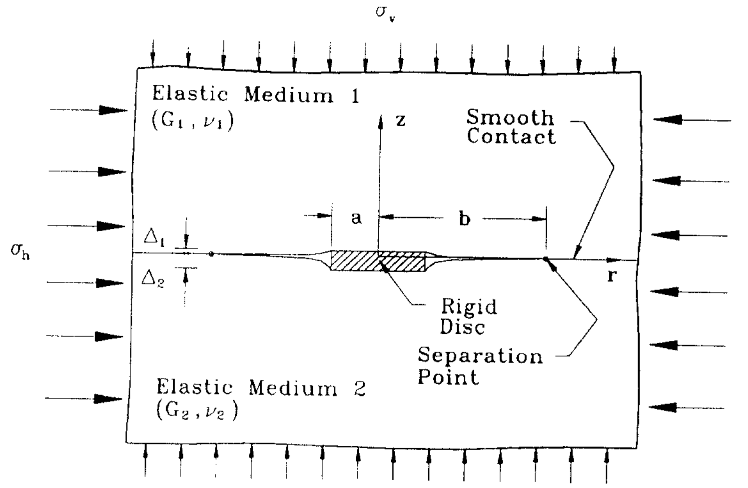

Figure 1. Interface Separation at a Smooth Bi-Material Interface.

Considering the two halfspace regions we have

$$u_z^{(1)}(r, 0) = \Delta_1 \quad ; \quad 0 \le r \le a \tag{1}$$

$$u_z^{(2)}(r, 0) = -\Delta_2 \quad ; \quad 0 \le r \le a \tag{2}$$

$$\sigma_{zz}^{(1)}(r, 0) = 0 \quad ; \quad a < r < b \tag{3}$$

$$\sigma_{zz}^{(2)}(r, 0) = 0 \quad ; \quad a < r < b \tag{4}$$

$$u_z^{(1)}(r, 0) = u_z^{(2)}(r, 0) \quad ; \quad b \le r < \infty \tag{5}$$

$$\sigma_{zz}^{(1)}(r, 0) = \sigma_{zz}^{(2)}(r, 0) \quad ; \quad b < r < \infty \tag{6}$$

$$\sigma_{rz}^{(1)}(r, 0) = \sigma_{rz}^{(2)}(r, 0) \quad ; \quad 0 < r < \infty \tag{7}$$

where Δ_1 and Δ_2 are, respectively, the indentations of the disc in the geological regions 1 and 2 respectively. Also $\Delta_1 + \Delta_2 = \Delta$. In addition to these boundary conditions, the stress field should correspond to σ_v as $|z| \to \infty$ and σ_h as $r \to \infty$. The mixed boundary value problem as posed in (1) - (7) can be solved in a rigorous fashion; however, the objective of this paper is to outline a convenient simplification which results from enforcing the condition that the contact region beyond the zone of separation $(b < r < \infty)$ does not experience any displacement in the axial direction. This is a plausible approximation for situations where the interface is pre-fractured and there is no "fracture" type process initiated at $r = b$, during the indentation. Accordingly, the boundary condition (5) is *replaced* by the constraint

$$u_z^{(1)}(r, 0) = u_z^{(2)}(r, 0) = 0 \quad ; \quad b \le r < \infty \tag{8}$$

The resulting problem is well posed and has a unique solution.

GOVERNING EQUATIONS

For the analysis of the axisymmetric problem, we employ the strain potential approach of Love [1], where in the absence of body forces, the solution of the displacement equations of equilibrium can be expressed in terms of a single function $\varphi^{(\alpha)}(r,z)$ which satisfies

$$\nabla^2\nabla^2\varphi^{(\alpha)}(r,z) = 0 \tag{9}$$

where ∇^2 is Laplace's operator referred to the cylindrical polar coordinate system. The solutions of (9) appropriate to regions 1 and 2 are given by

$$\varphi^{(1)}(r,z) = \int_0^\infty [A(\zeta) + B(\zeta)z]\, e^{-\zeta z} J_0(\zeta r)\, d\zeta \tag{10}$$

$$\varphi^{(2)}(r,z) = \int_0^\infty [C(\zeta) + D(\zeta)z]\, e^{\zeta z} J_0(\zeta r)\, d\zeta \tag{11}$$

where $A(\zeta), \ldots D(\zeta)$ are arbitrary functions to be determined by satisfying appropriate interface conditions. The relevant displacement and stress components are given by

$$2G_\alpha u_z^{(\alpha)} = 2(1-\nu_\alpha)\nabla^2\varphi^{(\alpha)} - \frac{\partial^2\varphi^{(\alpha)}}{\partial z^2} \tag{12}$$

$$\sigma_{zz}^{(\alpha)} = \frac{\partial}{\partial z}\left\{(2-\nu_\alpha)\nabla^2\varphi^{(\alpha)} - \frac{\partial^2\varphi^{(\alpha)}}{\partial z^2}\right\} \tag{13}$$

$$\sigma_{rz}^{(\alpha)} = \frac{\partial}{\partial r}\left\{(1-\nu_\alpha)\nabla^2\varphi^{(\alpha)} - \frac{\partial^2\varphi^{(\alpha)}}{\partial z^2}\right\} \tag{14}$$

THE SEPARATION PROBLEM

The objective of the study is to determine the extent of the separation zone $r = b$. This is achieved by determining the stress intensity factors at the location $r = b+$, associated with the following two *auxiliary* problems.

(i) The first problem considers the internal indentation of a penny-shaped crack of radius b located in an infinite solid with properties consistent with the region 1 and smoothly indented by a rigid disc of radius a and thickness $2\Delta_1$.

The mixed boundary value problem governing the problem takes the form

$$u_z^{(1)}(r,0) = \Delta_1 \quad ; \quad 0 \le r \le a \tag{15}$$

$$u_z^{(1)}(r,0) = 0 \quad ; \quad b \le r < \infty \tag{16}$$

$$\sigma_{zz}^{(1)}(r,0) = 0 \quad ; \quad a < r < b \tag{17}$$

$$\sigma_{rz}^{(1)}(r,0) = 0 \quad ; \quad 0 \le r < \infty \tag{18}$$

(It may be noted that for the region 2, the analogous mixed boundary value problem can be posed in terms of Δ_2).

Employing the results (10) - (14), the mixed boundary conditions (15) - (18) can be reduced to a single Fredholm integral equation of the second kind for an unknown function $\psi(\eta)$ i.e.

$$\psi(\eta) = 1 + \int_0^1 \psi(\omega) K(\omega, \eta) d\omega \quad ; \quad 0 \leq \eta \leq 1 \tag{19}$$

where

$$K(\omega, \eta) = \frac{-2\omega\left(1 - c^2\omega^2\right)^{1/2}}{\pi^2\left(1 - \omega^2\right)^{1/2}}\left[\frac{\Phi(\omega) - \Phi(\eta)}{\omega^2 - \eta^2}\right] \tag{20}$$

$$\Phi(\alpha) = \tau\, ln\left\{\frac{1 + c\tau}{1 - c\tau}\right\} \quad ; \quad \tau = \left[\frac{1 - \alpha^2}{1 - c^2\alpha^2}\right]^{1/2} \quad ; \quad (\alpha = \omega, \eta) \tag{21}$$

and $c = a/b$. The integral equation (19) can be solved [2,3] to obtain a power series approximation for the crack opening mode stress intensity factor at $r = b$, in terms of the parameter c. The resulting solution takes the form

$$\left(K_I^1\right)_{\Delta_1} = \frac{\Delta_1 G_1}{\pi(1 - \nu_1)\sqrt{b}} F_1(c) \tag{22}$$

where $F_1(c)$ can be represented in the following form

$$\begin{aligned} F_1(c) \;=\; & \left[\frac{4c}{\pi} + \frac{16c^2}{\pi^3} + c^3\left\{\frac{64}{\pi^5} + \frac{4}{3\pi}\right\} + c^4\left\{\frac{80}{9\pi^3} + \frac{256}{\pi^7}\right\}\right. \\ + \;& \left. c^5\left\{\frac{448}{9\pi^5} + \frac{1024}{\pi^9} + \frac{4}{5\pi}\right\} + 0\left(c^6\right)\right] \end{aligned} \tag{23}$$

(ii) The second problem considers the evaluation of the stress intensity factor at the outer boundary $r = b$ of an annular crack located in a homogeneous elastic infinite space (region 1) and subjected to uniform *tensile* stress σ_v in the region $r \epsilon (a, b)$. The associated crack opening mode stress intensity factor is denoted by $(K_I^1)_{\sigma_v}$. The solution to this problem was discussed, among others by, Selvadurai and Singh [4].

The mixed boundary value problem governing the annular crack problem is

$$\sigma_{zz}^{(1)}(r, 0) = \sigma_v \quad ; \quad a < r < b \tag{24}$$

$$u_z^{(1)}(r, 0) = 0 \quad ; \quad 0 \leq r \leq a \tag{25}$$

$$u_z^{(1)}(r, 0) = 0 \quad ; \quad b \leq r < \infty \tag{26}$$

$$\sigma_{rz}^{(1)}(r, 0) = 0 \quad ; \quad 0 \leq r < \infty \tag{27}$$

The resulting three-part boundary value problem can be reduced to a set of coupled integral equations which can be solved to generate a power series approximation for the stress intensity factor at the crack tip $r = b$. This result can be evaluated in the form

$$\left(K_I^1\right)_{\sigma_v} = \frac{2\sigma_v\sqrt{b}}{\pi} F_2(c) \tag{28}$$

where

$$\begin{aligned} F_2(c) \;=\; & 1 - \frac{4c}{\pi^2} - \frac{16c^2}{\pi^4} - c^3\left(\frac{1}{8} + \frac{64}{\pi^6}\right) \\ - \;& c^4\left\{\frac{16}{3\pi^4} + \frac{4}{\pi^2}\left(\frac{1}{24} - \frac{8}{9\pi^2} + \frac{64}{\pi^6} + \frac{4}{9\pi^3}\right)\right\} \\ - \;& c^5\left\{\frac{16}{\pi^4}\left(\frac{1}{24} + \frac{64}{\pi^6} - \frac{8}{9\pi^3} + \frac{8}{9\pi^2}\right) + \frac{256}{9\pi^6} - \frac{4}{15\pi^2}\right\} + 0\left(c^6\right) \end{aligned} \tag{29}$$

As is evident $(K_I^1)_{\sigma_v}$ is independent of the elastic constants of the elastic medium. Hence $(K_I^1)_{\sigma_v} = (K_I^2)_{\sigma_v}$. The quantity Δ_1 represents the indentation of the disc into the region 1. It is evident that if an infinite space region 2 is considered $(K_I^1)_{\Delta_2}$ will have a form similar to (22) except that Δ_1, ν_1 and G_1 will be replaced by Δ_2, ν_2 and G_2 respectively. Since $(K_I^1)_{\Delta_1} = (K_I^2)_{\Delta_2} = (K_I)_\Delta$, it can be shown that

$$(K_I)_\Delta = \frac{\Delta G_1 G_2 \, F_2(c)}{\pi \left\{G_1(1-\nu_2) + G_2(1-\nu_1)\right\}\sqrt{b}} \tag{30}$$

The characteristic equation required for the determination of the radius of the zone of separation (b) is given by

$$\left(K_I^{(\alpha)}\right)_\Delta + \left(K_I^{(\alpha)}\right)_{\sigma_v} = 0 \quad ; \quad (\alpha = 1, 2) \tag{31}$$

The result (31) can be solved to obtain the influence of σ_v/G_1, G_2/G_1, Δ/a, ν_1 and ν_2 on the extent of the zone of separation.

NUMERICAL RESULTS AND CONCLUSIONS

The characteristic equation can be rewritten as

$$\left(\frac{\Delta}{a}\right) \frac{\Gamma F_1(c)}{\{1 - \nu_2 + (1-\nu_1)\Gamma\}} - 2\left(\frac{\sigma_v}{G_1}\right)\frac{F_2(c)}{c} = 0 \tag{32}$$

where $\Gamma = G_2/G_1$. Equation (32) can be numerically evaluated to determine b/a. The single positive root of (32) which satisfies $b/a > 1$ constitutes the zone of separation consistent with specified values of (Δ/a), Γ, (σ_v/G_1), ν_1 and ν_2. The Figures 2 and 3 illustrate the influence of the modulus mismatch (Γ) and the normalized pre-compression (σ_v/G_1) on the extent of the separation zone (b/a). It is evident that both σ_v/G_1 and Γ have an important influence on the separation zone.

The methodology presented in this paper employs an *a priori* assumption pertaining to a zero axial displacement field at the contact zone beyond the separation zone. This assumption is a plausible approximation for the study of the separation problem and the "reduced analysis" effectively utilizes solutions to penny-shaped and annular crack problems. The assumption is identically satisfied when the elasticity properties of the halfspace regions are equal.

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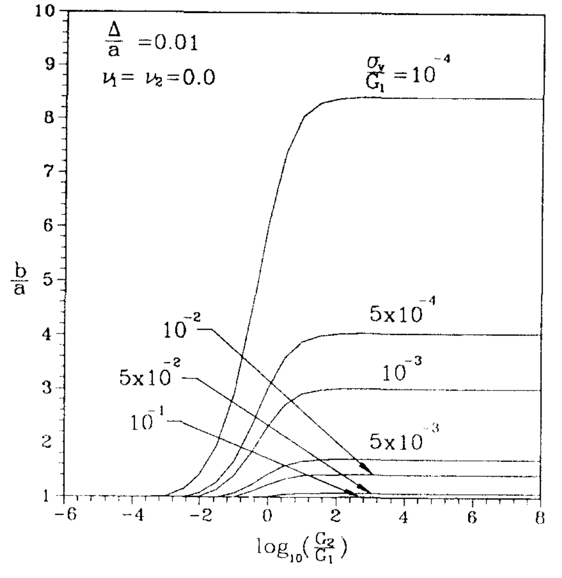

Figure 2. Influence of the pre-compression and elasticity mis-match on the radius of the zone of separation.

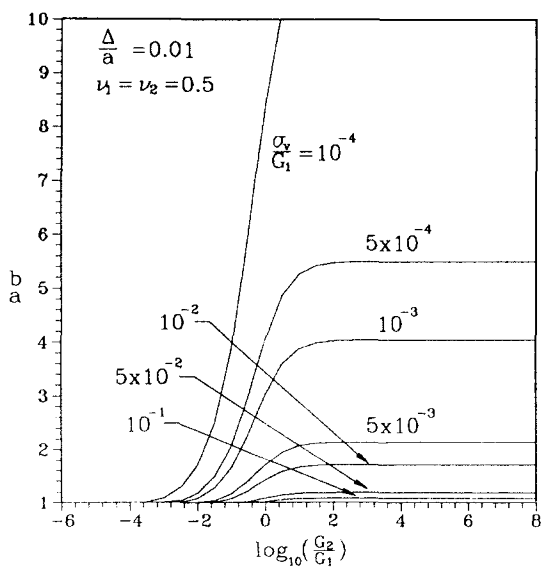

Figure 3. Influence of the pre-compression and elasticity mis-match on the radius of the zone of separation.