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Asymmetric loading of an externally cracked elastic solid by an in-plane penny-shaped inclusion

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This paper examines the elastostatic problem related to the interaction between an externally cracked solid and a centrally placed penny-shaped rigid inclusion located in the plane of the crack. The inclusion is subjected to an in-plane force. The integral equations governing the problem are solved in a numerical fashion to generate the lateral translational stiffness of the inclusion and the stress intensity factor at the boundary of the external circular crack.

1. Introduction

The category of problems which deal with elastic media reinforced with either elastic or rigid inclusions is important to the study of composite multiphase solids. Extensive accounts of inclusion and related problems in classical elasticity are given in [1–3]. Disc shaped inclusions are a particular limiting case of the general class of three-dimensional ellipsoidal or spheroidal inclusions. The disc inclusion problem has received extensive attention owing to the mathematical simplicity associated with the formulation of such problems. A number of investigators have examined disc inclusion problems related to an elastic solid to evaluate effects such as transverse isotropy of the elastic solid, annular and elliptical configuration of the inclusion, flexural behaviour of the inclusion, influence of externally applied loads, traction free boundaries, bimaterial regions and delaminations at the inclusion-elastic medium interface. A comprehensive account of the disc inclusion problem in classical elasticity will be given in a forthcoming communication [4].

A review of the literature on disc inclusion problems indicates that the group of problems which focus on the interaction between cracks and inclusions has received only limited attention. Examined in [5] is the problem of partial bonding between the inclusion and the elastic medium while others [6–8] have investigated the class of problems in which a penny-shaped crack is internally indented by a disc inclusion.

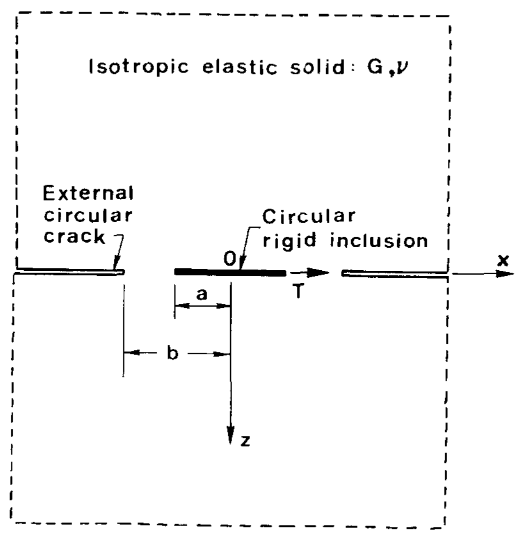

Fig. 1. Penny-shaped inclusion embedded at a cracked plane.

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This latter group of problems has applications in the study of thermally and environmentally induced fracture and degradation of multiphase composites.

This paper is concerned with the problem of the interaction between an external circular crack and a rigid disc inclusion which is located at the centre of the intact elastic region. The inclusion, which is in perfect bonded contact with the elastic solid, is subjected to an in-plane force T (Fig. 1). The inclusion problem is examined in relation to a set of mixed boundary conditions associated with a half space region. The mixed boundary value problem is formulated via solutions derived from a Hankel-transform development of the governing field equations. The integral equations associated with the mixed boundary conditions are solved in a numerical fashion to generate the translational stiffness value for the inclusion and the crack-opening mode stress intensity factor at the boundary of the external crack. The numerical results are also compared with certain exact analytical results derived for the inclusion embedded in an uncracked elastic solid or a completely crack elastic solid.

2. Fundamental equations

In examining the stated asymmetric problem in classical elasticity, it is convenient to employ the functions $\varphi(r, \theta, z)$ and $\psi(r, \theta, z)$ as proposed in [9]. These functions are special reductions of the more general class of Neuber–Papkovich functions [10] and satisfy the differential equations

$$\nabla^2 \nabla^2 \varphi(r, \theta, z) = 0 \tag{1}$$

and

$$\nabla^2 \psi(r, \theta, z) = 0 \tag{2}$$

where ∇^2 is Laplace's operator referred to the generalized cylindrical polar coordinate system. The displacement and stress components referred to the cylindrical polar coordinate system can be expressed in terms of $\varphi(r, \theta, z)$ and $\psi(r, \theta, z)$ in the following forms:

$$2Gu_r = -\frac{\partial^2\varphi}{\partial r\,\partial z} + \frac{2}{r}\frac{\partial\psi}{\partial\theta} \tag{3}$$

$$2Gu_\theta = -\frac{1}{r}\frac{\partial^2\varphi}{\partial\theta\,\partial z} - 2\frac{\partial\psi}{\partial r} \tag{4}$$

$$2Gu_z = 2(1-\nu)\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2} \tag{5}$$

and

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\varphi + \frac{\partial}{\partial\theta}\left(\frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\right)\psi \tag{6}$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi - \frac{1}{r}\frac{\partial}{\partial\theta}\left(2\frac{\partial}{\partial r} - \frac{2}{r}\right)\psi \tag{7}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\varphi \tag{8}$$

$$\sigma_{\theta z} = \frac{1}{r}\frac{\partial}{\partial\theta}\left\{(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\varphi - \frac{\partial^2\psi}{\partial r\,\partial z} \tag{9}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\varphi + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta\,\partial z} \tag{10}$$

$$\sigma_{r\theta} = \frac{1}{r}\frac{\partial^2}{\partial\theta\,\partial z}\left(\frac{1}{r} - \frac{\partial}{\partial r}\right)\varphi - \left(2\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial z^2}\right)\psi \tag{11}$$

where G and ν are the linear elastic shear modulus and Poisson's ratio, respectively.

3. The interaction between the disc inclusion and the external crack

Attention will be focused on the problem of a penny-shaped rigid inclusion of radius a which is embedded in bonded contact with an isotropic elastic medium. The plane containing the disc inclusion is weakened by an external circular crack of radius b ($b > a$) (Fig. 1). The disc inclusion is subjected to a central in-plane force T which acts in the x-direction. The in-plane displacement of the rigid circular inclusion is denoted by δ. Examining the state of symmetry and asymmetry associated with the deformation induced by the displaced inclusion, it is evident that the problem could be formulated in relation to single half space region occupying $z \geqslant 0$. The relevant displacement and traction boundary conditions associated with the inclusion problem are as follows

$$u_r(r, \theta, 0) = \delta \cos\theta, \quad 0 \leqslant r \leqslant a \tag{12}$$

$$u_\theta(r, \theta, 0) = -\delta \sin\theta, \quad 0 \leqslant r \leqslant a \tag{13}$$

$$u_z(r, \theta, 0) = 0, \quad 0 \leqslant r \leqslant b \tag{14}$$

and

$$\sigma_{rz}(r, \theta, 0) \sin\theta + \sigma_{\theta z}(r, \theta, 0) \cos\theta = 0, \quad a < r < \infty \tag{15}$$

$$\sigma_{rz}(r, \theta, 0) \cos\theta - \sigma_{\theta z}(r, \theta, 0) \sin\theta = 0, \quad a < r < \infty \tag{16}$$

$$\sigma_{zz}(r, \theta, 0) = 0, \quad b < r < \infty. \tag{17}$$

For the integral equation formulation of the mixed boundary value problem posed by eqs. (12)–(17), solutions of eqs. (1) and (2) will be found by application of Hankel transforms. Also, the displacement and stress fields derived from $\varphi(r, \theta, z)$ and $\psi(r, \theta, z)$ should reduce to zero as $(r^2 + z^2)^{1/2} \to \infty$. Following the works in [9,11], the relevant solutions take the form

$$\varphi(r, \theta, z) = \left[\int_0^\infty \zeta[A(\zeta) + zB(\zeta)]\, e^{-\zeta z} J_1(\zeta r)\, d\zeta\right] \cos\theta \tag{18}$$

and

$$\psi(r, \theta, z) = \left[\int_0^\infty \zeta A^*(\zeta)\, e^{-\zeta z} J_1(\zeta r)\, d\zeta\right] \sin\theta \tag{19}$$

where $A(\zeta)$, $B(\zeta)$ and $A^*(\zeta)$ are arbitrary functions which are to be determined by satisfying the boundary conditions in eqs. (12)–(17) on the plane $z = 0$. Using the integral representations for $\varphi(r, \theta, z)$ and $\psi(r, \theta, z)$ given by eqs. (18) and (19) in eqs. (3)–(11) for the displacements and stresses, it can be shown that the mixed boundary conditions in eqs. (12)–(17) yield the following system of integral equations

$$\int_0^\infty [\zeta\{\zeta A(\zeta) - B(\zeta)\}\{r\zeta J_0(\zeta r) - J_1(\zeta r)\} + 2\zeta A^*(\zeta) J_1(\zeta r)]\, d\zeta = 2G\delta r, \quad 0 \leqslant r \leqslant a \tag{20}$$

$$\int_0^\infty [\zeta\{-\zeta A(\zeta) + B(\zeta)\} J_1(\zeta r) - 2\zeta A^*(\zeta)\{r\zeta J_0(\zeta r) - J_1(\zeta r)\}]\, d\zeta = -2G\delta r, \quad 0 \leqslant r \leqslant a \tag{21}$$

$$\int_0^\infty \zeta^2[\zeta A(\zeta) + 2(1 - 2\nu) B(\zeta)]\, J_1(\zeta r)\, d\zeta = 0, \quad 0 \leqslant r \leqslant b \tag{22}$$

$$2\int_0^\infty \zeta^2[\zeta A(\zeta) - 2\nu B(\zeta)]\, J_1(\zeta r)\, d\zeta + \int_0^\infty \zeta^2 A^*(\zeta)[\zeta r J_0(\zeta r) - 2J_1(\zeta r)]\, d\zeta$$
$$+ r\int_0^\infty \zeta^3[-\zeta A(\zeta) + 2\nu B(\zeta)]\, J_0(\zeta r)\, d\zeta = 0, \quad a < r < \infty \tag{23}$$

$$\cos 2\theta\left[\int_0^\infty \zeta^2\{\zeta A(\zeta)-2\nu B(\zeta)\}J_1(\zeta r)\,\mathrm{d}\zeta-\int_0^\infty \zeta^2 A^*(\zeta)J_1(\zeta r)\,\mathrm{d}\zeta\right]$$
$$+r\cos^2\theta\int_0^\infty \zeta^3\{-\zeta A(\zeta)+2\nu B(\zeta)\}J_0(\zeta r)\,\mathrm{d}\zeta$$
$$-r\sin^2\theta\int_0^\infty \zeta^3 A^*(\zeta)J_0(\zeta r)\,\mathrm{d}\zeta=0,\quad a<r<\infty \tag{24}$$

$$\int_0^\infty \zeta^3[\zeta A(\zeta)+(1-2\nu)B(\zeta)]\,J_1(\zeta r)\,\mathrm{d}\zeta=0,\quad b<r<\infty. \tag{25}$$

From eq. (23), there results

$$\int_0^\infty \zeta^2[\zeta A(\zeta)-2\nu B(\zeta)-A^*(\zeta)]\,J_1(\zeta r)\,\mathrm{d}\zeta=\frac{r}{2}\int_0^\infty \zeta^3[\zeta A(\zeta)-2\nu B(\zeta)-A^*(\zeta)]\,J_0(\zeta r)\,\mathrm{d}\zeta,$$
$$a<r<\infty. \tag{26}$$

Using eq. (26), eq. (24) can be reduced to the form

$$r\int_0^\infty \zeta^3[\{\zeta A(\zeta)-2\nu B(\zeta)\}+A^*(\zeta)]\,J_0(\zeta r)\,\mathrm{d}\zeta=0,\quad a<r<\infty. \tag{27}$$

Now using the relationship

$$\zeta r J_2(\zeta r)=2J_1(\zeta r)-\zeta r J_0(\zeta r) \tag{28}$$

the result for eq. (26) can be rewritten as

$$r\int_0^\infty \zeta^3[\zeta A(\zeta)-2\nu B(\zeta)-A^*(\zeta)]\,J_2(\zeta r)\,\mathrm{d}\zeta=0,\quad a<r<\infty. \tag{29}$$

Performing other algebraic manipulation, the system of integral eqs. (20)–(25) can be rewritten in the form

$$H_0[\zeta\{\zeta A(\zeta)-B(\zeta)+2A^*(\zeta)\};\ r]=4G\delta,\quad 0\leqslant r\leqslant a \tag{30}$$
$$H_2[\zeta\{\zeta A(\zeta)-B(\zeta)-2A^*(\zeta)\};\ r]=0,\quad 0\leqslant r\leqslant a \tag{31}$$
$$H_1[\zeta\{\zeta A(\zeta)+2(1-2\nu)B(\zeta)\};\ r]=0,\quad 0<r<b \tag{32}$$
$$H_0[\zeta^2\{\zeta A(\zeta)-2\nu B(\zeta)+A^*(\zeta)\};\ r]=0,\quad a<r<\infty \tag{33}$$
$$H_2[\zeta^2\{\zeta A(\zeta)-2\nu B(\zeta)-A^*(\zeta)\};\ r]=0,\quad a<r<\infty \tag{34}$$
$$H_1[\zeta^2\{\zeta A(\zeta)+(1-2\nu)B(\zeta)\};\ r]=0,\quad b<r<\infty \tag{35}$$

where $H_n[f(r);\ r]$ is the Hankel transform of order n which is defined by

$$H_n[f(\zeta);\ r]=\int_0^\infty \zeta f(\zeta)J_n(\zeta r)\,\mathrm{d}\zeta. \tag{36}$$

Introduce the functions $L(\zeta)$, $M(\zeta)$ and $N(\zeta)$ such that

$$L(\zeta)=\zeta^2[\zeta A(\zeta)-2\nu B(\zeta)+A^*(\zeta)] \tag{37}$$
$$M(\zeta)=\zeta^2[\zeta A(\zeta)-2\nu B(\zeta)-A^*(\zeta)] \tag{38}$$
$$N(\zeta)=\zeta^2[\zeta A(\zeta)+2(1-2\nu)B(\zeta)]. \tag{39}$$

Alternatively, note that

$$\zeta A(\zeta)=[2\nu N(\zeta)+(1-2\nu)\{L(\zeta)+M(\zeta)\}]/(2\zeta^2(1-\nu)) \tag{40}$$
$$B(\zeta)=[2N(\zeta)-L(\zeta)-M(\zeta)]/(4\zeta^2(1-\nu)) \tag{41}$$
$$A^*(\zeta)=[L(\zeta)-M(\zeta)]/(2\zeta^2) \tag{42}$$

With the aid of the substitution indicated by eqs. (37)–(42), the integral eqs. (30)–(35) can be written as

$$H_0\left[\zeta^{-1}\left\{L(\zeta)-\frac{M(\zeta)}{(7-8\nu)}-\frac{2(1-2\nu)}{(7-8\nu)}N(\zeta)\right\};\ r\right]=\frac{16G\delta(1-\nu)}{(7-8\nu)},\quad 0\leqslant r\leqslant a \tag{43}$$

$$H_2\left[\zeta^{-1}\left\{M(\zeta)-\frac{L(\zeta)}{(7-8\nu)}-\frac{2(1-2\nu)}{(7-8\nu)}N(\zeta)\right\};\ r\right]=0,\quad 0<r<a \tag{44}$$

$$H_1\left[\zeta^{-1}N(\zeta);\ r\right]=0,\quad 0<r<b \tag{45}$$

$$H_0\left[L(\zeta);\ r\right]=0,\qquad a<r<\infty \tag{46}$$

$$H_2\left[M(\zeta);\ r\right]=0,\qquad a<r<\infty \tag{47}$$

$$H_1\left[\left\{L(\zeta)+M(\zeta)+\frac{2N(\zeta)}{(1-2\nu)}\right\};\ r\right]=0,\quad b<r<\infty. \tag{48}$$

Following the work in [11], introduce the representations

$$L(\zeta)=\int_0^a\varphi_1(t)\cos(\zeta t)\,\mathrm{d}t=\frac{\varphi_1(a)\sin(\zeta a)}{\zeta}-\frac{1}{\zeta}\int_0^a\varphi_1'(t)\sin(\zeta t)\,\mathrm{d}t \tag{49}$$

$$M(\zeta)=\int_0^a t\varphi_2(t)J_2(\zeta t)\,\mathrm{d}t \tag{50}$$

$$N(\zeta)=\int_b^\infty\varphi_3(t)\sin(\zeta t)\,\mathrm{d}t \tag{51}$$

where the prime denotes the derivative with respect to t and φ_3 satisfies the constraint $\varphi_3(\infty)=0$. It is evident that eqs. (49)–(51) identically satisfy eqs. (45)–(47) and the eqs. (43) and (48) yield the following Volterra-type integral equations

$$\int_0^r\frac{\varphi_1(t)\,\mathrm{d}t}{(r^2-t^2)^{1/2}}=\frac{16G\delta(1-\nu)}{(7-8\nu)}+\int_0^\infty\left[\frac{M(\zeta)+2(1-2\nu)N(\zeta)}{(7-8\nu)}\right]J_0(\zeta r)\,\mathrm{d}\zeta,\quad 0\leqslant r\leqslant a \tag{52}$$

$$\frac{\partial}{\partial r}\int_r^\infty\frac{\varphi_3(t)\,\mathrm{d}t}{(t^2-r^2)^{1/2}}=\frac{(1-2\nu)}{2}\int_0^\infty\zeta\left[L(\zeta)+M(\zeta)\right]J_1(\zeta r)\,\mathrm{d}\zeta,\quad b<r<\infty. \tag{53}$$

Equations (52) and (53) can be inverted to give

$$\varphi_1(t)=\frac{32G\delta(1-\nu)}{(7-8\nu)\pi}+\frac{2}{\pi}\int_0^\infty\left[\frac{M(\zeta)+2(1-2\nu)N(\zeta)}{(7-8\nu)}\right]\cos(\zeta t)\,\mathrm{d}\zeta,\quad 0\leqslant t\leqslant a \tag{54}$$

$$\varphi_3(t)=-\frac{(1-2\nu)}{\pi}\int_0^\infty\left[L(\zeta)+M(\zeta)\right]\sin(\zeta t)\,\mathrm{d}t,\quad b<t<\infty. \tag{55}$$

Using eqs. (49) and (50), it can be deduced that

$$\varphi_1(t)=\frac{32G\delta(1-\nu)}{(7-8\nu)\pi}+\frac{2}{\pi}\left[\frac{1}{(7-8\nu)}\int_0^a\frac{\varphi(u)(u^2-2t^2)H(u-t)\,\mathrm{d}u}{u(u^2-t^2)^{1/2}}+\frac{2(1-2\nu)}{(7-8\nu)}\int_b^\infty\frac{u\varphi_3(u)\,\mathrm{d}u}{(u^2-t^2)}\right],\quad 0\leqslant t\leqslant a \tag{56}$$

$$\varphi_3(t)=\frac{(1-2\nu)}{\pi}\left[\int_0^a\frac{u^3\varphi_2(u)\,\mathrm{d}u}{(t^2-u^2)^{1/2}\left\{t+(t^2-u^2)^{1/2}\right\}^2}-t\int_0^a\frac{\varphi_1(u)\,\mathrm{d}u}{(t^2-u^2)}\right],\quad b<t<\infty \tag{57}$$

where $H(u-t)$ is the Heaviside step function. In obtaining these results we make use of the integral relationships

$$\int_0^\infty \frac{\sin(\zeta t)\sin(\zeta u)\,d\zeta}{\zeta} = \tfrac{1}{2}\ln\left|\frac{u+t}{u-t}\right| \tag{58}$$

$$\int_0^\infty \cos(\zeta t)J_2(\zeta u)\,d\zeta = \frac{(u^2-2t^2)H(u-t)}{u^2(u^2-t^2)^{1/2}} \tag{59}$$

$$\int_0^\infty \sin(\zeta t)J_2(\zeta u)\,d\zeta = \frac{-u^2}{(t^2-u^2)^{1/2}\left\{t+(t^2-u^2)^{1/2}\right\}^2}. \tag{60}$$

Now, using eq. (50) in (44) it can be shown that

$$\int_0^a t\varphi_2(t)K(r,t)\,dt = \int_0^\infty \left[\frac{L(\zeta)+2(1-2\nu)N(\zeta)}{(7-8\nu)}\right]J_2(\zeta r)\,d\zeta, \quad 0<r<a \tag{61}$$

where

$$K(r,t) = \int_0^\infty J_2(\zeta r)J_2(\zeta t)\,d\zeta = \frac{2}{\pi(rt)^2}\int_0^{\min(r,t)} \frac{s^4\,ds}{\left[(t^2-s^2)(r^2-s^2)\right]^{1/2}}. \tag{62}$$

Using (62), eq. (61) can be expressed as an integral equation of the Abel-type

$$\int_0^r \frac{s^4\,ds}{(r^2-s^2)^{1/2}}\int_s^a \frac{\varphi_2(t)\,dt}{t(t^2-s^2)^{1/2}} = \frac{\pi r^2}{2}\int_0^\infty \left[\frac{L(\zeta)+2(1-2\nu)N(\zeta)}{(7-8\nu)}\right]J_2(\zeta r)\,d\zeta, \quad 0<r<a \tag{63}$$

which in turn can be reduced to the form

$$\int_s^a \frac{\varphi_2(t)\,dt}{t(t^2-s^2)^{1/2}} = (\pi/2)^{1/2}s^{-3/2}\int_0^\infty \zeta^{1/2}\left[\frac{L(\zeta)+2(1-2\nu)N(\zeta)}{(7-8\nu)}\right]J_{3/2}(\zeta s)\,d\zeta, \quad 0<s<a. \tag{64}$$

Inverting eq. (64), an integral expression for $\varphi_2(t)$ can be obtained in terms of $L(\xi)$ and $N(\xi)$. Substituting the expressions for $L(\xi)$ and $N(\xi)$ defined in terms of eqs. (49) and (50) in the inverted result for eq. (64), an integral equation governing $\varphi_1(t)$, $\varphi_2(t)$ and $\varphi_3(t)$ is obtained:

$$\varphi_2(t) = \int_0^a \varphi_1(u)K_1(u,t)\,du + \int_b^\infty \varphi_3(u)K_2(u,t)\,du, \quad 0<t<a \tag{65}$$

where

$$K_1(u,t) = \frac{1}{(7-8\nu)}(\pi/2)^{-1/2}\int_0^\infty \zeta^{1/2}I(t,\zeta)\cos(\zeta u)\,d\zeta \tag{66}$$

$$K_2(u,t) = \frac{2(1-2\nu)}{(7-8\nu)}(\pi/2)^{-1/2}\int_0^\infty \zeta^{1/2}I(t,\zeta)\sin(\zeta u)\,d\zeta \tag{67}$$

and

$$I(t,\zeta) = t\frac{d}{dt}\int_a^t \frac{J_{3/2}(\zeta s)\,ds}{s^{1/2}(s^2-t^2)^{1/2}} \tag{68}$$

The system of coupled integral eqs. (56), (57) and (65) need to be solved in a numerical fashion to generate results of technical interest. The details of the numerical procedures are outlined in the Appendix.

4. In-plane translational stiffness of the inclusion

The load-displacement relationship for the in-plane translation of the disc inclusion located at the cracked plane can be obtained by evaluating the integral

$$T = -2\int_0^{2\pi}\int_0^a [\sigma_{rz}(r, \theta, 0)\cos\theta - \sigma_{\theta z}(r, \theta, 0)\sin\theta]\, r\, \mathrm{d}r\, \mathrm{d}\theta. \tag{69}$$

Using the developments presented in Section 3, it can be shown that eq. (69) reduces to

$$T = 2\pi\int_0^a r\, \mathrm{d}r\int_0^\infty \zeta L(\zeta) J_0(\zeta r)\, \mathrm{d}\zeta. \tag{70}$$

Using eq. (49), it is noted that

$$\int_0^\infty \zeta L(\zeta) J_0(\zeta r)\, \mathrm{d}\zeta = \left[\frac{\varphi_1(a)}{(a^2 - r^2)^{1/2}} - \int_r^a \frac{\varphi_1'(t)\, \mathrm{d}t}{(t^2 - r^2)^{1/2}}\right]. \tag{71}$$

From eqs. (70) and (71), the in-plane load-displacement relationship is obtained from the result:

$$T = 2\pi\int_0^a \varphi_1(t)\, \mathrm{d}t. \tag{72}$$

As shown in [5,12,13], the in-plane translational stiffness of a rigid circular inclusion located in an uncracked (i.e. $b \to \infty$) elastic solid takes the form

$$T = \frac{64G\delta a(1-\nu)}{(7-8\nu)}. \tag{73}$$

Consequently, a non-dimensional result for eq. (72) can be established:

$$\bar{T} = \frac{T(7-8\nu)}{64G\delta a(1-\nu)}. \tag{74}$$

5. Stress intensity factor at the boundary of the externally cracked region

From the state of symmetry of the deformation about the plane $z = 0$ and from the traction free nature of the cracked region ($b < r < \infty$), it is evident that the nonzero stress intensity factor at the crack boundary is of the Mode I type, which is induced by the normal stress σ_{zz}. From the results presented earlier, it can be shown that

$$\sigma_{zz}(r, \theta, 0) = \cos\theta\int_0^\infty \zeta\left[\frac{(1-2\nu)}{4(1-\nu)}L(\zeta) + \frac{(1-2\nu)}{4(1-\nu)}M(\zeta) + \frac{N(\zeta)}{2(1-\nu)}\right] J_1(\zeta r)\, \mathrm{d}\zeta, \quad a < r < b. \tag{75}$$

From eq. (51), note that

$$N(\zeta) = \frac{\varphi_3(b)\cos(\zeta b)}{\zeta} + \frac{1}{\zeta}\int_b^\infty \varphi_3'(t)\cos(\zeta t)\, \mathrm{d}t \tag{76}$$

where the prime denotes differentiation with respect to t. Upon substituting the values of $L(\zeta)$, $M(\zeta)$ and $N(\zeta)$ from eqs. (49), (50) and (76) in (75), eq. (75) can be rewritten as

$$\begin{aligned}\sigma_{zz}(r,\theta,0) &= \cos\theta\left[\frac{(1-2\nu)}{4(1-\nu)}\left\{\frac{a\varphi_1(a)}{r(r^2-a^2)^{1/2}} - \frac{1}{r}\int_0^a \frac{t\varphi_1'(t)\,\mathrm{d}t}{(r^2-t^2)^{1/2}}\right\}\right. \\ &\quad + \frac{1}{2(1-\nu)}\left\{\frac{\varphi_3(b)}{r}\left(1-\frac{b}{(b^2-r^2)^{1/2}}\right) + \frac{1}{r}\int_b^\infty \varphi_3'(t)\left(1-\frac{t}{(t^2-r^2)^{1/2}}\right)\mathrm{d}t\right\} \\ &\quad \left. + \frac{(1-2\nu)}{4(1-\nu)}\frac{\partial}{\partial r}\frac{1}{r}\int_0^\infty u\varphi_2(u)F\left(\tfrac{3}{2}, -\tfrac{1}{2}; 1; \frac{u^2}{r^2}\right)\mathrm{d}u\right], \quad a < r < b \end{aligned} \tag{77}$$

where F denotes the hypergeometric function. The result of interest to fracture mechanics concerns the flaw opening mode stress intensity factor at the boundary of the externally crack region, which is defined by

$$K_1 = \lim_{r\to b^-} [2(b-r)]^{1/2}\sigma_{zz}(r,\theta,0). \tag{78}$$

From eqs. (77) and (78), it is found that

$$K_1 = -\frac{\varphi_3(b)\cos\theta}{2(1-\nu)\sqrt{b}}. \tag{79}$$

A non-dimensional estimate for the stress intensity factor can be written in the form

$$\overline{K}_1 = \frac{K_1(7-8\nu)\pi^2 b^{3/2}}{16G\delta a(1-2\nu)\cos\theta} \tag{80}$$

6. Numerical results and conclusions

The numerical procedures outlined in the Appendix have been used to evaluate the in-plane translational stiffness of the embedded disc inclusion and the flaw opening mode stress intensity factor at the crack tip. Figure 2 illustrates the manner in which the in-plane translational stiffness of the inclusion is influenced by the extent of the cracked region (i.e. the aspect ratio a/b). As indicated previously, when

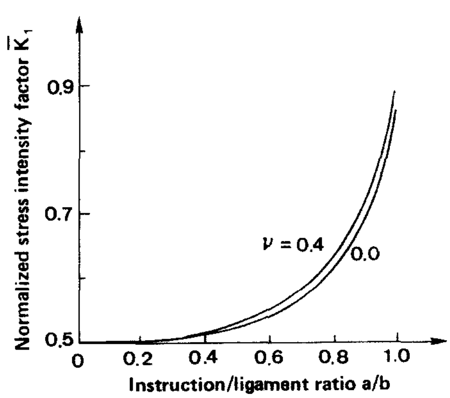

Fig. 2. In-plane translational stiffness of the penny-shaped inclusion located at the cracked plane.

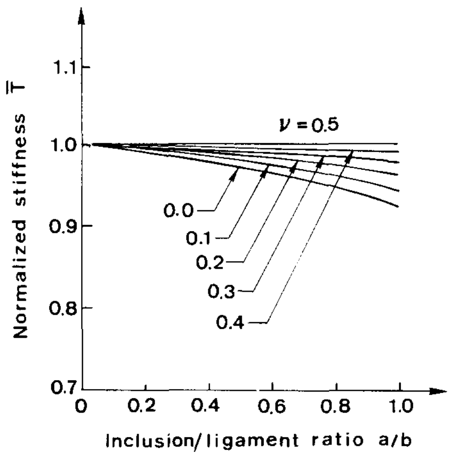

Fig. 3. Crack opening mode stress intensity factor at the boundary of the external circular crack.

$a/b \to 0$, the problem reduces to that of the translation of the disc inclusion in an uncracked elastic solid of infinite extent. The numerical results presented in Fig. 2 have been normalized with respect to the axial stiffness result for the uncracked solid. As $a/b \to 1$, the problem reduces to that of the translation of an inclusion which is bonded to two discrete half space regions. It may be noted that when $a/b = 1$, the analysis of the problem must take into consideration the effects of oscillatory stress singularities which can occur at the boundary of the disc inclusion. The method of analysis developed here does not account for such effects and consequently the results are valid for $a/b < 1$. The exact solution for the in-plane stiffness of a rigid disc inclusion embedded in bonded contact with two half spaces can be, however, obtained by making use of the results in [14]. It can be shown that

$$T = 16G\delta a \Bigg/ \left[1 + \frac{(1-2\nu)}{\ln(3-4\nu)} \right] \tag{81}$$

The results derived from the numerical scheme have also been compared with results derived from eq. (81). It is found that the maximum discrepancy of approximately 0.03 percent occurs when $\nu = 0$. As $\nu \to 1/2$, the oscillatory singular behaviour in the stress field at the inclusion boundary disappears and the exact closed form result for the in-plane translational stiffness takes the form

$$T = 32G\delta a/3 \tag{82}$$

In view of the comments outlined above, the crack opening mode stress intensity factor K_1 at the crack boundary can be evaluated for values of a/b in the range $0 < a/b < 1$. Figure 3 illustrates the manner in which the stress intensity factor K_1 is influenced by the inclusion/ligament ratio (a/b). As is evident from eq. (80), in the limit of material incompressibility the flaw opening mode stress intensity factor K_1 reduces to zero for all choices of $(a/b) \in (0, 1)$. Consequently, results are presented for values of $\nu = 0$ and 0.4.

The integral transform technique and the associated numerical developments presented in the paper form convenient techniques for the treatment of this class of asymmetric problems in the theory of elasticity. The analysis can be extended to include the asymmetric problem in which the inclusion is embedded within a penny-shaped crack. In these circumstances it is important to consider the influences of oscillatory singular stress fields which can be encountered at the inclusion boundary. The results of this investigation indicates that while such oscillatory phenomena have an influence on the stress intensity factors which are evaluated at the inclusion-elastic medium boundary, they have virtually no influence on the load-displacement relationship for the inclusion.

In conclusion, it should be noted that in the problem examined here, the force resultant at infinity has a non-vanishing component of magnitude T in the x-direction. Recent studies [15,16] show the nature of the regularity conditions imposed at infinity influences the magnitude of the stress intensity factor at the boundary of the external crack. When the solution to the problem of the externally cracked solid with a zero force resultant at infinity is required, then certain auxiliary solutions which satisfy the regularity criteria need to be added to the results presented in this paper to recover the stress intensity factor. The evaluation of the translational stiffness of the disc inclusion is uninfluenced by these auxiliary solutions.

Appendix

Integrating, by parts, eq. (68) and by making use of the relation

$$\frac{d}{ds}\left[s^{-3/2}J_{3/2}(\zeta s)\right] = -\zeta s^{-3/2}J_{5/2}(\zeta s) \tag{83}$$

$I(t, \zeta)$ can be written in the form

$$I(t, \zeta) = J_{3/2}(\zeta a)\frac{t^2}{a^{3/2}(a^2-t^2)^{1/2}} + \zeta t^2\int_t^a \frac{J_{5/2}(\zeta s)\,ds}{s^{3/2}(s^2-t^2)^{1/2}} \tag{84}$$

for values of t in the range $0 < t < a$. Upon substituting the function $I(t, \zeta)$ into eqs. (66) and (67), the following expressions for the kernel functions K_1 and K_2 are obtained:

$$K_1(u, t) = \frac{1}{(7-8\nu)}\left(\frac{2}{\pi}\right)^{1/2}\left\{\frac{t^2}{a^{3/2}(a^2-t^2)^{1/2}}I_1(u, a) + \int_t^a \frac{t^2}{s^{3/2}(s^2-t^2)^{1/2}}I_2(u, s)\,ds\right\},\quad 0 < u,\ t < a \tag{85}$$

and

$$K_2(u, t) = \frac{2(1-2\nu)}{(7-8\nu)}\left(\frac{2}{\pi}\right)^{1/2}\left\{\frac{t^2}{a^{3/2}(a^2-t^2)^{1/2}}I_3(u, a) + \int_t^a \frac{t^2}{s^{3/2}(s^2-t^2)^{1/2}}I_4(u, s)ds\right\},\quad b < u < \infty,\ 0 < t < a. \tag{86}$$

In eqs. (85) and (86), the integral functions I_i $(i = 1, 2, 3, 4)$ are given by

$$I_1(u, a) = \int_0^\infty \zeta^{1/2}J_{3/2}(\zeta a)\cos(\zeta u)\,d\zeta \tag{87}$$

$$I_2(u, a) = \int_0^\infty \zeta^{3/2}J_{5/2}(\zeta a)\cos(\zeta u)\,d\zeta \tag{88}$$

$$I_3(u, a) = \int_0^\infty \zeta^{1/2}J_{3/2}(\zeta a)\sin(\zeta u)\,d\zeta \tag{89}$$

$$I_4(u, a) = \int_0^\infty \zeta^{3/2}J_{5/2}(\zeta a)\sin(\zeta u)\,d\zeta. \tag{90}$$

It is known from [17] that

$$\int_0^\infty \zeta^{-1/2}J_{3/2}(\zeta a)\cos(\zeta u)\,d\zeta = \left(\frac{2}{a\pi}\right)^{1/2}\left[1 - \frac{u}{2a}\ln\left(\frac{u+a}{u-a}\right)\right] \tag{91}$$

Differentiating eq. (91) with respect to u we obtain

$$I_3(u, a) = \frac{1}{a}\left(\frac{2}{a\pi}\right)^{1/2}\left[-\frac{ua}{(u^2-a^2)} + \tfrac{1}{2}\ln\left(\frac{u+a}{u-a}\right)\right] \tag{92}$$

for $a < b < u$. Also, differentiating the integral

$$\int_0^\infty \zeta^{-3/2} J_{5/2}(\zeta s) \cos(\zeta u)\,\mathrm{d}\zeta = -\left(\frac{s}{2\pi}\right)^{1/2} \left\{\frac{1}{3} + \frac{(u^2 - s^2)}{s^2}\left[1 - \frac{u}{2s}\ln\left(\frac{u+s}{u-s}\right)\right]\right\} \tag{93}$$

three times with respect to u, there results

$$I_4(u, s) = \frac{1}{s^3}\left(\frac{s}{2\pi}\right)^{1/2} \left\{\frac{2us(5s^2 - 3u^2)}{(u^2 - s^2)^2} + 3\ln\left(\frac{u+s}{u-s}\right)\right\} \tag{94}$$

for $s < a < b < u$. Similarly, explicit expressions for the functions $I_1(u, a)$ and $I_2(u, a)$ are found in the forms

$$I_1(u, a) = \frac{1}{a}\left(\frac{\pi}{2a}\right)^{1/2}, \quad 0 < u < a \tag{95}$$

and

$$I_2(u, a) = \frac{3}{s^2}\left(\frac{\pi}{2s}\right)^{1/2} H(s - u), \quad 0 < u,\ s < a. \tag{96}$$

Substituting eqs. (95) and (96) into (85), the kernel function $K_1(u, t)$ can be expressed in the form

$$K_1(u, t) = \frac{1}{(7 - 8\nu)}\left\{\frac{(2a^2 - t^2)}{at^2(a^2 - t^2)^{1/2}} - \frac{(u^2 - t^2)^{1/2}(2u^2 + t^2)}{t^2 u^3} H(u - t)\right\} \tag{97}$$

where $u > 0$, $t < a$. Also by substituting eqs. (92) and (94) into (86), the kernel function $K_2(u, t)$ can be written as

$$K_2(u, t) = \frac{4(1 - 2\nu)}{(7 - 8\nu)\pi}\left\{\frac{t^2}{a^3(a^2 - t^2)^{1/2}}\left[-\frac{ua}{(u^2 - a^2)} + \tfrac{1}{2}\ln\left(\frac{u+a}{u-a}\right)\right]\right.$$
$$\left. + \int_t^a \frac{t^2}{2s^4(s^2 - t^2)^{1/2}}\left[\frac{2us(5s^2 - 3u^2)}{(u^2 - s^2)^2} + 3\ln\left(\frac{u+s}{u-s}\right)\right]\mathrm{d}s\right\} \tag{98}$$

where $b < u < \infty$ and $0 < t < a$.

The complete system of Fredholm integral eqs. (56), (57) and (65) can be written respectively, as

$$\varphi_1(t) + \int_0^a K_3(u, t)\varphi_2(u)\,\mathrm{d}u + \int_b^\infty K_4(u, t)\varphi_3(u)\,\mathrm{d}u = \frac{32G\delta(1 - \nu)}{(7 - 8\nu)\pi}, \quad 0 \leqslant t \leqslant a, \tag{99}$$

$$\varphi_2(t) - \int_0^a K_1(u, t)\varphi_1(u)\,\mathrm{d}u - \int_b^\infty K_2(u, t)\varphi_3(u)\,\mathrm{d}u = 0, \quad 0 < t < a, \tag{100}$$

$$\varphi_3(t) + \int_0^a K_5(u, t)\varphi_1(u)\,\mathrm{d}u + \int_0^a K_6(u, t)\varphi_2(u)\,\mathrm{d}u = 0, \quad b < t < \infty. \tag{101}$$

The previously undefined kernel functions are

$$K_3(u, t) = \frac{-2}{(7 - 8\nu)\pi}\frac{(u^2 - 2t^2)}{u(u^2 - t^2)^{1/2}} H(u - t), \quad u > 0,\ t < a \tag{102}$$

$$K_4(u, t) = -\frac{4(1 - 2\nu)}{(7 - 8\nu)\pi}\frac{u}{(u^2 - t^2)}, \quad b < u < \infty,\ t < a \tag{103}$$

$$K_5(u, t) = -\frac{(1-2\nu)}{\pi} \frac{t}{(t^2-u^2)}, \quad 0<u<a,\ b<t<\infty \tag{104}$$

$$K_6(u, t) = \frac{(1-2\nu)}{\pi} \frac{u^3}{(t^2-u^2)^{1/2}\left[t+(t^2-u^2)^{1/2}\right]^2}, \quad 0<u<a,\ b<t<\infty. \tag{105}$$

Since the integral eqs. (99)–(101) are defined over $[0,a]$ and $[b,\infty]$, in a discretized solution of the system of equations, N_1 and N_2 segments are applied, respectively to each segment. Therefore, the ends of a segment can be expressed as

$$u_i = \begin{cases} (i-1)h, & 1 \leqslant i \leqslant N_1+1 \\ b+(i-1-N_1)h, & N_1+1<i \leqslant N_1+L \\ 3u_{i-1}-2u_{i-2}, & N_1+L<i \leqslant N_1+N_2+2 \end{cases} \tag{106}$$

where $h=a/N_1$ and L is an arbitrary integer. Equation (106) is used to generate a geometrically increasing segment size in the integral $[b+Lh,\infty]$. Then, the nodes of the segments are

$$t_i = \begin{cases} (u_i+u_{i+1})/2, & 1 \leqslant i \leqslant N_1 \\ (u_{i+1}+u_{i+2})/2, & N_1<i \leqslant N \end{cases} \tag{107}$$

where $N=N_1+N_2$. The complete discretized system of equations can be expressed as

$$[A_{ij}]\{\varphi_j\} = \{B_i\} \tag{108}$$

where $i,\ j=1, 2,\dots, M=N+N_1$. The coefficients of the equations are $A_{ij}=\delta_{ij}+\hat{A}_{ij}$, such that for $1 \leqslant l$, $m \leqslant N_1$

$$\hat{A}_{2l-1,2m} = -\frac{2}{(7-8\nu)\pi}\left[\left\{(u^2-t_l^2)^{1/2}-2t_l\cos^{-1}\left(\frac{t_l}{u}\right)\right\}H(u-t_l)\right]_{u_m}^{u_{m+1}} \tag{109}$$

and

$$\hat{A}_{2l-1,2m-1} = \frac{1}{(7-8\nu)}\cdot\frac{1}{t_l^2}\left[\frac{(2a^2-t_l^2)u}{a(a^2-t_l^2)^{1/2}} - \frac{(4u^2-t_l^2)(u^2-t_l^2)^{1/2}}{2u^2} + \tfrac{3}{2}t_l\cos^{-1}\left(\frac{t_l}{u}\right)\right]_{u_m}^{u_{m+1}} \tag{110}$$

For $1 \leqslant l \leqslant N_1$ and $N_1<m \leqslant N$

$$\hat{A}_{2l-1,m+N_1} = -\frac{2(1-2\nu)}{(7-8\nu)\pi}\left[\ln(u^2-t_l^2)\right]_{u_{m+1}}^{u_{m+2}} \tag{111}$$

and

$$\begin{aligned}\hat{A}_{2l,m+N_1} = -&\frac{4(1-2\nu)}{(7-8\nu)\pi} \\ &\times\left[\frac{t_l^2}{2a^3(a^2-t_l^2)^{1/2}}\{(u+a)\ln(u+a)-(u-a)\ln(u-a)-a\ln(u^2-a^2)\}\right. \\ &+\int_{t_l}^{u}3\left\{(u+s)\ln(u+s)-(u-s)\ln(u-s)\right. \\ &\left.\left.-s\ln(u^2-s^2)-\frac{2s^3}{(u^2-s^2)}\right\}\frac{t_l^2\,\mathrm{d}s}{2s^4(s^2-t_l^2)^{1/2}}\right]_{u_{m+1}}^{u_{m+2}}\end{aligned} \tag{112}$$

For $N_1 < l \leqslant N$ and $1 \leqslant m \leqslant N_1$

$$\hat{A}_{l+N_1,2m-1} = -\frac{(1-2\nu)}{2\pi}\left[\ln\left(\frac{t_l+u}{t_l-u}\right)\right]_{u_m}^{u_{m+1}} \tag{113}$$

and

$$\hat{A}_{l+N_1,2m} = \frac{(1-2\nu)}{\pi}\left[t_l + \left(t_l^2-u^2\right)^{1/2} - 2t_l \ln\left\{\frac{t_l+\left(t_l^2-u^2\right)^{1/2}}{2t_l}\right\}\right]_{u_m}^{u_{m+1}}. \tag{114}$$

We note that all other $\hat{A}_{ij} = 0$. The right hand side of eq. (108) takes the form

$$B_{2l-1} = \frac{32G\delta(1-\nu)}{(7-8\nu)\pi}, \quad 1 \leqslant l \leqslant N_1 \tag{115}$$

and all other $B_i = 0$.

Upon solving eq. (108), the required results for the in-plane stiffness and the stress intensity factor can be expressed in terms of the discretized values for φ_1.

The non-dimensional in-plane stiffness of the embedded inclusion defined by eq. (74) can be expressed in the form

$$\overline{T} = \frac{(7-8\nu)\pi}{32G\delta(1-\nu)}\left\{\frac{1}{N_1}\sum_{i=1}^{N_1}\varphi_{2i-1}\right\}. \tag{116}$$

Similarly, the non-dimensional form of the flaw opening mode stress intensity factor defined by eq. (80) can be expressed in the form

$$\overline{K}_1 = \frac{\pi^2(7-8\nu)}{32G\delta(1-\nu)(1-2\nu)}\left(\frac{b}{a}\right)\varphi_{2N_1+1}. \tag{117}$$

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