Matrix Crack Extension at a Frictionally Constrained Fiber

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The paper presents the application of a boundary element scheme to the study of the behavior of a penny-shaped matrix crack which occurs at an isolated fiber which is frictionally constrained. An incremental technique is used to examine the progression of self similar extension of the matrix crack due to the axial straining of the composite region. The extension of the crack occurs at the attainment of the critical stress intensity factor in the crack opening mode. Iterative techniques are used to determine the extent to crack enlargement and the occurrence of slip and locked regions in the frictional fiber-matrix interface. The studies illustrate the role of fiber-matrix interface friction on the development of stable cracks in such frictionally constrained zones. The methodologies are applied to typical isolated fiber configurations of interest to fragmentation tests.

1 Introduction

The integrity of bond between fibers and the surrounding matrix is of fundamental importance to the development of adequate reinforcing action in fiber reinforced solids. Debonding and cracking at either the matrix or the fibre or at the fiber-matrix interface (Fig. 1) can be initiated by a variety of factors including non-uniformities in the geometry of the fibers, mismatch in the thermo-elastic properties of the fiber-matrix system, dynamic fracture of individual fibers and environmental effects associated with moisture migration and adhesion degradation at the fibre-matrix interface. The conventional approach to the study of micro-mechanics of fiber reinforced solids invariably assumes that there is perfect continuity between the reinforcing fiber and the surrounding matrix (see e.g., Christensen, 1980; Selvadurai, 1981; Mura, 1982; Hashin and Herakovich, 1983; Kelly and Rabotnov, 1983; Dvorak, 1991). An alternative to the continuity of bonding at the fiber-matrix interface assumes that there is complete detachment of the fiber from the matrix. This basically results in a matrix with partially constrained cylindrical voids and the approximation becomes realistic only in situations involving short fibres embedded within a matrix. Problems involving sliding contact and bilateral contact between embedded inclusions and surrounding elastic media have been examined in the literature (see e.g., Ghahremani, 1980; Mura and Furuhashi, 1984; Mura et al. 1985; Selvadurai and Au, 1985; Jasiuk et al., 1987; Selvadurai and Dasgupta, 1990). In reality, the interface conditions are much more complex than the extreme idealizations described previously. These can include nonlinear and dissipative processes including Coulomb friction, finite friction, dilatancy with or without interface degradation and other experimentally derived nonlinear phenomena. These nonlinear phenomena can be coupled with processes such as interface separation and slip. The study of nonlinear processes occurring at the interfaces of fiber reinforced elastic media is a nonroutine exercise. Such processes can occur in a random fashion even in unidirectionally reinforced solids. The complexities associated with fiber geometry and localized nature of the interface processes makes it difficult to develop purely analytical solutions to even relatively straightforward unidirectionally reinforced solids.

Numerical methods based on either finite element methods or boundary element methods offer both accurate and efficient techniques for the investigation of nonlinear processes which can occur at fractured or delaminated fiber-matrix interfaces. The objectives of this paper are twofold; first to examine the influence of fiber-matrix interface nonlinear phenomena on the behavior of both an embedded isolated matrix-fiber crack and a bridged crack and second to illustrate the application of boundary element techniques to the study of micromechanics of nonlinear phenomena at fiber-matrix interfaces. To illustrate the methodology, attention is restricted to the study of two specific problems related to an isolated fiber, where the matrix fracture is accompanied with either fibre fracture or fibre continuity (Figs. 2 and 3).The numerical procedure illustrates the influence of interface nonlinear phenomena on

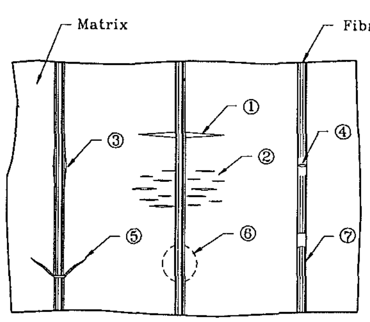

Fig. 1 Micromechanical processes of damage in fiber reinforced solids

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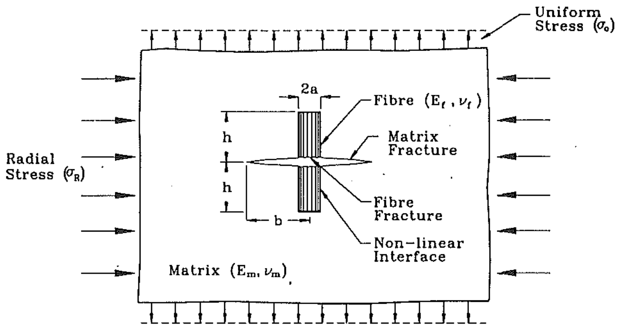

Fig. 2 Frictionally constrained cracked short fiber at a fiber-matrix crack

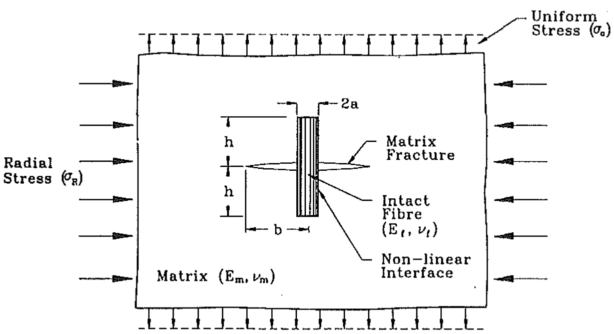

Fig. 3 Frictionally constrained intact short fiber at a matrix crack

the potential for the extension of the penny-shaped matrix crack in a self similar manner. Numerical results presented in the paper for a typical fiber reinforced material illustrate the influence of Coulomb friction at the fiber-matrix interface and fiber continuity on the development of self-similar stable matrix cracking. The results of the studies have applications to the modeling of "Fragmentation Tests" used to examine nonlinear interface effects at a single fibre embedded in a matrix (Berg, 1990).

2 The Boundary Element Method

Considering the nonlinear nature of the matrix-fiber interaction, it is necessary to adopt an incremental formulation of the boundary integral equation. For isotropic bimaterial regions, the axisymmetric form of the boundary element integral equation for the location 'o' takes the form

$$c_{lk}\dot{u}_k^{(\alpha)} + \int_{\Gamma_{(\alpha)}} \{ T_{lk}^{*(\alpha)} \dot{u}_k^{(\alpha)} - u_{lk}^{*(\alpha)} \dot{T}_k^{(\alpha)} \} \frac{r}{r_0}\, d\Gamma = 0 \quad (2.1)$$

where $T_{lk}^{*(\alpha)}$ and $u_{lk}^{*(\alpha)}$ are, respectively, the traction and displacement fundamental solutions (see e.g. Brebbia et al., 1984; Selvadumi and Au, 1985). In (2.1) $c_{lk} = 0$ if the field point in outside the body; $c_{lk} = \delta_{lk}$ if the point is inside the body; $c_{lk} = \delta_{lk}/2$ if the point is located at a smooth boundary and δ_{lk} is Kronecker's delta function. The notation $(\dot{\ })$ refers to an incremental value of the variable concerned and the Greek index (α) refers to either the matrix (m) or fiber (f) regions.

Considering (2.1), the boundaries $\Gamma_{(\alpha)}$ can be discretized into boundary elements and the integral equation can be replaced by its discretized equivalent. For an isoparametric boundary element, the geometric, displacement, and traction variations can be represented in the form

$$[x_i;\ u_i;\ T_i] = \sum_{\gamma=1}^{S} N^{\gamma}(\xi)[x_i^{\gamma};\ u_i^{\gamma};\ T_i^{\gamma}] = N(\xi)[\{x_i\};\ \{u_i\};\ \{T_i\}] \quad (2.2)$$

For a quadratic element, $S = 3$ and the shape functions $N(\xi)$ are given by

$$N^{(1)}(\xi) = \frac{\xi(\xi+1)}{2};\ N^{(2)}(\xi) = (\xi^2 - 1);\ N^{(3)}(\xi) = \frac{\xi(\xi+1)}{2} \quad (2.3)$$

with $-1 \le \xi \le 1$. The discretized form of (2.1) can now be written as

$$c_{lk}\dot{u}_i^{(\alpha)} + \sum_e \int_{-1}^{1} T_{lk}^{*(\alpha)}[N(\xi)]\,|J|\,\frac{r}{r_0}\,d\xi\ \{\dot{u}_k^{(\alpha)}\}^e = \sum_e \int_{-1}^{1} u_{lk}^{*(\alpha)}[N(\xi)]\,|J|\,\frac{r}{r_0}\,d\xi\ \{\dot{T}_k^{(\alpha)}\}^e \quad (2.4)$$

where e is the element number and $|J|$ is the boundary Jacobian matrix, which for an axisymmetric problem reduces to

$$|J| = \left[\left(\frac{\partial r}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 \right]^{1/2} \quad (2.5)$$

The fiber-matrix region can be subjected to the conventional traction and displacement boundary conditions. In addition, the interface between the fibre and the matrix can be subjected to constraints of the type

$$\dot{T}_i = \dot{R}_i + \tilde{K}_{ij}\tilde{\dot{u}}_j \quad (2.6)$$

where $\tilde{\dot{u}}_j$ are the incremental *relative* displacements at an interface, $\dot{R}_i$ are the incremental residual or initial tractions and $\tilde{K}_{ij}$ are stiffness coefficients which are derived through nonlinear constitutive responses at the fiber-matrix interface. The evaluation of $\tilde{K}_{ij}$ for an interface which exhibits Coulomb frictional phenomena will be presented in a subsequent section. From the boundary element discretization (2.4), we can formulate a boundary element matrix equation of the form

$$[\mathbf{H}]\{\dot{\mathbf{u}}\} = [\mathbf{G}]\{\dot{\mathbf{T}}\} \quad (2.7)$$

where $[\mathbf{H}]$ and $[\mathbf{G}]$ are the boundary element influence coefficients matrices which are obtained by an integration of the fundamental solutions $u_{ij}^{*(\alpha)}$ and $T_{ij}^{*(\alpha)}$ appropriate to the fiber and matrix regions.

3 Interface Behavior

The mechanical response of the fiber matrix interface is influenced by the material and surface characteristics of the fiber, the mechanical properties of the matrix including deformability and fracture phenomena, surface agents used to enhance bond action, and the topography of the delaminated surface. The interface processes can include friction, slip, separation, yield, dilatancy, asperity degradation etc., which can be described by generalized elasto-plastic interface constitutive assumptions. For completeness of the presentation we assume that the incremental relative displacements $\tilde{\dot{u}}_i$ at an interface are composed of elastic and plastic components $\tilde{\dot{u}}_i^{(e)}$ and $\tilde{\dot{u}}_i^{(p)}$, respectively, i.e.,

$$\tilde{\dot{u}}_i = \tilde{\dot{u}}_i^{(e)} + \tilde{\dot{u}}_i^{(p)} \quad (3.1)$$

The elastic component of the increment relative displacement is related to the incremental interface tractions by the linear interface constitutive relationship

$$\dot{T}_i = K_{ij}^{(e)}\tilde{\dot{u}}_j^{(e)} \quad (3.2)$$

where $K_{ij}^{(e)}$ are the linear elastic stiffness coefficients of the interface. In order to establish the irreversible components of (3.1) it is necessary to define the stress level at which yielding occurs at the interface. For example, for an interface which exhibits Coulomb friction, the yield function F is given by

$$F = (T_t T_t)^{1/2} - \mu T_n = 0 \quad (3.3)$$

where μ is the coefficient of friction at the interface and T_n and T_t are, respectively, the components of the total traction at the interface in the normal and tangential directions. At the limit (3.3), the interface will slip and irreversible slip displacements can occur. These can be obtained from a flow/slip rule identical that used in the classical theory of plasticity, i.e.,

$$\tilde{\dot{u}}_i^{(p)} = \dot{\lambda}\,\frac{\partial \Phi}{\partial T_i} \tag{3.4}$$

where $\dot{\lambda}$ is a proportionality factor defined as the plastic/slip multiplier, and Φ is the plastic/slip potential. For an interface with Coulomb friction

$$\Phi(T_i) = T_t \tag{3.5}$$

From (3.2) and (3.5) we have

$$\dot{T}_i = K_{ij}^{(e)} \left[\tilde{\dot{u}}_j - \dot{\lambda}\,\frac{\partial \Phi}{\partial T_j} \right] \tag{3.6}$$

where for problems with axial symmetry, $i, j = r, z$. When slip occurs at the interface between the inclusion and the elastic medium

$$\frac{\partial F}{\partial T_i}\, dT_i = 0 \tag{3.7}$$

and

$$\dot{\lambda} = \frac{1}{\Omega}\,\frac{\partial F}{\partial T_i}\, K_{ij}^{(e)}\, \tilde{\dot{u}}_j; \quad (i, j = r, z) \tag{3.8}$$

$$\Omega = \frac{\partial F}{\partial T_l}\, K_{lm}^{(e)}\, \frac{\partial \Phi}{\partial T_m}; \; (i, j = r, z) \tag{3.9}$$

Using (3.8) in (3.7) we obtain the elasto-plastic interface constitutive relationship as

$$\dot{T}_i = K_{ij}^{(ep)}\, \tilde{\dot{u}}_j \tag{3.10}$$

where

$$K_{ij}^{(ep)} = K_{ij}^{(e)} - \frac{1}{\Omega}\,\frac{\partial \Phi}{\partial T_l}\, K_{il}^{(e)} K_{mj}^{(e)}\, \frac{\partial F}{\partial T_m} \tag{3.11}$$

Therefore, as in conventional numerical implementation of classical associative or nonassociative plasticity phenomena, once the yield function F and the slip potential Φ are known, it is possible to define $K_{ij}^{(ep)}$.

4 Modeling of Crack Tip Behavior

In the boundary element discretization discussed previously, quadratic elements will be employed to model the boundaries of the matrix and fiber regions. That is, the variation of the displacements and tractions within an element can be described by

$$\left. \begin{array}{l} u_i^{(\alpha)} \\ T_i^{(\alpha)} \end{array} \right\} = \sum_{n=0}^{2} a_n \zeta^n \tag{4.1}$$

where ζ is the local coordinate of the element and a_n are constants of interpolation. However, in the context of linear elastic fracture mechanics, the stress field at the crack tip located in a homogeneous solid should incorporate a $1/\sqrt{r}$ type singularity. Cruse and Wilson (1977) proposed a so-called singular traction quarter-point boundary element where the displacement and traction variations can be expressed in the forms

$$u_i^{(\alpha)} = \sum_{n=0}^{2} b_n r^{n/2} \tag{4.2}$$

$$T_i^{(\alpha)} = \sum_{n=0}^{2} c_n r^{(n-1)/2} \tag{4.3}$$

where b_i and c_i are constants. The performance of the quarter point element has been extensively studied and documented (Blandford et al., 1981; Smith and Mason, 1982; Selvadurai and Au, 1987, 1988, 1989; Selvadurai, 1991). In the crack-fibre interaction problem examined in this paper, the axial straining induces a state of axial symmetry in the fibre-matrix composite region. Consequently, in general, only the Mode I and Mode II stress intensity factors are present at the tips of a penny-shaped crack region. The incremental values of the stress intensity factors are given by

$$\dot{K}_I = \frac{G_\alpha}{(k_\alpha + 1)} \sqrt{\frac{2\pi}{l_0}}\, \{4[\dot{u}_y(B) - \dot{u}_y(D)] + \dot{u}_y(E) - \dot{u}_y(A)\} \tag{4.4}$$

$$\dot{K}_{II} = \frac{G_\alpha}{(k_\alpha + 1)} \sqrt{\frac{2\pi}{l_0}}\, \{[\dot{u}_x(B) - \dot{u}_x(D)] + \dot{u}_x(E) - \dot{u}_x(A)\} \tag{4.5}$$

where l_0 is the length of the crack tip element and $k_\alpha = (3 - 4\nu_\alpha)$.

5 The Fiber-Crack Interaction Problem

We apply the basic procedures outlined in the preceding sections to the study of problems associated with an isolated fibre of finite length which is embedded in an isotropic elastic matrix of infinite extent. We focus attention on two specific problems in which a penny-shaped crack is initiated in the matrix region. The plane of the crack is normal to the axis of the fibre and the axis of the fiber is assumed to coincide with the axis of the penny-shaped crack. Also for the purposes of illustration; the matrix crack is assumed to occur at the mid-section of the fiber. Two specific problems are examined to assess the influence of the fiber-matrix interface frictional phenomena on the performance of penny-shaped matrix cracks. The first (Fig. 2) considers the problem where fibre fracture has also occured at the matrix crack location. In the second problem (Fig. 3), fiber continuity exists in the presence of the matrix crack. The cylindrical surface corresponding to the fibre-matrix interface is debonded but exhibits Coulomb frictional phenomena which are characterized by a friction coefficient μ. The composite region is subjected to a far field radial stress σ_R as indicated in Figs. 2 and 3. It is assumed that this radial stress is directly transmitted to the debonded fiber-matrix interface. This is admittedly an approximation which can be eliminated by incrementally applying the radial stress to its specified value σ_R. In such a case regions of frictional slip and locking could occur at the interface depending upon the value of μ and the elasticity mismatch between the fibre and the matrix. Alternatively, it could be assumed that σ_R represents stresses associated with the shrinkage of the matrix during curing (Busschen, 1991). For the purposes of this paper we assume that the uniform normal stress at the interface is σ_R. In the presence of the radial stress, the composite region is subjected to an axial stress σ_0, in an incremental fashion.

The computational modeling first focusses on the examination of the influence of fiber-matrix interface friction and fibre continuity across the matrix crack on the crack opening mode stress intensity factor at the matrix crack. In this instance crack extension phenomena are not considered. The objective of the analysis is to establish the relative influences of crack bridging by the fibre and the coefficient of friction μ, on the stress intensity factor at the tip of the matrix crack. The second aspect of the computational modeling focusses on the examination of influences of interface friction and fibre continuity on the extension of the matrix crack. In order to conduct such an analysis, we assume the existence of a starter crack within the matrix region. In the case of an intact fibre, the annular matrix crack surrounding the fibre is assumed to have an outer radius of $r^* = 1.01a$ where a is the radius of the fibre. With a cracked fiber, the penny-shaped "fibre-matrix crack" is assumed to occupy an incremental starter crack configuration with a crack radius of $r^* = 1.01a$. The crack is assumed to extend beyond r^* at the application of an incremental stress $\dot{\sigma}_0$. The analysis is performed in an iterative manner to ascertain the regions of the fiber-matrix interface which exhibits frictional slip at the attainment of the failure condition (3.3). The crack opening mode stress intensity factor at the matrix crack

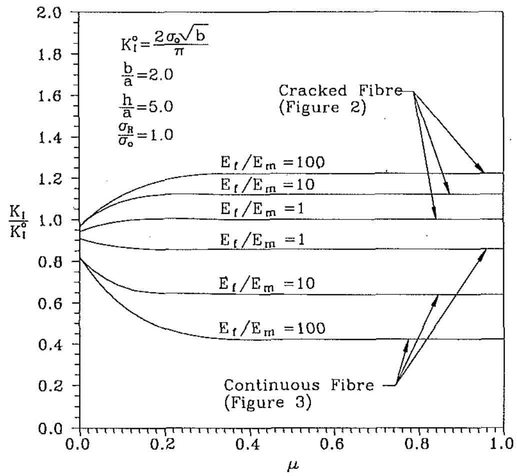

Fig. 4 Influence of interface friction and fiber continuity on the amplification or attenuation of K_I

boundary is also evaluated and the crack is allowed to extend quasi-statically to a new location $r = \hat{r}$ when the stress intensity factor at the crack tip satisfies the condition

$$K_I^{(m)} = K_{IC}^{(m)} \tag{5.1}$$

For a given increment of $\dot{\sigma}_0$, the iterations are performed to determine values of $\hat{r}^*/a$ for which the crack extension ceases and stable regions of the fibre-matrix interface which experience frictional slip are identified. The procedure is repeated with the application of a further increment of axial stress with the application of a final level of stress σ_0, the stable location of the tip penny-shaped crack b/a for which $[K_I^{(m)}]_{r=b} < K_{IC}^{(m)}$ can be identified. In the current investigations it is explicitly assumed that the crack in the matrix region extends in a self-similar fashion. More advanced analyses can be made to examine situations where multiple conical and penny-shaped starter cracks can extend into the matrix region.

6 Numerical Results

For the purposes of presentation of appropriate numerical results, attention is focussed on a specific isolated fibre-matrix composite region which consists of an E-Glass fibre of finite length embedded in a polyester matrix of infinite extent. The elastic properties of the fiber and the matrix are as follows (see e.g., Busschen, 1991).

E-Glass Fiber:

Elastic modulus	(E_f)	$=70$ GPa
Poisson's ratio	(ν_f)	$=0.20$
Tensile strength	(σ_T^f)	$=2.5$ GPa
Diameter of fiber	$(2a)$	$=15\ \mu$m

Polyester Matrix:

Elastic modulus	(E_m)	$=1.5$ GPa
Poisson's ratio	(μ_m)	$=0.35$
Tensile strength	(σ_T^m)	$=87$ MPa
Critical stress intensity factor	(K_{IC}^m)	$=1$ MPa $\sqrt{m}$

The coefficient of friction (μ) at the fiber-matrix interface is assumed to vary between 0 to 1.0, which represents normal stress independent friction angles of 0 deg and 45 deg, respectively.

The first set of numerical computations deals with a general evaluation of the mode I stress intensity factor at the tip of a stable matrix crack. The factors influencing the problem are the following:

(a) geometric aspect ratio of the fibre $(h/a) = 5.0$
(b) matrix crack radius-fibre radius ratio $(b/a) = 2.0$
(c) radial stress to axial stress ratio $(\sigma_R/\sigma_0) = 1.0$
(c) coefficient of fibre-matrix interface friction $(\mu) \in (0,1)$
(e) fibre-matrix modular ratio $(E_f/E_m) = (1,10,10^2)$

Figure 4 illustrates the manner in which the stress intensity factor at the tip of the matrix crack is influenced by the fibre-matrix modular ratio and the coefficient of friction at the fibre-matrix interface. The mode I stress intensity factor at the matrix crack in the composite is normalized with respect to the mode-stress intensity factor at the tip of a penny-shaped crack of radius b located in a homogeneous matrix (Sneddon, 1946), i.e.,

$$K_I^0 = \frac{2\sigma_0\sqrt{b}}{\pi} \tag{6.1}$$

Figure 4 presents results for the two situations where the fiber either exhibits continuity across the faces of the crack or is cracked at the plane of the matrix crack. The results of the computations, albeit for a specific fiber-matrix system, indicate that the interface friction, the fiber-matrix modular ratio and either the presence or absence of fiber continuity have important influences on the amplification or attenuation of the mode I stress intensity at the matrix crack.

The analysis is now extended to the consideration of the matrix crack extension due to the attainment of the crack extension criterion (5.1). In this case, the computations are also specifically related to the examination of the E-Glass fibre-polyester matrix system described previously. Figures 5 and 6 illustrate the influence of σ_0/σ_R and the fiber-matrix interface friction coefficient μ on the dimensions of the stable matrix crack b/a. As is evident, the extent of stable crack development is considerably influenced by the relative magnitude of the axial stress and the presence or absence of fibre continuity across the faces of the crack.

Conclusions

Classical studies of inclusion problems invariably consider idealizations of interface characteristics between the embedded inclusion and the surrounding elastic medium. The limiting cases of either perfect continuity or sliding conditions are useful models which provide bounds for the relevant results pertaining to embedded inclusion problems. In fiber reinforced materials, the interface between reinforcing fibers and the surrounding matrix can exhibit frictional characteristics particularly at regions of the composite which experience interface debonding. This paper examines a specific problem in which a crack extends within the matrix at a detached frictional interface location. The boundary element technique can be effectively applied to examine the self-similar extension of a matrix crack in the presence of Coulomb friction at the interface. The procedure can be used to examine the dimensions of stable matrix cracks that could exist at debonded but frictionally constrained interface regions. The analysis indicates that the presence of a nominal amount of friction and confining stress is sufficient to influence the matrix crack extension process. As the frictional constraint increases, the crack extension is suppressed in instances where the fiber exhibits continuity across the faces of the crack. When the fiber is cracked, the frictional constraint at the interface tends to amplify the stress transfer which accentuates matrix crack extension. This observation is expected to be generally true in most situations where the fibre inclusion has a higher elastic stiffness than the surrounding matrix. The boundary element technique is advocated as an efficient method for the study of this class of micro-mechanics problem. The efficiency of the method rests on the a priori identification of the interface on which the irreversible phenomena occur in the form the frictional effects. The iterative procedures can then

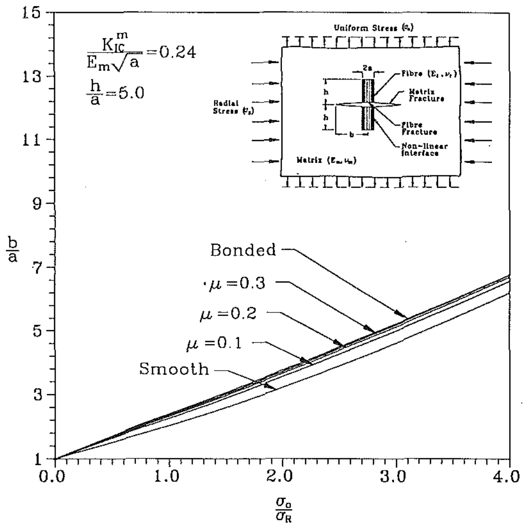

Fig. 5 Influence of interface friction on the extent of stable crack development in a E-glass-polyester system (cracked fiber at the matrix crack)

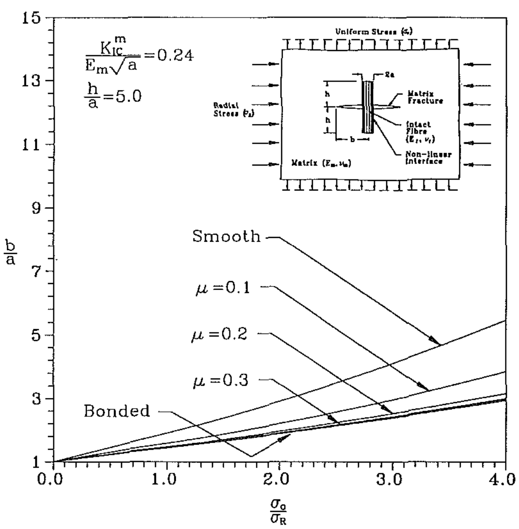

Fig. 6 Influence of interface friction on the extent of stable crack development in a E-glass-polyester system (continuous fiber at the matrix crack)

be effectively employed for the identification of both the location of slip (and frictional locking) and quasi-static crack extension under applied axial strain. The methodologies discussed in the paper can be extended to cover other forms of interface characteristics including dilatancy and degradation. Such effects can have an important influence on the modeling of micro crack extension under quasi-static load cycling.

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