INTERNATIONAL JOURNAL FOR NUMERICAL AND ANALYTICAL METHODS IN GEOMECHANICS, VOL. 17, 343–353 (1993)

THE AXIAL LOADING OF A RIGID CIRCULAR ANCHOR PLATE EMBEDDED IN AN ELASTIC HALF-SPACE

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SUMMARY

The present paper examines the axisymmetric problem related to the loading of a rigid circular anchor plate which is embedded in bonded contact with an isotropic elastic half-space. A Hankel transform development of the governing equations is used to reduce the associated mixed boundary value problem to a set of coupled Fredholm integral equations of the second kind. These equatons are solved in a numerical fashion to generate results of engineering interest. In particular, the results indicate the influence of the depth of embedment on the axial stiffness of the rigid anchor plate.

INTRODUCTION

Solutions for plate-shaped objects embedded in elastic media provide a valuable basis for the estimation of the short-term working load range performance of anchor plates. The solution to the problem of a rigid disc anchor embedded in bonded contact in an isotropic elastic medium of infinite extent was first developed by Collins.[1] The solution to the plate anchor problem was extended by Selvadurai[2–4] to include flexibility of the plate, transverse isotropic elastic behaviour of the surrounding infinite medium and eccentric loading of the anchor plate. The plate anchor problem also occurs as a limiting case of the spheroidal anchor region which is subjected to axial loads. Such solutions were developed by Kanwal and Sharma[5] and Selvadurai[6] by using singularity methods and spheroidal harmonic function techniques. These studies have been extended to include transversely isotropic behaviour of the elastic medium of infinite extent (see e.g. Reference 7). In situations where the anchor is embedded in a saturated soil, Selvadurai[8] has shown that the axial stiffness of the anchor plate can be obtained by considering an appropriate slow viscous flow analogy. Other non-classical effects pertaining to partial debonding at the interface of a plate anchor has been examined by Keer[9] and Selvadurai.[10] Similar effects related to anchor plates embedded in cracked regions located in elastic media of infinite extent have been examined by Selvadurai and Singh[11] and Selvadurai.[12] The generalized stiffness properties of anchor plates embedded in transversely isotropic elastic and bimaterial elastic regions of infinite extent have been evaluated by Selvadurai[13] and Selvadurai and Au.[14] By virtue of the simplicity of the mathematical treatment, a majority of problems related to the analytical solution of anchor plates in elastic media focus on situations where the plates are embedded in elastic media of infinite extent. This assumption is satisfactory only when the anchor plate is deeply embedded in the elastic medium.

In situations where the anchor plate is located in the vicinity of the boundary of an elastic half-space region, the traction-free constraint influences the stiffness characteristics of the embedded plate anchor. In addition to ground anchors with shallow embedment, the solution to a disc embedded within a half-space region has been utilized in the evaluation of *in situ* tests such as

0363–9061/93/050343–11$10.50

Received 3 August 1991
Revised 27 September 1991

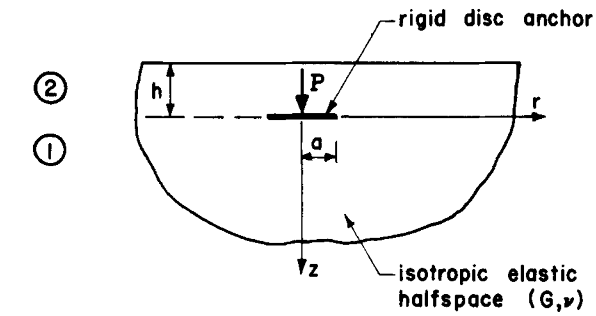

Figure 1. Axial loading of a rigid circular anchor plate embedded in an elastic half-space

plate load tests and screw plate tests (see e.g., References 15 and 16). The mathematical analysis of the problem of a rigid disc anchor embedded within a half-space region was first examined by Hunter and Gamblen.[17] Although attention was restricted to incompressible elastic behaviour of the surrounding soil medium, these authors did examine the influence of debonding at the soil–plate interface. Rowe and Booker[18] examined problems related to horizontally embedded anchors in an elastic half-space by using a numerical scheme. Rajapakse[19] and Rajapakse and Selvadurai[20] have investigated problems dealing with solid and plate anchors embedded in homogeneous and non-homogeneous elastic half-space regions by employing a variational scheme and associated discretization procedures. Pak and Gobert[21] have examined the axisymmetric problem of a rigid disc embedded in an elastic half-space where the displacement boundary conditions at the anchor–elastic medium interface are only partially satisfied. Other aspects of disc-shaped inclusions or anchors embedded in an elastic half-space are reviewed by Mura[22,23] and in a forthcoming text by Selvadurai.[24] More recently Selvadurai *et al.*[25] have examined the problem of the in-plane loading of a disc-shaped anchor embedded within a half-space region.

In this paper we examine the elastostatic problem related to a rigid disc anchor which is embedded in bonded contact with an isotropic elastic half-space region and subjected to an axial load (Figure 1). The assumption of perfect bonding at the anchor–elastic medium interface is an idealization which could be modified by incorporating the possible effects of delamination and fracture at the base of the rigid circular anchor plate. The mathematical analysis of the debonded anchor problem is somewhat more complex than the problem discussed in the present paper. The basic model of the fully bonded shallow anchor plate is a reasonable approximation for the examination of the working load behaviour of a flat anchor region formed in, for example, a grout layer where adhesive forces could maintain the interface in a bonded condition. Also in the context of near-surface anchors of this type, the self weight of the medium exerts only a minor influence on the axial stiffness of the embedded anchor. The axisymmetric problem is formulated by adopting a Hankel transform development of the governing equations. The integral equations associated with the mixed boundary value problem are reduced to a set of coupled Fredholm integral equations of the second kind. These are solved in a numerical fashion to develop results for the axial stiffness of the disc anchor embedded in a half-space region.

FUNDAMENTAL EQUATIONS

We consider the axisymmetric elasticity problem in which a disc anchor of radius a is embedded in bonded contact with an isotropic elastic half-space at a depth h. The rigid disc anchor is subjected to a central axial force P which induces an axial displacement of Δ in the z-direction

(Figure 1). The boundary conditions governing the problem can be specified in relation to the layer region (denoted by superscript (1) and $-h \le z \le 0$) and the half-space region (denoted by superscript (2) and $0 \le z < \infty$). The boundary and continuity conditions governing the problem are

$$u_z^{(1)} = u_z^{(2)} = \Delta, \qquad z = 0, \quad r \in (0, a) \tag{1}$$

$$u_r^{(1)} = u_r^{(2)} = 0, \qquad z = 0, \quad r \in (0, a) \tag{2}$$

$$u_r^{(1)} = u_r^{(2)}, \qquad z = 0, \quad r \in (a, \infty) \tag{3}$$

$$u_z^{(1)} = u_z^{(2)}, \qquad z = 0, \quad r \in (a, \infty) \tag{4}$$

$$\sigma_{zz}^{(1)} = \sigma_{zz}^{(2)}, \qquad z = 0, \quad r \in (a, \infty) \tag{5}$$

$$\sigma_{rz}^{(1)} = \sigma_{rz}^{(2)}, \qquad z = 0, \quad r \in (a, \infty) \tag{6}$$

$$\sigma_{rz}^{(1)} = 0, \qquad z = -h, \quad r \in (0, \infty) \tag{7}$$

$$\sigma_{zz}^{(1)} = 0, \qquad z = -h, \quad r \in (0, \infty) \tag{8}$$

For rotationally symmetric problems, $u_\theta = 0$ and the equations of equilibrium can be satisfied if u_r and u_z are expressed in terms of the functions $f(r, z)$ and $g(r, z)$ which satisfy

$$\nabla^2 f = 0, \qquad \nabla^2 g = 0 \tag{9}$$

where ∇^2 is Laplace's operator in axisymmetric cylindrical polar co-ordinates.

The displacements and stresses can be expressed in terms of $f(r, z)$ and $g(r, z)$ in the following forms (see e.g. References 26 and 27):

$$u_r = \frac{\partial f}{\partial r} - z\frac{\partial g}{\partial r} \tag{10}$$

$$u_z = (3 - 4\nu)g + \frac{\partial f}{\partial z} - z\frac{\partial g}{\partial z} \tag{11}$$

and

$$\sigma_{rr} = 2G\left[\frac{\partial^2 f}{\partial r^2} + 2\nu\frac{\partial g}{\partial z} - z\frac{\partial^2 g}{\partial r^2}\right] \tag{12}$$

$$\sigma_{\theta\theta} = 2G\left[\frac{1}{r}\frac{\partial f}{\partial r} + 2\nu\frac{\partial g}{\partial z} - \frac{z}{r}\frac{\partial g}{\partial r}\right] \tag{13}$$

$$\sigma_{zz} = 2G\left[2(1 - \nu)\frac{\partial g}{\partial z} + \frac{\partial^2 f}{\partial z^2} - z\frac{\partial^2 g}{\partial z^2}\right] \tag{14}$$

$$\sigma_{rz} = 2G\frac{\partial}{\partial r}\left[(1 - 2\nu)g + \frac{\partial f}{\partial z} - z\frac{\partial g}{\partial z}\right] \tag{15}$$

The integral expressions for the displacement and stress components in the layer $-h \le z \le 0$ and the half-space region $z \ge 0$ can be derived by application of Hankel integrals to represent the potential functions $f(r, z)$ and $g(r, z)$. The relevant integral expressions for $u_r^{(2)}$, $u_z^{(2)}$, $\sigma_{zz}^{(2)}$, etc. take the forms

$$u_r^{(2)}(r, z) = \int_0^\infty [A(\xi) + \xi z B(\xi)]\,\mathrm{e}^{-\xi z} J_1(\xi r)\,\mathrm{d}\xi \tag{16}$$

$$u_z^{(2)}(r,z) = \int_0^\infty [A(\xi) + (3-4\nu)B(\xi) + \xi z B(\xi)] e^{-\xi z} J_0(\xi r)\, d\xi \tag{17}$$

$$\sigma_{zz}^{(2)}(r,z) = -2G \int_0^\infty \xi[A(\xi) + 2(1-\nu)B(\xi) + \xi z B(\xi)] e^{-\xi z} J_0(\xi r)\, d\xi \tag{18}$$

$$\sigma_{rz}^{(2)}(r,z) = -2G \int_0^\infty \xi[A(\xi) + (1-2\nu)B(\xi) + \xi z B(\xi)] e^{-\xi z} J_1(\xi r)\, d\xi \tag{19}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions. Following Kuz'Min and Uflyand[28] and Srivastava and Singh[29] it can be shown that the relevant solutions of (9) which satisfy boundary conditions (7) and (8) can be evaluated in the form

$$\begin{aligned} u_r^{(1)}(r,z) = \int_0^\infty \{ & [\xi(z+h)C(\xi) - 2(1-\nu)D(\xi)] \cosh[\xi(z+h)] \\ & + [(1-2\nu)C(\xi) - \xi(z+h)D(\xi)] \sinh[\xi(z+h)]\} \frac{J_1(\xi r)}{\sinh(\xi h)}\, d\xi \end{aligned} \tag{20}$$

$$\begin{aligned} u_z^{(1)}(r,z) = \int_0^\infty \{ & [2(1-\nu)C(\xi) + \xi(z+h)D(\xi)] \cosh[\xi(z+h)] \\ & - [\xi(z+h)C(\xi) + (1-2\nu)D(\xi)] \sinh[\xi(z+h)]\} \frac{J_0(\xi r)}{\sinh(\xi h)}\, d\xi \end{aligned} \tag{21}$$

$$\begin{aligned} \sigma_{zz}^{(1)} = 2G \int_0^\infty \{ & [C(\xi) + \xi(z+h)D(\xi)] \sinh[\xi(z+h)] \\ & - \xi(z+h)C(\xi) \cosh[\xi(z+h)]\} \frac{\xi J_0(\xi r)}{\sinh(\xi h)}\, d\xi \end{aligned} \tag{22}$$

$$\begin{aligned} \sigma_{rz}^{(1)}(r,z) = -2G \int_0^\infty \{ & [D(\xi) - \xi(z+h)C(\xi)] \sinh[\xi(z+h)] \\ & + \xi(z+h)D(\xi) \cosh[\xi(z+h)]\} \frac{\xi J_1(\xi r)}{\sinh(\xi h)}\, d\xi \end{aligned} \tag{23}$$

where $C(\xi)$ and $D(\xi)$ are arbitrary functions.

THE CIRCULAR ANCHOR PLATE PROBLEM

From the displacement boundary conditions applicable to the embedded anchor problem (see equations (1)–(4)) it is evident that

$$u_r^{(1)}(r,0) = u_r^{(2)}(r,0), \quad r \in (0, \infty) \tag{24}$$

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0), \quad r \in (0, \infty) \tag{25}$$

From these conditions and the results (16), (17), (20) and (21), we obtain

$$A(\xi) = C(\xi)\{\xi h \coth(\xi h) + 1 - 2\nu\} - D(\xi)\{\xi h + 2(1-\nu)\coth \xi h\} \tag{26}$$

$$\begin{aligned} B(\xi) = \frac{1}{(3-4\nu)} & [C(\xi)\{\coth(\xi h)[2(1-\nu) - \xi h] - \xi h - (1-2\nu)\} \\ & + D(\xi)\{\xi h(1+\coth(\xi h)) - (1-2\nu) + 2(1-\nu)\coth(\xi h)\}] \end{aligned} \tag{27}$$

The boundary conditions (1), (2), (5) and (6) can now be represented in terms of the following integral equations:

$$\int_0^\infty \{C(\xi)[\xi h\coth(\xi h) + (1-2\nu)] - D(\xi)[2(1-\nu)\coth(\xi h) + \xi h]\} J_1(\xi r)\,\mathrm{d}\xi = 0, \quad 0 \le r \le a \tag{28}$$

$$\int_0^\infty \{C(\xi)[2(1-\nu)\coth(\xi h) - \xi h] + D(\xi)[\xi h\coth(\xi h) - (1-2\nu)]\} J_0(\xi r)\,\mathrm{d}\xi = \Delta, \quad 0 \le r \le a \tag{29}$$

$$\int_0^\infty \xi[1+\coth(\xi h)]\{C(\xi)(\xi h + 1 - 2\nu) - D(\xi)(\xi h + 2(1-\nu))\} J_1(\xi r)\,\mathrm{d}\xi = 0, \quad a < r < \infty \tag{30}$$

$$\int_0^\infty \xi[1+\coth(\xi h)]\{(2\nu - 1 + \xi h)D(\xi) + (2 - 2\nu - \xi h)C(\xi)\} J_0(\xi r)\,\mathrm{d}\xi. = 0, \quad a < r < \infty \tag{31}$$

Introducing the substitutions

$$R_1(\xi) = C(\xi)\{\xi h + 1 - 2\nu\} - D(\xi)\{\xi h + 2(1-\nu)\} \tag{32}$$

$$R_2(\xi) = C(\xi)\{2(1-\nu) - \xi h\} + D(\xi)\{2\nu - 1 + \xi h\} \tag{33}$$

and solving (32) and (33), we obtain

$$C(\xi) = \frac{1}{(3-4\nu)}[R_1(\xi)\{\xi h + 2\nu - 1\} + R_2(\xi)\{\xi h + 2(1-\nu)\}] \tag{34}$$

$$D(\xi) = \frac{-1}{(3-4\nu)}[R_1(\xi)\{2(1-\nu) - \xi h\} - R_2(\xi)\{1 + \xi h - 2\nu\}] \tag{35}$$

Using these representations, the integral equations (28)–(31) can be rewritten in the following forms:

$$\begin{aligned}\int_0^\infty \{R_2(\xi)[&-(\xi h + 2 - 2\nu)(\xi h\coth(\xi h) + 1 - 2\nu)\\ &+ (1 + \xi h - 2\nu)(2(1-\nu)\coth(\xi h) + \xi h)]\\ &- R_1(\xi)[(\xi h + 2\nu - 1)(\xi h\coth(\xi h) + 1 - 2\nu)\\ &+ (2(1-\nu) - \xi h)(2(1-\nu)\coth(\xi h) + \xi h)]\} J_1(\xi r)\,\mathrm{d}\xi = 0,\\ &\qquad 0 \le r \le a\end{aligned} \tag{36}$$

$$\begin{aligned}\int_0^\infty \bigg\{&\frac{R_1(\xi)}{(3-4\nu)}[(2(1-\nu) - \xi h)(\xi h\coth(\xi h) - (1-2\nu))\\ &- (\xi h + 2\nu - 1)(2(1-\nu)\coth(\xi h) - \xi h)]\\ &- \frac{R_2(\xi)}{(3-4\nu)}[(\xi h + 2(1-\nu))(2(1-\nu)\coth(\xi h) - \xi h)\\ &+ (\xi h\coth(\xi h) - (1-2\nu))(1 + \xi h - 2\nu)]\bigg\} J_0(\xi r)\,\mathrm{d}\xi = \Delta,\\ &\qquad 0 \le r \le a\end{aligned} \tag{37}$$

$$\int_0^\infty \xi(1+\coth(\xi h))R_1(\xi)J_1(\xi r)\mathrm{d}\xi = 0, \quad a < r < \infty \tag{38}$$

$$\int_0^\infty \xi(1+\coth(\xi h))R_2(\xi)J_0(\xi r)\ \mathrm{d}\xi = 0, \quad a < r < \infty \tag{39}$$

Introduce the transformation

$$[1+\coth(\xi h)]R_1(\xi) = \xi\int_0^a \phi(t)\cos(\xi t)\mathrm{d}t \tag{40}$$

$$[1+\coth(\xi h)]R_2(\xi) = \int_0^a \psi(t)\cos(\xi t)\,\mathrm{d}t \tag{41}$$

such that the equations (38) and (39) are identically satisfied. Equations (36) and (37) can be reduced to the following Fredholm integral equations:

$$\phi(t) + \int_0^a \phi(u)K_1(u,t)\,\mathrm{d}u + \int_0^a \psi(u)K_2(u,t)\,\mathrm{d}u = 0, \quad 0 \le t \le a \tag{42}$$

$$\psi(t) + \int_0^a \psi(u)K_3(u,t)\ \mathrm{d}u + \int_0^a \psi(u)K_4(u,t)\mathrm{d}u = -\frac{4\Delta}{\pi}, \quad 0 \le t \le a \tag{43}$$

where the kernel functions $K_i(u,t)$ $(i = 1,2,3,4)$ are given by

$$\begin{aligned} K_1(u,t) = \frac{-4t}{(3-4\nu)\pi}\int_0^\infty \frac{\cos(\xi u)\{1-\cos(\xi t)\}}{\{1+\coth(\xi h)\}}\Bigg[&-\frac{(3-4\nu)}{2}\{1+\coth(\xi h)\} \\ &+ (\xi h + 2\nu - 1)(\xi h\coth(\xi h) + 1 - 2\nu) \\ &+ \{2(1-\nu) - \xi h\}\{2(1-\nu)\coth(\xi h) + \xi h\}\Bigg]\mathrm{d}\xi \end{aligned} \tag{44}$$

$$\begin{aligned} K_2(u,t) = \frac{4t}{(3-4\nu)\pi}\int_0^\infty &\frac{\cos(\xi u)\{1-\cos(\xi t)\}}{\xi(1+\coth(\xi h))} \\ &\times[(1+\xi h - 2\nu)\{2(1-\nu)\coth(\xi h) + \xi h\} \\ &-(\xi h + 2(1-\nu))\{\xi h\coth(\xi h) + 1 - 2\nu\}]\,\mathrm{d}\xi \end{aligned} \tag{45}$$

$$\begin{aligned} K_3(u,t) = \frac{4}{(3-4\nu)\pi}\int_0^\infty \frac{\xi\cos(\xi u)\cos(\xi t)}{(1+\coth(\xi h))}\Bigg[&-\frac{(3-4\nu)}{2}(1+\coth(\xi h)) \\ &+ (\xi h + 2(1-\nu))\{2(1-\nu)\coth(\xi h) - \xi h\} \\ &+ (1+\xi h - 2\nu)\{\xi h\coth(\xi h) - (1-2\nu)\}\Bigg]\mathrm{d}\xi \end{aligned} \tag{46}$$

$$\begin{aligned} K_4(u,t) = \frac{-4}{(3-4\nu)\pi}\int_0^\infty &\frac{\cos(\xi u)\cos(\xi t)}{(1+\coth(\xi h))}[(2(1-\nu) - \xi h)\{\xi h\coth(\xi h) \\ &-(1-2\nu)\} - (\xi h - 2\nu - 1)\{2(1-\nu)\coth(\xi h) - \xi h\}]\,\mathrm{d}\xi \end{aligned} \tag{47}$$

This formally completes the analysis of the axial loading of a rigid disc anchor which is located in bonded contact with an isotropic elastic half-space region. The system of coupled integral equations (42) and (43) can be solved in a numerical fashion to generate results of engineering interest.

LOAD–DISPLACEMENT RELATIONSHIP FOR THE ANCHOR PLATE

A result of primary interest to the paper concerns the evaluation of the axial stiffness of the embedded anchor plate. From equation (41), we have

$$\{1 + \coth(\xi h)\}\, R_2(\xi) = -\frac{\psi(a)\sin(\xi a)}{\xi} + \frac{1}{\xi}\int_0^a \psi'(t)\sin(\xi t)\,\mathrm{d}t \tag{48}$$

where the prime denotes the derivative with respect to t.

Considering the tractions acting on the rigid anchor plate, we obtain

$$P = \frac{8\pi G(1-\nu)}{(3-4\nu)}\int_0^a [\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0)]\,\mathrm{d}r \tag{49}$$

Using the expressions (18), (22), (33) and the result (48) it can be shown that

$$P = \frac{8\pi G(1-\nu)}{(3-4\nu)}\int_0^a \psi(t)\,\mathrm{d}t \tag{50}$$

NUMERICAL SOLUTION OF THE INTEGRAL EQUATIONS

An inspection of the governing integral equations (42) and (43) would suggest that it is not feasible to obtain solutions for these integral equations in any compact or exact closed forms. Recourse must therefore be made to develop numerical schemes for the solution of these equations. The ensuing deals with such a numerical procedure.

The kernel functions $K_i (i = 1,2,3,4)$ defined by (44)–(47) can be rewritten as

$$K_1(u,t) = \frac{-4t}{(3-4\nu)\pi}\int_0^\infty \mathrm{e}^{-2\xi h}\left\{(\xi h)^2 - (3-4\nu)\xi h + \frac{(5-12\nu+8\nu^2)}{2}\right\} \cos(\xi u)[1-\cos(\xi t)]\,\mathrm{d}\xi \tag{51}$$

$$K_2(u,t) = \frac{-4t}{(3-4\nu)\pi}\int_0^\infty \mathrm{e}^{-2\xi h}\{(\xi h)^2 - 2(1-\nu)(1-2\nu)\}\,\frac{\cos(\xi u)[1-\cos(\xi t)]}{\xi}\,\mathrm{d}\xi \tag{52}$$

$$K_3(u,t) = \frac{4}{(3-4\nu)\pi}\int_0^\infty \mathrm{e}^{-2\xi h}\left\{(\xi h)^2 + (3-4\nu)\xi h + \frac{(5-12\nu+8\nu^2)}{2}\right\} \cos(\xi u)\cos(\xi t)\xi\,\mathrm{d}\xi \tag{53}$$

$$K_4(u,t) = \frac{4}{(3-4\nu)\pi}\int_0^\infty \mathrm{e}^{-2\xi h}\{(\xi h)^2 - 2(1-\nu)(1-2\nu)\}\cos(\xi u)\cos(\xi t)\,\mathrm{d}\xi \tag{54}$$

For the solution of the coupled integral equations (42) and (43), the interval $[0, a]$ is divided into N segments with limits given by $u_i = (i-1)(a/N)$, $(i = 1, 2, \ldots, N+1)$. The corresponding collocation points are

$$t_i = \left\{\frac{u_i + u_{i+1}}{2}\right\}; \quad (i = 1, 2, \ldots, N) \tag{55}$$

The integral equations (42) and (43) can be expressed in the form of a single matrix equation

$$[\mathbf{A}_{ij}]\{\chi_j\} = \{\mathbf{B}_i\} \tag{56}$$

with $i, j = 1, 2, \ldots, 2N$. Also

$$A_{(2l-1),(2m-1)} = \delta_{lm} + \frac{a}{N} K_1(t_m, t_l) \tag{57}$$

$$A_{(2l-1)(2m)} = \frac{a}{N} K_2(t_m, t_l) \tag{58}$$

$$A_{(2l)(2m-1)} = \frac{a}{N} K_4(t_m, t_l) \tag{59}$$

$$A_{(2l)(2m)} = \delta_{lm} + \frac{a}{N} K_3(t_m, t_l) \tag{60}$$

$$B_{(2l-1)} = 0, \qquad B_{(2l)} = 1{\cdot}0 \tag{61}$$

$$\chi_{(2l-1)} = \varphi(t_l); \qquad \chi_{2l} = \psi(t_l) \tag{62}$$

where $l, m = 1, 2, 3, \ldots, N$.

Upon solution of the matrix equation (56), the load on the rigid inclusion can be obtained by making use of (50). The axial stiffness of the anchor plate can be expressed in the form

$$\bar{P} = \frac{P(3-4\nu)}{32(1-\nu)G\Delta a} = \frac{1}{N}\sum_{l=1}^{N} \psi(t_l) \tag{63}$$

The procedure outlined above represents an accurate and effective technique for the solution of the coupled system of integral equations given by (42) and (43). Details of alternative procedures that can be used for solution of integral equations of the Fredholm-type are given by Baker[30] and Delves and Mohamed.[31]

NUMERICAL RESULTS AND CONCLUSIONS

Prior to examining the results derived from the numerical analysis of the coupled integral equations governing the disc anchor plate problem, it is useful to establish solutions to certain limiting cases associated with the depth of embedment of the anchor plate.

(a) In the limiting case when $h/a \to \infty$, the problem reduces to that of the axial loading of a rigid circular anchor plate which is embedded in bonded contact within an isotropic elastic medium of infinite extent. The exact closed-form solution for the axial stiffness of the embedded rigid plate was derived by Collins,[1] Selvadurai[6] and Kanwal and Sharma[5] by using integral equation techniques, spheroidal harmonic function techniques and singularity methods, respectively. As $h/a \to \infty$, we have

$$\frac{P}{16G\Delta a} = \frac{2(1-\nu)}{(3-4\nu)} \tag{64}$$

(b) As the relative depth of embedment $h/a \to 0$, the problem reduces to the case where the axially loaded rigid anchor plate is bonded to the surface of a half-space region. The exact analytical solution to the problem was obtained by Mossakovskii[32] and Ufliand[33] (see also

Reference 27) by solving the mixed boundary value problem defined by the following:

$$u_z(r, 0) = \Delta, \quad 0 \le r \le a \tag{65}$$

$$u_r(r, 0) = 0, \quad 0 \le r \le a \tag{66}$$

$$\sigma_{zz}(r, 0) = 0, \quad a < r < \infty \tag{67}$$

$$\sigma_{rz}(r, 0) = 0, \quad a < r < \infty \tag{68}$$

The solution of the above mixed boundary value problem developed by Mossakovskii[32] and Ufliand[33] incorporates the oscillatory form of the stress singularity at the boundary of the rigid plate. The result of particular interest to the present discussion concerns the load–displacement behaviour of the rigid plate. Using the formulation based on the Hilbert problem, the result can be evaluated in the exact closed form

$$\frac{P}{4G\Delta a} = \frac{\ln(3 - 4\nu)}{(1 - 2\nu)} \tag{69}$$

It may be noted that the mixed boundary value problem defined by (65)–(68) can also be

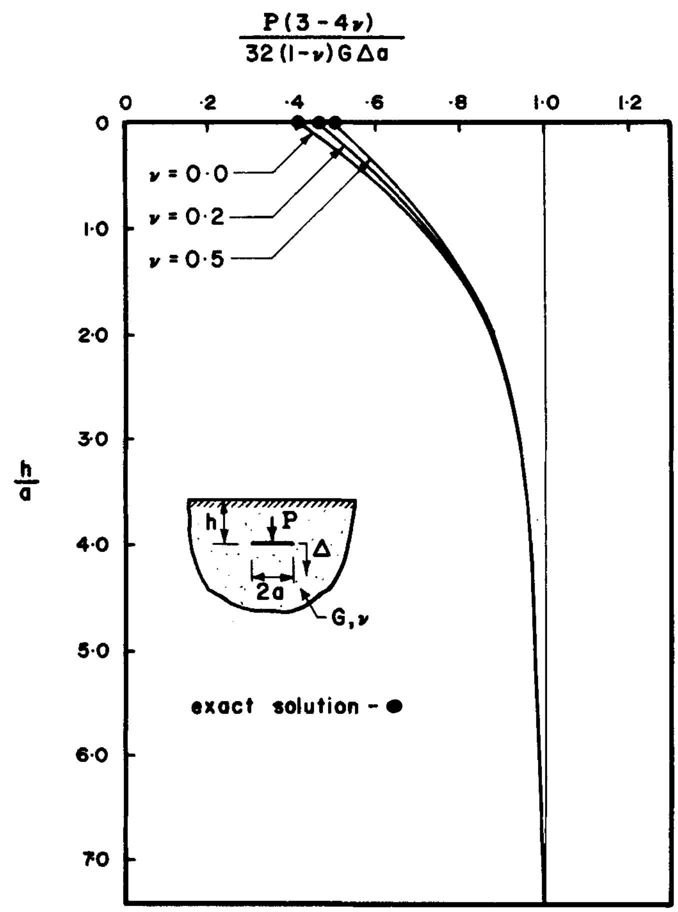

Figure 2. Influence of the depth of embedment on the axial stiffness of the embedded anchor plate

solved by using a Hankel integral transform approach in which the oscillatory stress singularities are suppressed and the probem is effectively reduced to the solution of a single Fredholm integral equation of the second kind. Selvadurai[12] has shown that the incorporation of the oscillatory form of the stress singularity does not significantly influence the axial stiffness of the rigid circular plate bonded to a half-space region. Comparison of the results obtained by the two schemes indicate that the maximum difference between the two solution schemes does not exceed 0.5 per cent. In the limit of material incompressibility, the oscillatory singular behaviour is absent and the two solution schemes yield the same result.

The results for the axial stiffness of the anchor plate embedded in bonded contact with an elastic half-space, derived by using the numerical analysis presented in the previous section, are shown in Figure 2. It is evident that the stiffness of the anchor plate increases as the depth of embedment increases. The influence of the traction-free surface becomes important as $h/a \rightarrow 0$. For all practical purposes, the axial stiffness of the anchor plate embedded in a half-space region approaches the result for a rigid plate embedded in an infinite space when $h/a > 4$. In situations where the self weight of the material or adhesive strength at the anchor plate–soil interface is sufficient to prevent debonding at its base, the stiffness of the anchor plate can be estimated by utilizing the results presented in this study. The axial stiffness of the anchor is considered to be the result of primary interest to engineering applications. Other results concerning the state of stress at the anchor–soil interface can be obtained by using the developments presented in the paper.

ACKNOWLEDGEMENTS

The work described in this paper was supported by Natural Sciences and Engineering Research Council of Canada Research Grant A3866. The author is indebted to M. C. Au for assistance with the numerical computations.

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