A UNILATERAL CONTACT PROBLEM FOR A RIGID DISC INCLUSION EMBEDDED BETWEEN TWO DISSIMILAR ELASTIC HALF-SPACES

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SUMMARY

The present paper examines the axisymmetric elasto-static problem where the contact between two smoothly compressed dissimilar elastic half-spaces is perturbed by a smooth disc inclusion of finite thickness. The paper develops the governing integral equation and the additional constraint which is associated with the separation condition. The analysis focuses on the estimation of the influence of the precompression and the elasticity mismatch between the bimaterial half-space regions on the radius of the separation zone. The paper also records a useful analytical approximation to the unilateral contact problem which is derived by prescribing a priori the displacements in the contact region beyond the separation zone.

1. Introduction

UNILATERAL contact problems in the classical theory of elasticity usually deal with the development of either advancing or receding contact regions between elastic bodies in smooth contact. Comprehensive accounts of developments in this area are given by Dundurs and Stippes (**1**), de Pater and Kalker (**2**), Duvaut and Lions (**3**) and Gladwell (**4**). A number of such studies have been examined in the literature on contact problems dealing with elastic layers, elastic half-spaces, etc., where the regions are subjected to localized concentrated forces, rigid indenters and gravity loads. References to these studies are given by Selvadurai (**5**). Non-classical problems with unilateral constraints consider the influence of frictional effects on the interfaces which can exhibit separation and slip (**6**; **7**, Chapter 5, pp. 112–128; **8**; **9**). The study of unilateral contact between structural elements such as beams and plates and elastic media has important applications in structural mechanics and geomechanics (**4**, **10** to **13**).

A paper by Gladwell and Hara (**14**) is of particular importance to the basic theme of this paper. They examined the problem of a smooth precompressed bimaterial elastic interface which is subjected to separation by an axisymmetric rigid inclusion with an oblate spheroidal shape non-symmetric about the plane $z = 0$. The solution procedure is based on the analysis of annular problems which uses oblate spheroidal coordinates, described by Gladwell and Gupta (**15**). Implicit in the analysis of Gladwell and Hara (**14**) is the conventional Hertzian assumption which implies that in the local contact region the displacements induced by the oblate spheroidal obstacle, which has a finite geometry, can be imposed as boundary conditions applicable to the undeformed surface of a half-space region. These authors also present numerical results for the case of a symmetric obstacle compressed between two dissimilar half-spaces.

In this paper we examine a related problem pertaining to the separation at a precompressed smooth bimaterial elastic interface, which is induced by a rigid disc-shaped inclusion. The problem can be visualized as either the situation where the smooth disc inclusion is compressed between the smooth surfaces of two dissimilar half-space regions or as the situation where the smooth disc inclusion is inserted at a compressed smooth interface. It may be noted that since both the disc inclusion and the interface are smooth, the extent of the zone of separation is unaffected by the mode of introduction of the separation zone. In the formulation adopted in this paper, it is assumed that the smooth interface is first subjected to the compressive stress field, and the introduction of the disc inclusion at the interface results in a separation zone which is traction free. This problem is if some peripheral interest to the examination of the extent of separation induced by a *proppant* (a granular porous material) injected at a prefractured resource-bearing geological formation. The idealization of a proppant region as a rigid disc inclusion is an extreme simplification. Nonetheless the analysis of the problem is intended to serve as a useful and convenient approximation to be employed in preliminary studies. The methodology employed in the paper utilizes a Hankel-transform development of the governing mixed-boundary-value problem associated with displacements that are prescribed at the inclusion region, the traction-free conditions at the separation zone and the continuity conditions for normal tractions and displacements at the zone of smooth contact beyond the separation region. When the dimensions (thickness to radius ratio) of the inclusion region are specified, the single unknown in the problem corresponds to the radius of the separation zone. The single governing integral equation together with the constraint necessary for the determination of the zone of separation are solved numerically to generate results of interest to engineering applications.

2. Basic equations

For the analysis of the axisymmetric problem related to separation at the precompressed interface we employ the strain-potential approach proposed by Love (**16**). In the absence of body forces, the solution of the displacement equations of equilibrium can be represented in terms of a biharmonic function $\Phi^{(\alpha)}(r, z)$, that is,

$$\nabla^2\nabla^2\Phi^{(\alpha)}(r, z) = 0, \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

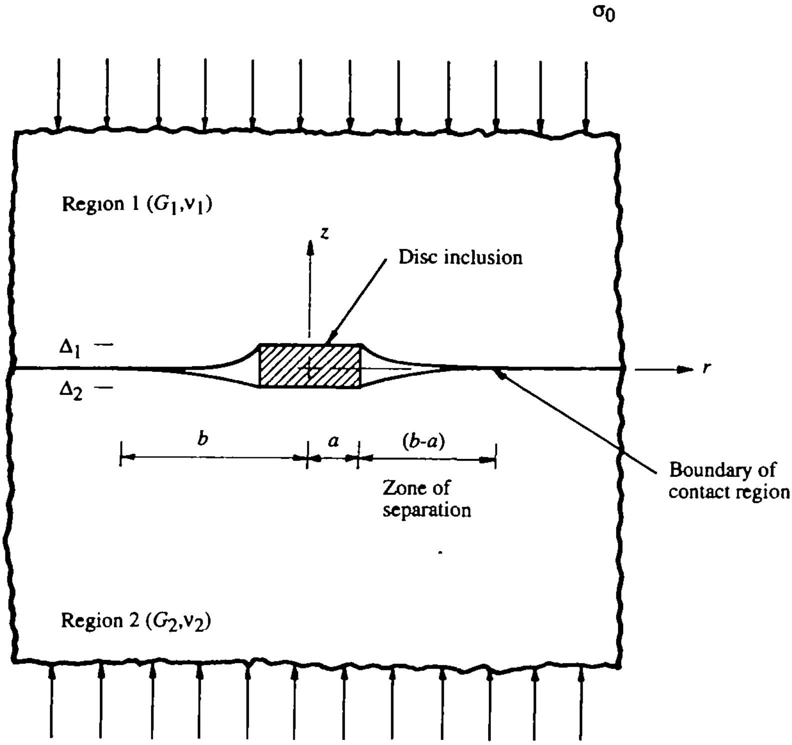

FIG. 1. The disc inclusion embedded between two dissimilar elastic half-spaces

is the axisymmetric form of Laplace's operator referred to the cylindrical polar-coordinate system. The superscript or subscript α takes the value 1 or 2 corresponding to the half-space region 1 where $z \in (0, \infty)$ and 2 where $z \in (0, -\infty)$ respectively (Fig. 1). The components of the displacement vectors $\mathbf{u}^{(\alpha)}$ and the Cauchy stress tensor $\boldsymbol{\sigma}^{(\alpha)}$ referred to the cylindrical polar-coordinate system can be expressed in terms of the derivatives of $\Phi^{(\alpha)}$. We have

$$2G_\alpha u_r^{(\alpha)} = \frac{\partial^2 \Phi^{(\alpha)}}{\partial r\, \partial z}, \tag{3}$$

$$2G_\alpha u_z^{(\alpha)} = 2(1 - \nu_\alpha)\nabla^2 \Phi^{(\alpha)} - \frac{\partial^2 \Phi^{(\alpha)}}{\partial z^2}, \tag{4}$$

where G_α and ν_α are the linear elastic-shear moduli and Poisson's ratios respectively. Similarly, the components of the stress tensor are given by

$$\sigma_{rr}^{(\alpha)} = \frac{\partial}{\partial z}\left\{\nu_\alpha \nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\Phi^{(\alpha)}, \tag{5}$$

$$\sigma_{\theta\theta}^{(\alpha)} = \frac{\partial}{\partial z}\left\{\nu_\alpha \nabla^2 - \frac{1}{r}\frac{\partial^2}{\partial r}\right\}\Phi^{(\alpha)}, \tag{6}$$

$$\sigma_{zz}^{(\alpha)} = \frac{\partial}{\partial z}\left\{(2-\nu_\alpha)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\Phi^{(\alpha)}, \tag{7}$$

$$\sigma_{rz}^{(\alpha)} = \frac{\partial}{\partial r}\left\{(1-\nu_\alpha)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\Phi^{(\alpha)}. \tag{8}$$

The solutions of (1) appropriate for the half-space regions 1 and 2 should satisfy regularity conditions pertaining to stresses and displacements at infinity. A Hankel-transform development of (1) yields the following solutions for $\Phi^{(\alpha)}$:

$$\Phi^{(1)}(r, z) = \int_0^\infty \xi[A(\xi) + zB(\xi)]e^{-\xi z}J_0(\xi r)\,d\xi, \qquad z \in (0, \infty), \tag{9}$$

$$\Phi^{(2)}(r, z) = \int_0^\infty \xi[C(\xi) + zD(\xi)]e^{\xi z}J_0(\xi r)\,d\xi, \qquad z \in (0, -\infty), \tag{10}$$

where $A(\xi), \ldots, D(\xi)$ are arbitrary functions to be determined by satisfying the displacement and traction boundary conditions applicable to the smooth interface between the bimaterial regions.

3. The embedded inclusion problem

We refer to the problem illustrated in Fig. 1, where the smooth contact between two dissimilar half-spaces, with precompression stress of σ_0, is perturbed by a rigid circular disc inclusion of thickness Δ and radius a. The radius of the zone of separation is denoted by b. The mixed boundary conditions at the interface region are the following:

$$u_z^{(1)}(r, 0) = \Delta_1, \qquad 0 \leqslant r \leqslant a, \tag{11}$$

$$\sigma_{zz}^{(1)}(r, 0) = \sigma_0, \qquad a < r < b, \tag{12}$$

$$\sigma_{rz}^{(1)}(r, 0) = \sigma_{rz}^{(2)}(r, 0) = 0, \qquad 0 \leqslant r < \infty, \tag{13}$$

$$u_z^{(2)}(r, 0) = -\Delta_2, \qquad 0 \leqslant r \leqslant a, \tag{14}$$

$$\sigma_{zz}^{(2)}(r, 0) = \sigma_0, \qquad a < r < b, \tag{15}$$

$$u_z^{(1)}(r, 0) = u_z^{(2)}(r, 0), \qquad b \leqslant r < \infty, \tag{16}$$

$$\sigma_{zz}^{(1)}(r, 0) = \sigma_{zz}^{(2)}(r, 0), \qquad b < r < \infty, \tag{17}$$

where Δ_1 and Δ_2 are the indentations of disc inclusion into the half-space regions 1 and 2 respectively and

$$\Delta_1 + \Delta_2 = \Delta. \tag{18}$$

The strain potentials (9) and (10) together with the expressions (3) to (8) can be used to develop integral relationships corresponding to the interface conditions (11) to (17). Avoiding details we have

$$H_0[B(\xi); r] = -\frac{\Delta_1 G_1}{(1-\nu_1)}, \qquad 0 \leqslant r \leqslant a, \tag{19}$$

$$H_0[\xi B(\xi); r] = \sigma_0, \qquad a < r < b, \tag{20}$$

$$H_0[D(\xi); r] = \frac{-\Delta_2 G_2}{(1-\nu_2)}, \qquad 0 \leqslant r \leqslant a, \tag{21}$$

$$H_0[\xi D(\xi); r] = \sigma_0, \qquad a < r < b, \tag{22}$$

$$H_0\big[\{G_2(1-\nu_1)B(\xi) + G_1(1-\nu_2)D(\xi)\}; r\big] = 0, \qquad b \leqslant r < \infty, \tag{23}$$

$$H_0\big[\{\xi B(\xi) - \xi D(\xi)\}; r\big] = 0, \qquad b < r < \infty, \tag{24}$$

where the Hankel transform of order zero is given by

$$H_0[F(\xi); r] = \int_0^\infty \xi F(\xi) J_0(\xi r)\, d\xi. \tag{25}$$

Considering (20) we assume that

$$H_0[\xi B(\xi); r] = \begin{cases} f_1(r), & 0 < r < a, \\ f_3(r), & b < r < \infty. \end{cases} \tag{26}$$

Using the Hankel inversion theorem we have

$$\xi B(\xi) = \int_0^a r f_1(r) J_0(\xi r)\, dr + \sigma_0 \int_a^b r J_0(\xi r)\, dr + \int_b^\infty r f_3(r) J_0(\xi r)\, dr. \tag{27}$$

Substituting this result in (19) and (23) and using the results given by Selvadurai and Singh (17), we can obtain the following integral equations:

$$F_1(r) + \frac{2}{\pi}\int_b^\infty \frac{s F_3(s)\, ds}{(s^2 - r^2)} = -\frac{\Delta_1 G_1}{(1-\nu_1)} - \sigma_0\big\{(b^2 - r^2)^{\frac{1}{2}} - (a^2 - r^2)^{\frac{1}{2}}\big\}, \qquad 0 < r < a, \tag{28}$$

$$F_3(r) + \frac{2r}{\pi}\int_0^a \frac{F_1(s)\, ds}{(r^2 - s^2)} = \sigma_0\big\{(r^2 - b^2)^{\frac{1}{2}} - (r^2 - a^2)^{\frac{1}{2}}\big\} - \alpha \int_0^\infty \xi D(\xi) \sin(\xi r)\, d\xi, \qquad b < r < \infty, \tag{29}$$

where

$$F_1(r) = \int_r^a \frac{s f_1(s)\, ds}{(s^2 - r^2)^{\frac{1}{2}}}, \qquad 0 < r < a, \tag{30}$$

$$F_3(r) = \int_b^r \frac{s f_3(s)\, ds}{(r^2 - s^2)^{\frac{1}{2}}}, \qquad b < r < \infty \tag{31}$$

and

$$\alpha = \frac{G_1(1-\nu_2)}{G_2(1-\nu_1)}. \tag{32}$$

We note that the equations (30) and (31) are integral equations of the Abel type which give the following:

$$sf_1(s) = -\frac{2}{\pi}\frac{d}{ds}\int_s^a \frac{rF_1(r)\,dr}{(r^2-s^2)^{\frac{1}{2}}}, \qquad 0 < s < a, \tag{33}$$

$$sf_3(s) = \frac{2}{\pi}\frac{d}{ds}\int_b^s \frac{rF_3(r)\,dr}{(s^2-r^2)^{\frac{1}{2}}}, \qquad b < s < \infty. \tag{34}$$

Substituting (33) and (34) in (27) we have

$$\xi B(\xi) = \frac{2}{\pi}\left[\int_0^a F_1(s)\cos(\xi s)\,ds + \frac{\sigma_0\pi}{2}\int_a^b rJ_0(\xi r)\,dr + \int_b^\infty F_3(s)\sin(\xi s)\,ds\right]. \tag{35}$$

Considering (22) and (24) we assume that

$$H_0[\xi D(\xi); r] = f_2(r), \qquad 0 < r < a. \tag{36}$$

Using the Hankel inversion theorem, the value of $f_2(r)$, and the integral equation (21), we can show that

$$\int_0^r \frac{ds}{(r^2-s^2)^{\frac{1}{2}}}\int_s^a \frac{uf_2(u)\,ds}{(u^2-s^2)^{\frac{1}{2}}} + \frac{\pi\sigma_0}{2}\int_a^b u\,du\int_0^\infty J_0(\xi u)J_0(\xi r)\,d\xi$$
$$+\int_b^\infty \frac{F_3(s)\,ds}{(s^2-r^2)^{\frac{1}{2}}} = \frac{\pi\Delta_2 G_2}{2(1-\nu_2)}, \qquad 0 < r < a. \tag{37}$$

Introducing the function $F_2(s)$ such that

$$\int_s^a \frac{uf_2(u)\,du}{(u^2-s^2)^{\frac{1}{2}}} = F_2(s), \qquad 0 < s < a \tag{38}$$

the integral equation (37) can be written as

$$F_2(s) = \frac{\Delta_2 G_2}{(1-\nu_2)} - \sigma_0\left\{(b^2-s^2)^{\frac{1}{2}} - (a^2-s^2)^{\frac{1}{2}}\right\}$$
$$-\frac{2}{\pi}\int_b^\infty \frac{uF_3(u)\,du}{(u^2-s^2)}, \qquad 0 < s < a. \tag{39}$$

Also, by using the result

$$\int_0^\infty \xi D(\xi)\sin(\xi r)\,d\xi = \frac{1}{\pi}\left[r\int_0^a \frac{F_2(s)\,ds}{(r^2-s^2)} + \frac{\pi}{2}\sigma_0\left\{(r^2-a^2)^{\frac{1}{2}} - (r^2-b^2)^{\frac{1}{2}}\right\}\right.$$
$$\left.+\frac{\pi}{2}F_3(r)\right], \tag{40}$$

the integral equation (29) gives the following:

$$F_3(r)+\frac{2}{\pi}\frac{r}{(1+\alpha)}\int_0^a\frac{F_1(s)\,ds}{(r^2-s^2)}=\sigma_0\left\{(r^2-b^2)^{\frac{1}{2}}-(r^2-a^2)^{\frac{1}{2}}\right\}$$

$$-\frac{2}{\pi}\frac{r\alpha}{(1+\alpha)}\int_0^a\frac{F_2(s)\,ds}{(r^2-s^2)},\qquad b<r<\infty. \tag{41}$$

The location of the point of separation $r=b$ is an unknown in the problem. This can be evaluated by using the condition that the stress-intensity factor vanishes at $r=b$: that is,

$$K_I^b=\lim_{r\to b^+}\left(2(r-b)\right)^{\frac{1}{2}}[\sigma_{zz}(r,0)]=0,\qquad b<r<\infty. \tag{42}$$

It can be shown that (42) is equivalent to

$$F_3(b)=0. \tag{43}$$

The magnitude of the indentations Δ_1 and Δ_2 can be evaluated by considering the equilibrium of forces acting on the faces of the disc inclusion, that is,

$$\int_0^a\sigma_{zz}^{(1)}(r,0)r\,dr=\int_0^a\sigma_{zz}^{(2)}(r,0)r\,dr. \tag{44}$$

The result (44) is equivalent to the constraint

$$\int_0^a[F_1(r)-F_2(r)]\,dr=0. \tag{45}$$

Finally, we introduce the substitutions

$$S_i(r)=\frac{F_i(r)}{\sigma_0},\qquad i=1,2,3. \tag{46}$$

The integral equations and consistency conditions governing the embedded, smooth rigid disc-inclusion problem can be written in the following forms:

$$S_1(r)+\frac{2}{\pi}\int_b^\infty\frac{sS_3(s)\,ds}{(s^2-r^2)}=-\frac{\Delta_1G_1}{\sigma_0(1-\nu_1)}-\left\{(b^2-r^2)^{\frac{1}{2}}-(a^2-r^2)^{\frac{1}{2}}\right\},\qquad 0<r<a, \tag{47}$$

$$S_3(r)+\frac{2r}{\pi(1+\alpha)}\int_0^a\frac{S_1(s)\,ds}{(r^2-s^2)}=\left\{(r^2-b^2)^{\frac{1}{2}}-(r^2-a^2)^{\frac{1}{2}}\right\}$$

$$-\frac{2r\alpha}{\pi(1+\alpha)}\int_0^a\frac{S_2(s)\,ds}{(r^2-s^2)},\qquad b<r<\infty, \tag{48}$$

$$S_2(s) = \frac{-\Delta_2 G_2}{\sigma_0(1-\nu_2)} + \{(a^2-s^2)^{\frac{1}{2}} - (b^2-s^2)^{\frac{1}{2}}\} - \frac{2}{\pi}\int_b^\infty \frac{uS_3(u)\,du}{(u^2-s^2)}, \qquad 0 < s < a, \tag{49}$$

$$S_3(b) = 0, \tag{50}$$

$$\int_0^a [S_1(r) - S_2(r)]\,dr = 0. \tag{51}$$

Substituting (47) and (49) in (51) we can obtain a relationship between Δ_1 and Δ_2 as follows:

$$\Delta_1 = \frac{\Delta}{(1+\alpha)} = \frac{\Delta_2}{\alpha}. \tag{52}$$

Furthermore, by substituting Δ_1 and Δ_2 respectively in (47) and (49) it can be shown that

$$S_1(r) = S_2(r), \qquad 0 < r < a, \tag{53}$$

which implies that the forms of the stress distribution on both sides of the inclusion are identical. This gives, from (48),

$$S_3(r) = (r^2-b^2)^{\frac{1}{2}} - (r^2-a^2)^{\frac{1}{2}} - \frac{2r}{\pi}\int_0^a \frac{S_2(u)\,du}{(r^2-u^2)}. \tag{54}$$

Consequently, substituting (52) and (54) into (49) we obtain

$$S_2(r) + \frac{2}{\pi^2}\int_0^a \frac{S_2(\xi)}{(r^2-\xi^2)}\left\{r\ln\left|\frac{b-r}{b+r}\right| - \xi\ln\left|\frac{b-\xi}{b+\xi}\right|\right\}d\xi$$
$$= -\frac{G_1\Delta}{\sigma_0(1-\nu_1)(1+\alpha)} + \frac{2}{\pi}(a^2-r^2)^{\frac{1}{2}}\tan^{-1}\left(\frac{b^2-a^2}{a^2-r^2}\right)^{\frac{1}{2}}, \qquad 0 < r < a. \tag{55}$$

The condition (50) which is required for the determination of b, can be written as

$$\frac{2b}{\pi}\int_0^a \frac{S_2(\xi)\,d\xi}{(b^2-\xi^2)} + (b^2-a^2)^{\frac{1}{2}} = 0. \tag{56}$$

Let us now examine the axial displacement field on the smooth interface beyond the separation region $b < r < \infty$. We have

$$u_z^{(1)}(r,0) = -\frac{2\sigma_0(1-\nu_1)}{\pi G_1}\int_0^a \frac{S_2(\xi)\,d\xi}{(r^2-\xi^2)^{\frac{1}{2}}} + \frac{2}{\pi}\int_r^\infty \frac{S_3(\xi)\,d\xi}{(\xi^2-r^2)^{\frac{1}{2}}}$$
$$+ \int_a^b u\,du\int_0^\infty J_0(\xi u)J_0(\xi r)\,d\xi, \qquad b \leqslant r < \infty. \tag{57}$$

For the second integral in the right-hand side of (57) we apply (54) and obtain

$$\int_r^\infty \frac{S_3(\xi)\,d\xi}{(\xi^2-r^2)^{\frac{1}{2}}} = \int_r^\infty \frac{(\xi^2-b^2)^{\frac{1}{2}}-(\xi^2-a^2)^{\frac{1}{2}}}{(\xi^2-r^2)^{\frac{1}{2}}}\,d\xi - \int_0^a \frac{S_2(\xi)\,d\xi}{(r^2-\xi^2)^{\frac{1}{2}}}, \qquad b<r<\infty. \tag{58}$$

Considering a change in the order of the integration, the third integral on the right-hand side of (57) can be evaluated in the following form for $r > u$:

$$\int_a^b u\,du \int_0^\infty J_0(\xi u)J_0(\xi r)\,d\xi = -\frac{2}{\pi}\int_r^\infty \left\{\frac{(u^2-b^2)^{\frac{1}{2}}-(u^2-a^2)^{\frac{1}{2}}}{(u^2-r^2)^{\frac{1}{2}}}\right\} du. \tag{59}$$

Substituting (58) and (59) into (57) and considering (16) it is evident that

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0) \equiv 0, \qquad b \leqslant r < \infty. \tag{60}$$

4. Numerical results

The single integral equation (55) governing the embedded disc-inclusion problem is not amenable to exact solution. The equation, however, can be numerically evaluated by adopting a discretization procedure. Accounts of the generalized procedures that can be employed for the solution of Fredholm-type integral equations of the second kind are given by Atkinson (**18**), Baker (**19**) and Delves and Mohamed (**20**).

Using the substitutions

$$\varphi(\eta) = S_2(r), \qquad \eta = \frac{r}{a}, \qquad c = \frac{b}{a}, \tag{61}$$

the integral equation can be written as

$$\varphi(\eta) + \frac{2}{\pi^2}\int_0^1 \frac{\varphi(\xi)}{(\eta^2-\xi^2)}\left\{\eta \ln\left|\frac{c-\eta}{c+\eta}\right| - \xi \ln\left|\frac{c-\xi}{c+\xi}\right|\right\} d\xi = \frac{\Delta G_1}{a\sigma_0(1-\nu_1)(1+\alpha)} + \frac{2}{\pi}(1-\eta^2)^{\frac{1}{2}} \tan^{-1}\left\{\frac{c^2-1}{1-\eta^2}\right\}, \qquad 0<\eta<1, \tag{62}$$

with the additional condition that

$$\frac{2c}{\pi}\int_0^1 \frac{\varphi(\xi)\,d\xi}{(c^2-\xi^2)} + (c^2-1)^{\frac{1}{2}} = 0 \tag{63}$$

for the determination of c.

The interval [0, 1] is divided into N equal segments. The equation (62) can be replaced by its discretized equivalent

$$[A_{ij}]\{\varphi_j\} = \{\lambda_i\}, \tag{64}$$

where the coefficients A_{ij} are given by

$$A_{ij} = \delta_{ij} + \frac{2}{\pi N(\xi_i^2 - \xi_j^2)}\left\{\xi_i \ln\left|\frac{c - \xi_i}{c + \xi_i}\right| - \xi_j \ln\left|\frac{c - \xi_j}{c + \xi_j}\right|\right\}. \tag{65}$$

When $i = j$, the appropriate values of A_{ij} should be obtained by considering the limiting values. Also

$$\varphi_j = \varphi(\xi_j), \qquad \xi_j = \frac{(2j - 1)}{2N}, \qquad j = 1, 2, \ldots, N \tag{66}$$

and

$$f_i = -\frac{(\Delta/a)}{\{\sigma_0(1 - \nu_1)/G_1\}}\frac{1}{(1 + \alpha)} + \frac{2}{\pi}(1 - \xi_i^2)^{\frac{1}{2}} \tan^{-1}\left[\frac{c^2 - 1}{1 - \xi_i^2}\right]^{\frac{1}{2}}. \tag{67}$$

The condition (63) can also be replaced by its discretized equivalent

$$\varepsilon = \frac{2c}{\pi N}\sum_{j=1}^{N} \frac{\varphi_j}{(c^2 - \xi_j^2)} + (c^2 - 1)^{\frac{1}{2}}, \tag{68}$$

where ε denotes the error. Since the characteristic equation for the determination of c is nonlinear, the solution of that equation has to be obtained with the constraint that the value of ε in (68) is a minimum. Since the problem has a unique value for the extent of the separation zone ($0 \leqslant c \leqslant 1$), a standard iterative secant method can be applied with an error bound of $|\varepsilon| < 10^{-6}$. The solution converges to a unique result within 30 iterations.

The extent of the normalized separation region b/a is influenced by the following non-dimensional parameter groups including the relative thickness of the disc inclusion (Δ/a), the relative magnitude of the precompression ($\sigma_0(1 - \nu_1)/G_1$) and the elasticity mismatch between the two half-space regions $\big(G_1(1 - \nu_2)/G_2(1 - \nu_1)\big)$. Figures 2 to 6 illustrate the manner in which the extent of the zone of separation is influenced by non-dimensional factors influencing the problem.

5. An approximate analytical result

A further aspect of the problem is the observation (60) that the axial displacements $u_z^{(1)}(r, 0)$ and $u_z^{(2)}(r, 0)$ are identically zero in the region where contact is re-established beyond the separation zone. In this instance an alternative approach can be applied to examine the problem. This involves the combination of solutions to two auxiliary problems. The first involves the internal indentation of a penny-shaped crack of radius b by a rigid disc inclusion of radius a. The analysis can be referred to an elastic infinite-space region composed of either material 1 or 2. The indentation displacements referred to the respective half-space regions are denoted by Δ_1 and Δ_2. The second auxiliary problem involves the application of internal tensile tractions σ_0 to the surfaces of an annular crack of internal radius a and external radius b. Again, the analysis can be referred to an elastic infinite space composed of either material 1 or 2.

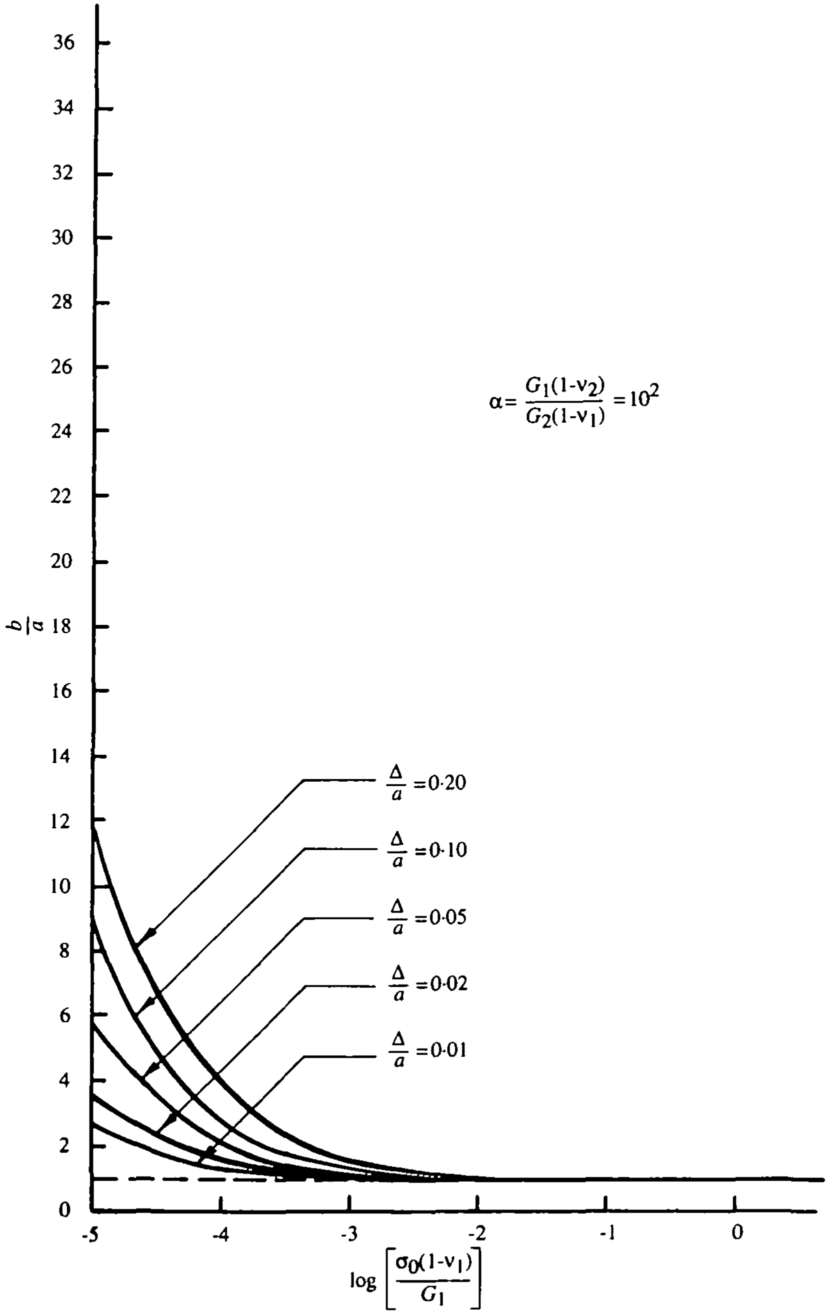

FIG. 2. Results for the extent of the zone of separation at the interface

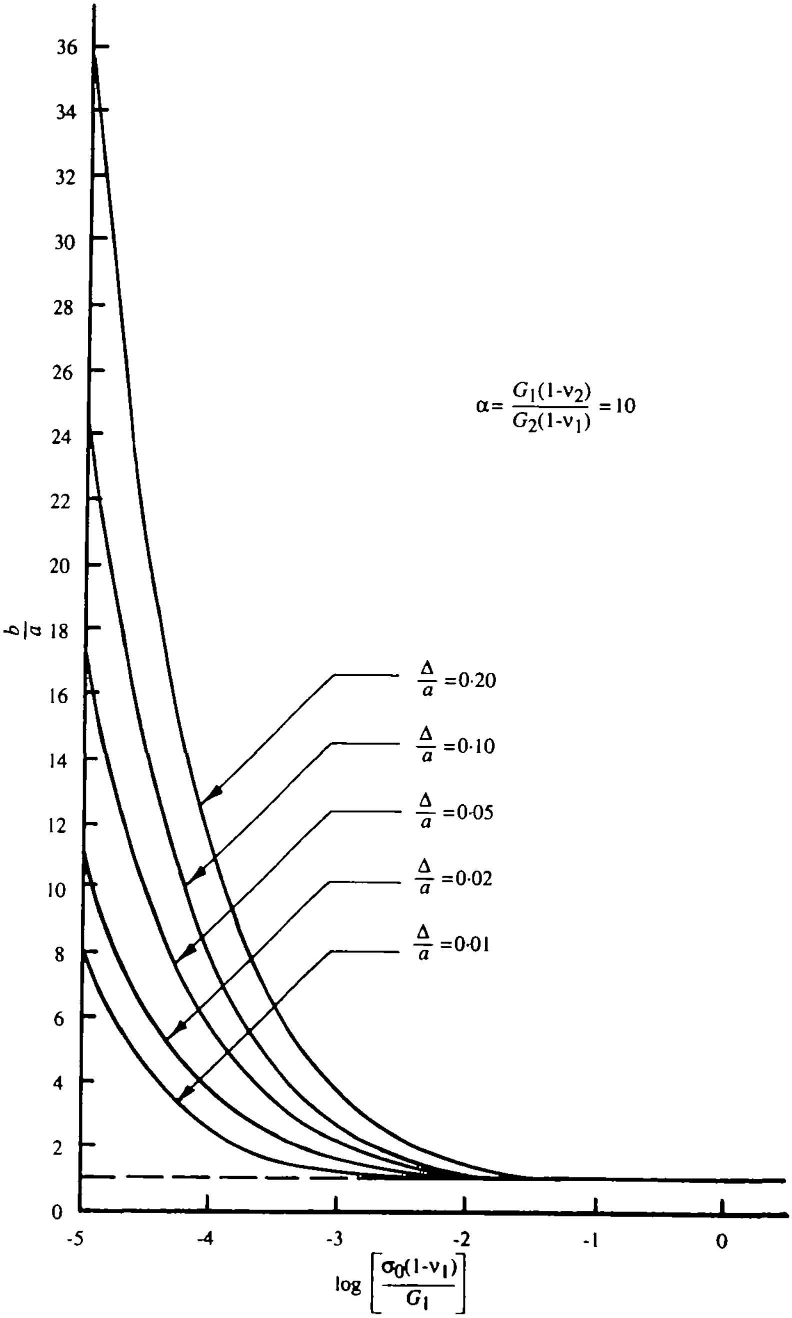

FIG. 3. Results for the extent of the zone of separation at the interface

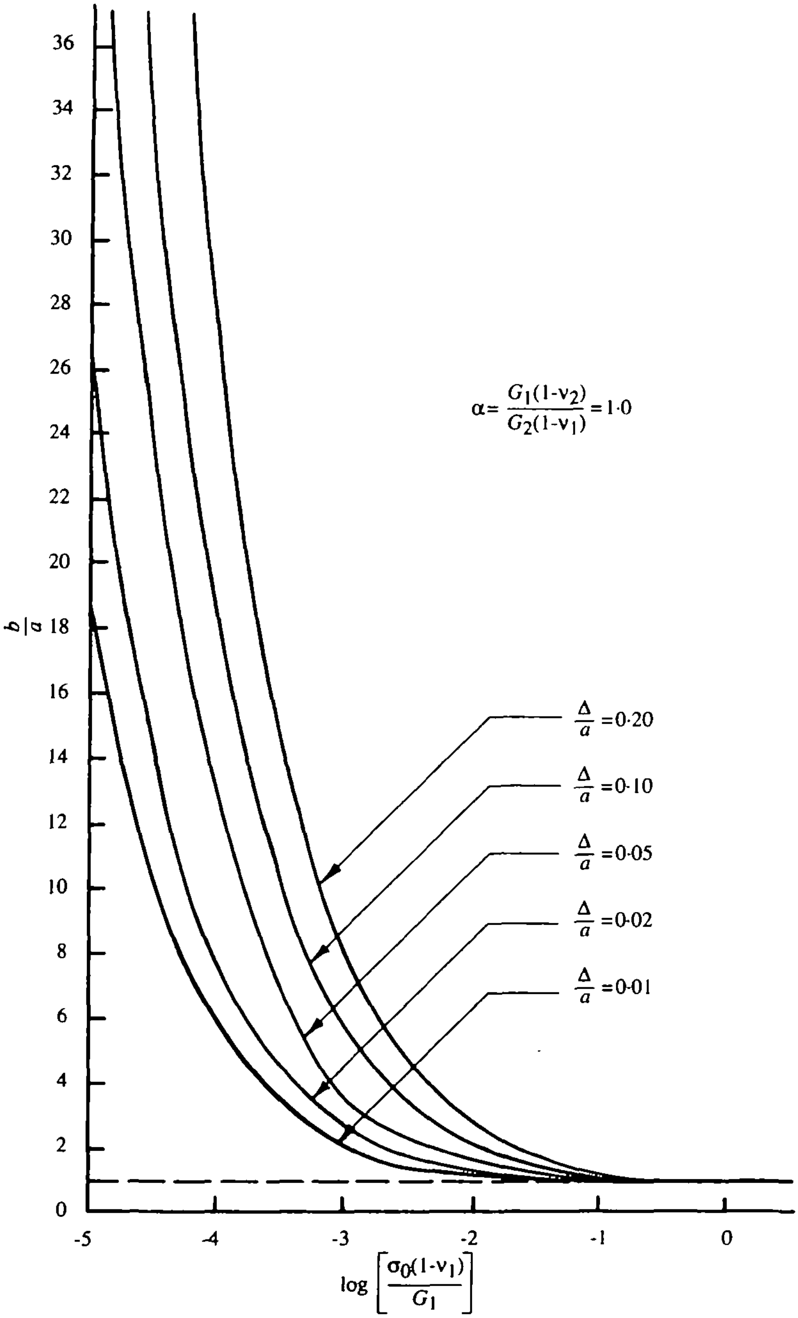

FIG. 4. Results for the extent of the zone of separation at the interface

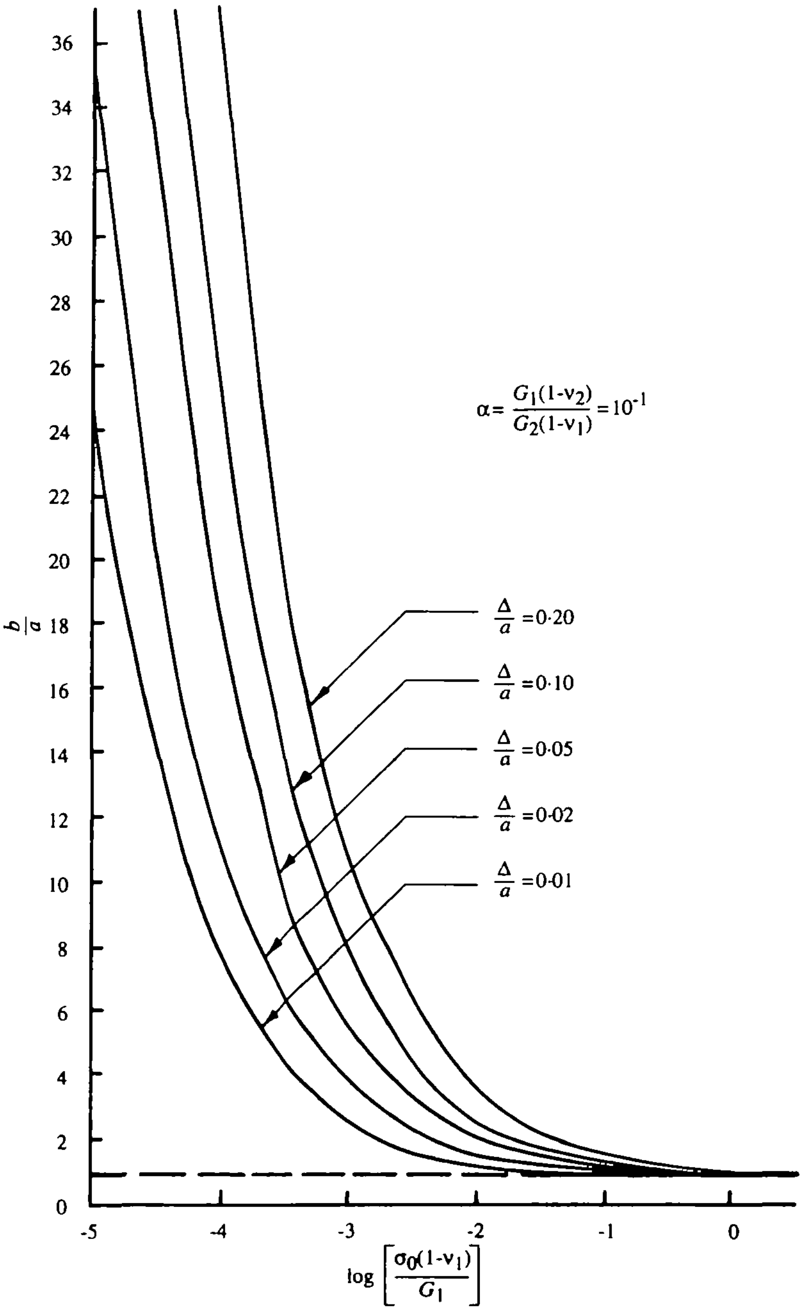

FIG. 5. Results for the extent of the zone of separation at the interface

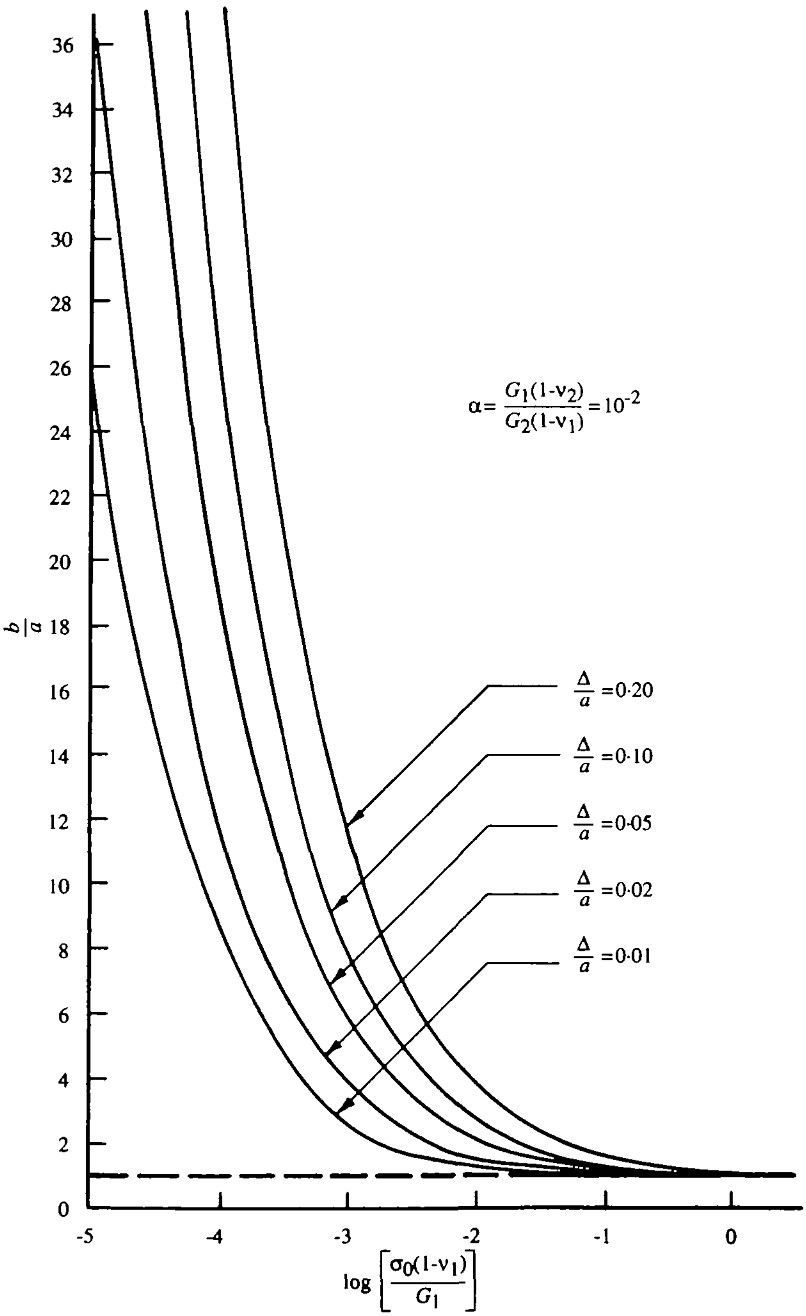

FIG. 6. Results for the extent of the zone of separation at the interface

For both problems the crack-opening mode stress-intensity factors at the boundary $r = b$ are evaluated; these are denoted by $(K_I)_{\Delta_1}$ and $(K_I)_{\sigma_0}$. These auxiliary problems have been examined in the literature (**17**, **21**, **22**) and the stress-intensity factors can only be evaluated either in a power-series form in terms of a/b or numerically. The radius of the zone of separation can be evaluated by invoking the conditions $(K_I)_{\Delta_1} = (K_I)_{\Delta_2} = (K_I)_{\Delta}$, $\Delta_1 + \Delta_2 = \Delta$ and $(K_I)_{\Delta} + (K_I)_{\sigma_0} = 0$. This gives a characteristic equation for the determination of the radius of the zone of separation, that is,

$$\frac{\Omega_1(\lambda)}{(1+\alpha)}\left(\frac{\Delta}{a}\right) - 2\left[\frac{\sigma_0(1-\nu_1)}{G_1}\right]\frac{\Omega_2(\lambda)}{\lambda} = 0, \tag{69}$$

where $\lambda = a/b = 1/c$ and $\Omega_i(\lambda)$ $(i = 1, 2)$ can be given in the form of power-series expansions in λ as follows:

$$\Omega_1(\lambda) = \frac{4}{\pi}\lambda + \frac{16\lambda^2}{\pi^3} + \lambda^3\left\{\frac{64}{\pi^5} + \frac{4}{3\pi}\right\} + \lambda^4\left\{\frac{80}{9\pi^3} + \frac{256}{\pi^7}\right\}$$
$$+ \lambda^5\left\{\frac{448}{9\pi^5} + \frac{1024}{\pi^9} + \frac{4}{5\pi}\right\} + O(\lambda^6), \tag{70}$$

$$\Omega_2(\lambda) = 1 - \frac{4\lambda}{\pi^2} - \frac{16\lambda^2}{\pi^4} - \lambda^3\left(\frac{1}{8} + \frac{64}{\pi^6}\right)$$
$$- \lambda^4\left\{\frac{16}{3\pi^4} + \frac{4}{\pi^2}\left(\frac{1}{24} - \frac{8}{9\pi^2} + \frac{64}{\pi^6} + \frac{4}{9\pi^3}\right)\right\}$$
$$- \lambda^5\left\{\frac{16}{\pi^4}\left(\frac{1}{24} + \frac{64}{\pi^6} - \frac{8}{9\pi^3} + \frac{8}{9\pi^2}\right) + \frac{256}{9\pi^6} - \frac{4}{15\pi^2}\right\}$$
$$+ O(\lambda^4). \tag{71}$$

Further details of the formulation are given by Selvadurai (**23**). The characteristic equation (69) can be solved numerically to determine an approximate result for the radius of the zone of separation b. The solution (69) yields only one positive root $b/a > 1$. The results for b/a derived via the approximate procedure compare very accurately with results presented in Figs 2 to 6 for the range of non-dimensional parameters Δ/a, α and $\sigma_0(1 - \nu_1)/G_1$ indicated. The differences between the two sets of solutions are indiscernible in the graphical representations given in Figs 2 to 6.

6. Conclusions

It is shown that the unilateral contact problem associated with a disc inclusion embedded at a precompressed, bimaterial elastic interface can be reduced to the solution of a single Fredholm-type integral equation of the second kind. An important result of the analysis is that in the re-established smooth contact region beyond the separation zone, the axial displacements are identically zero. The general validity of this observation applicable to inclusions with an arbitrary axisymmetric or asymmetric profile requires further investigation. The results obtained by Gladwell and Hara (**14**) indicate that in the case of the axisymmetric spheroidal inclusion the displacements beyond the separation zone are zero. This observation also allows the development of an alternative approximate solution to the unilateral contact problem. The numerical results presented in the paper indicates that the radius of the zone of separation is strongly influenced by the relative magnitude of the precompression stress σ_0. As $\sigma_0(1-\nu_1)/G_1 \to 1$ the interface separation is suppressed for $(\Delta/a) \in (0, 0{\cdot}2)$ and $\alpha \in (10^{-3}, 10^3)$.

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