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AXISYMMETRIC DEFORMATION OF A WINKLER LAYER BY INTERNALLY LOADED ELASTIC HALFSPACES

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Introduction

The present work is concerned with the axisymmetric interaction of a Winkler medium contained between two homogeneous, isotropic elastic halfspaces. The Winkler medium, which is composed of a dense array of independent spring elements, is in bonded contact with the halfspaces of differing elastic characteristics. The bonded contact provides continuity of displacement at interfaces of the Winkler medium and the elastic halfspaces. The elastic halfspaces are subjected separately to a Mindlin force, namely, a concentrated force which acts at the interior of the halfspace and directed along the axis of symmetry (Fig. 1). The method of solution adopted here is based on the use of Hankel transform techniques outlined by Sneddon [1]. Formal integral relationships are developed for the surface displacements of the individual halfspaces and for the stress distribution in the Winkler layer. It is found that these particular integral representations give closed form results, when evaluated at the axis of symmetry. The basic problem discussed here is of interest in connection with certain problems in fracture mechanics [2].

Analysis

The axially symmetric problem relating to a homogeneous isotropic elastic halfspace which is subjected to an axisymmetric normal traction $q(r)$ on its plane boundary and a concentrated force P at a distance c from the origin of coordinates is examined. The internal concentrated force is assumed to act in the negative z-direction of the spatial coordinate system (Fig. 1). The solution to the surface load problem is generated by employing Hankel transform techniques. It can be shown that the transformed value of the surface displacement of the halfspace in the z-direction $[u_z(r,0) = w_q(r)]$,

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due to the external traction q(r) is given by

$$\bar{w}_q^0(\xi) = \frac{a(1-\nu)}{G\xi} \bar{q}^0(\xi) \tag{1}$$

where a is a typical length parameter of the problem and G and ν are respectively the linear elastic shear modulus and Poisson's ratio of the elastic material. In (1) $\bar{w}_q^0(\xi)$ denotes the zeroth-order Hankel transform defined as

$$\bar{w}_q^0(\xi) = H\{w_q(r);\xi\} = \int_0^\infty r w_q(r) J_0(\xi r/a)dr \tag{2}$$

and $\bar{q}^0(\xi)$ denotes the zeroth-order Hankel transform of the normal surface traction q(r). The corresponding Hankel inversion theorem is

$$w_q(r) = \frac{1}{a^2}\int_0^\infty \xi \bar{w}_q^0(\xi) J_0(\xi r/a)d\xi . \tag{3}$$

By considering the solution to the Mindlin force problem [3,4] it can be shown that the transformed value for the surface displacement $w_p(r)$ is given by

$$\bar{w}_p^0(\xi) = -\frac{a(1-\nu)}{G\xi} \bar{S}^0(\xi) \tag{4}$$

where

$$\bar{S}^0(\xi) = \frac{P}{4\pi} [2 + \frac{(\xi c/a)}{(1-\nu)}] e^{-\xi c/a}. \tag{5}$$

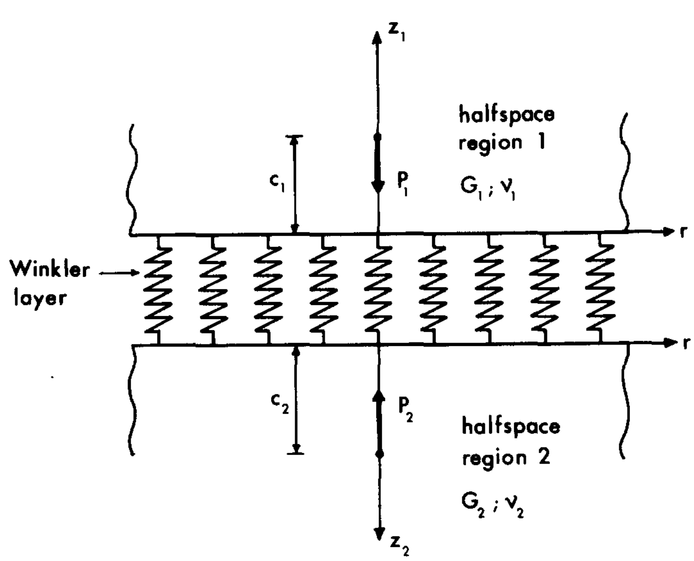

Fig. 1

Compression of The Winkler Layer

The results described above can now be employed to obtain expressions for the transformed values of the surface displacements of the halfspaces under the combined action of the stress in the Winkler layer and the internal loading. Identifying the stress in the Winkler layer as $q(r)$, the transformed expressions for the surface displacements, denoted by $\bar{w}_1^0(\xi)$ and $\bar{w}_2^0(\xi)$, can be reduced to the generalized form

$$\bar{w}_n^0(\xi) = \frac{a(1-\nu_n)}{G_n \xi} [\bar{q}^0(\xi) - \bar{S}_n^0(\xi)] \tag{6}$$

where G_n and ν_n, $(n = 1,2)$ are the elastic parameters of the halfspaces 1 and 2 respectively, and

$$\bar{S}_n^0(\xi) = \frac{P_n}{4\pi} [2 + \frac{\xi c_n/a}{(1-\nu_n)}] e^{-\xi c_n/a} \; ; \; (n = 1,2) \tag{7}$$

are the equivalent representations of (5) for the regions 1 and 2 respectively Also, the transformed constitutive relationship for the Winkler layer is

$$\bar{q}^0(\xi) = -k [\bar{w}_1^0(\xi) + \bar{w}_2^0(\xi)] \tag{8}$$

where k is a material parameter. The elimination of $\bar{w}_n^0(\xi)$ between (6) and (8), and the subsequent application of the inversion theorem (3) yields the following expression for the stress in the Winkler layer:

$$\frac{q(r)}{(ka)} = \int_0^\infty \frac{(1-\nu_1)}{G_1 a^2} \frac{\{\bar{S}_1^0(\xi) + \Gamma \bar{S}_2^0(\xi)\}}{\left[\frac{ka}{\xi G_1} (1-\nu_1) [1+\Gamma] + 1\right]} J_0(\xi r/a) \, d\xi \tag{9}$$

where

$$\Gamma = \frac{(1-\nu_2) G_1}{(1-\nu_1) G_2}$$

is a 'relative stiffness' parameter.

Similarly, the surface displacements of the respective halfspaces can be presented in the contracted form

$$\frac{w_m(r)}{[ka^2\ (1-\nu_m)/G_m]} = \int_0^\infty \frac{\left[\frac{(1-\nu_n)}{\xi a^2 G_n}\ \{\bar{S}_n^0(\xi) - \bar{S}_m^0(\xi)\} - \bar{S}_m^0(\xi)\right]}{\left[1 + \frac{ka}{\xi}\{\frac{(1-\nu_m)}{G_m} + \frac{(1-\nu_n)}{G_n}\}\right]}\ J_0(\xi r/a)\ d\xi\ . \tag{10}$$

The specific expressions for the surface displacements of the halfspace regions 1 and 2 are recovered by substituting $m = 1$, $n = 2$ and $m = 2$, $n = 1$ in (10), respectively. In addition, the contact stress $q(r)$ can be directly employed to derive appropriate expressions for the stresses and displacements in the two halfspace regions.

Evaluation of The Infinite Integrals

The general numerical evaluation of the integral expressions for the surface displacements $w_m(r)$ and the contact stress $q(r)$ can be performed by using a direct numerical integration technique. Briefly, such numerical integration is performed by representing the integrand as an infinite series bounded by subsequent zeros of $J_0(\xi r/a)$. The application of a Gauss-Legendre quadrature technique for the evaluation of each interval of the integrand yields rapidly convergent results. Alternative procedures are also discussed by Sneddon et al. [5].

It is, however, of interest to note that the integral representations (9) and (10) do infact reduce to very compact closed forms for specific values of r. For example, the expression for the contact stress (q) when evaluated at the origin (r = 0) gives

$$q(0) = \sum_{\alpha=1}^{2} \frac{kP_\alpha c_\alpha}{4\pi a^2 G_\alpha}\ [\lambda(\mu_\alpha - \lambda)e^{\lambda c_\alpha/a}\ \mathrm{Ei}(-\lambda c_\alpha/a) + \frac{a}{c_\alpha}\{\mu_\alpha - \lambda + \frac{a}{c_\alpha}\}]\ . \tag{11}$$

Similarly, the surface displacements $w_m(r)$ when evaluated at the origin give

$$w_m(0) = \frac{P_m c_m}{4\pi a^2 G_m}\ [\frac{a}{c_m}\{\lambda - \mu_m - \frac{a}{c_m}\} + \lambda\{\lambda - \mu_m\}e^{\lambda c_m/a}\ \mathrm{Ei}(-\lambda c_m/a)] +$$

$$+ \frac{(1-\nu_m)\ kP_n c_n}{4\pi G_m G_n a} [\{\frac{a}{c_n} + (\lambda-\mu_n) e^{\lambda c_n/a}\ \mathrm{Ei}(-\lambda c_n/a)\} + \tag{12}$$

$$- \{\frac{a}{c_m} + (\lambda-\mu_m) e^{\lambda c_m/a}\ \mathrm{Ei}(-\lambda c_m/a)\} (\frac{1-\nu_n}{1-\nu_m}\ \frac{P_m c_m}{P_n c_n})] \ .$$

In (11) and (12) $|\arg \lambda| < \pi$; $(c_i/a) > 0$ and

$$\mu_\alpha = \frac{2(1-\nu_\alpha)a}{c_\alpha} \ ; \quad \lambda = ka\ [\frac{(1-\nu_m)}{G_m} + \frac{(1-\nu_n)}{G_n}] \ ; \tag{13}$$

also $\mathrm{Ei}(-x)$ is the exponential integral, which is related to the function $E_1(x)$ according to $\mathrm{Ei}(-x) = - E_1(x)$; tabulated numerical values for $E_1(x)$ are given by Pagurova [6] and Abramowitz and Stegun [7]. For the special case of identical loading by identical halfspaces the equations (11) and (12) reduce to the convenient forms

$$\begin{aligned} \frac{q(0)}{[Pk/2\pi cG]} &= \lambda\{\mu-\lambda\}e^{\lambda}\mathrm{Ei}(-\lambda) + \{\mu - \lambda + 1\} \ , \\ \frac{w(0)}{[P/4\pi cG]} &= \lambda\{\lambda-\mu\}e^{\lambda}\mathrm{Ei}(-\lambda) + \{\lambda - \mu - 1\} \ , \end{aligned} \tag{14}$$

where the length parameter a has been set equal to $c_i = c$. It is evident that the relationships (14) are also in agreement with equivalent results derived from the constitutive relationship for the Winkler layer (8) evaluated at $r = 0$, i.e. $q(0) = - 2kw(0)$.

Acknowledgements

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References

[1] I.N. Sneddon, Fourier Transforms, McGraw-Hill, N.Y. (1951)

[2] J.N. Goodier, Fracture, II, Liebowitz, H., Ed. Academic Press, N.Y. (1968)

[3] W.R. Dean, H.W. Parsons and I.N. Sneddon, Proc. Camb. Phil. Soc., 40, 5 (1944)

[4] R.D. Mindlin, Physics, 7, 195 (1936)

[5] I.N. Sneddon, G.M.L. Gladwell and S. Coen, Letters Appl. and Enging. Sci 3, 1 (1975)

[6] V.I. Pagurova, Tables of the Exponential Integral $E_\nu(x) = \int_1^\infty e^{-xu} u^{-\nu} du$, Pergamon Press, Oxford (1961)

[7] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, Dover, N.Y. (1965)