Non-linear Interfaces and Fracture Mechanics

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Introduction

In the study of cracks in brittle solids such as composites, ceramic materials, concrete and rocks it is important to consider the conditions that are present at the interfaces of the defects. The classical analyses of such cracks invariably assume that they are open and smooth. This assumption is violated in most situations except in cases where the cracks are opened by tensile stresses normal to its plane. During compressive and shear loads, the interface behaviour is expected to exert a dominant influence on the behaviour of the crack. Owing to the non-linear phenomena associated with the frictional processes it is difficult to utilize classical analytical procedures for the solution of such crack problems. Crack problems with interface non-linearities have a great deal of similarity with contact problems with frictional interface constraints. Such problems have been examined by using incremental formulations of the coupled integral equations (see e.g. de Pater and Kalker [1], Gladwell [2]). Recently Ballarini and Plesha [3] have examined the shear loading of a plane crack with exhibits frictional constraints at the crack surfaces. These analytical investigations examine crack problems and loading configurations which are relatively simple. The boundary element method can be successfully applied to the solution of crack problems with interface non-linearities. Recently Selvadurai and Au [4, 5, 6] have presented several studies which deal with the boundary element analysis of elastic media with non-linear interface constraints. The boundary element technique is particularly effective for the class of problems in which predominantly elastic regions exhibit non-linear interaction through contact phenomena. The present paper applies an incremental boundary element approach to the study of the fracture mechanics aspect of a crack with a non-linear contact region.

The Boundary Element Method

The incremental formulation of the boundary element method is briefly outlined for completeness. Further details of the method are given by Brebbia [7] and Banerjee and Butterfield [8]. In the absence of body forces, the incremental form of the boundary integral equation applicable to an elastic region is given by

$$c_{ij}(P)\dot{u}_j(P) + \int_S T^*_{ij}(P,Q)\dot{u}_j(Q)\,dS = \int_S U^*_{ij}(P,Q)\dot{t}_j(Q)\,dS \tag{1}$$

where Q denotes a general field point, $\dot{u}_j(Q)$ and $\dot{t}_j(Q)$ and the j-th component of an incremental displacement and traction respectively; S is the boundary of the elastic region; c_{ij} is a constant; $i, j = x, y$ for two dimensional problems and $U^*_{ij}(P\,Q)$ and $T^*_{ij}(P,Q)$ are the displacement and traction fundamental solutions given by

$$U^*_{ij}(P,Q) = \frac{1}{8\pi G(1-v)}\{(3-4v)\delta_{ij}\ln(\frac{1}{r}) + r_{,i}r_{,j}\} \tag{2}$$

$$T^*_{ij}(P,Q) = -\frac{1}{4\pi(1-v)r}\{[(1-2v)\delta_{ij} + 2r_{,i}r_{,j}]r_{,n} - (1-2v)[r_{,i}n_j - r_{,j}n_i]\} \tag{3}$$

In (2) and (3) G and v are respectively the shear modulus and Poisson's ratio; r is the distance between (P) and (Q), δ_{ij} is Kronecker's delta function and n_j is the jth component of the outward unit normal on S. The BIE can be reduced to a matrix equation by discretizing S into elements with piecewise continuous variation of displacements and traction over the element. Considering all locations of P on S we can obtain a system matrix equation relating the incremental displacements and tractions on S. For a well posed problem, the boundary S should be prescribed as follows:

(a) boundary with known displacements (S_1) on which

$$\dot{u}_i = \bar{u}_i \tag{4a}$$

(b) boundary with known tractions (S_2) on which

$$\dot{t}_i = \bar{t}_i \tag{4b}$$

(c) boundary on which there is a displacement and traction coupling (S_3)

$$\dot{t}_i = k^{ep}_{ij}\dot{u}_j \tag{4c}$$

where k^{ep}_{ij} is a non-linear relationship at the boundary S_3. The conditions (4a–c) should make up the complete boundary. i.e. $S = S_1 \cup S_2 \cup S_3$. Applications of the boundary conditions of the type (4a) and (4b) into the boundary element stiffness matrix and omitting the boundary condition of the type (4c) we can write

$$\begin{bmatrix} -g_{11} & h_{12} \\ -g_{21} & h_{22} \\ -g_{31} & h_{32} \end{bmatrix} \begin{bmatrix} \dot{t}_1 \\ \\ \dot{u}_1 \end{bmatrix} + \begin{bmatrix} h_{13} & -g_{13} \\ h_{23} & -g_{23} \\ h_{33} & -g_{33} \end{bmatrix} \begin{bmatrix} \dot{u}_3 \\ \\ \dot{t}_3 \end{bmatrix} = \dot{f} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \tag{5}$$

where $\dot{f}$ is a loading factor; $\{\mathbf{B}\}$ is the vector from the prescribed boundary values: $[h_{ij}]$ and $[g_{ij}]$ are the coefficients matrices derived from the fundamental solutions; the values of $i, j = 1,2,3$ indicate the location of the boundary S_1, S_2, S_3. Equation (5) does not have a solution since the boundary conditions on S_3 have not been applied. However, this is the BEM system equation with the boundary condition on S_3 which needs to be determined.

Interface relationships

The boundary element formulation of the problem can be completed by prescribing the constitutive relationships which are applicable to the interface. There are a number of interface responses that can be adopted for this purpose. These can range from Coulomb friction, finite friction to dilative frictional phenomena. We shall consider the dilation model proposed by Plesha and Belytschko [9]. Assuming that the interface is located on the x-axis, a relative displacement can occur between both sides of the interface. Adopting the generalized treatment proposed by Fredriksson [10] the incremental form of the relative displacement can be written as

$$\dot{R}_i = \dot{u}_i(x, 0^+) - \dot{u}_i(x, 0^-) \tag{6}$$

where $+ve$ or $-ve$ signs refer to the regions corresponding to an interface. The incremental relative displacement can be decomposed into its elastic (e) and plastic (p) parts:

$$\dot{R}_i = \dot{R}_i^{(e)} + \dot{R}_i^{(p)} \tag{7}$$

The elastic displacement can be obtained from the relationship

$$\dot{t}_i = E_{ij}^{(e)} \ \dot{R}_j^{(e)} \tag{8}$$

where $E_{ij}^{(e)}$ is the elastic stiffness at the interface. This result is valid provided the tractions at the interface do not exceed the interface yield criterion

$$F(t_i) = \{(t_x \cos a + t_y \sin a)^2\}^{1/2} + \mu\{t_x \sin a - t_y \cos a\} \tag{9}$$

where μ is the friction coefficient at the interface and a is the asperity angle; where $a = 0$ the result (9) reduces to the case of Coulomb friction. Using conventional plasticity formulations applicable to continua, we assume that the incremental plastic deplacements $\dot{R}_i^{(p)}$ at the interface can be expressed as

$$\dot{R}_i^{(p)} = \begin{cases} 0 & \text{if } F(t_i) < 0 \text{ or } \dot{F}(t_i) < 0 \\ \dot{\lambda} \frac{\vartheta \Phi}{\vartheta t} & \text{if } F(t_i) = \dot{F}(t_i) = 0 \end{cases} \tag{10}$$

where $\dot{\lambda}$ is a plastic multiplier and Φ is an interface plastic potential i.e.,

$$\Phi = \{(t_x \cos a + t_y \sin a)^2\}^{1/2} \tag{11}$$

It can be shown that

$$\dot{t}_i = E_{ij}^{(ep)} \dot{R}_j \tag{12}$$

where the elastic-plastic stiffness at the interface is given by

$$E_{ij}^{(ep)} = E_{ij}^{(e)} - \frac{1}{\Omega} \frac{\vartheta F}{\vartheta t_i} E_{ij}^{(e)} E_{im}^{(e)} \frac{\vartheta \Phi}{\vartheta t_m} \tag{13}$$

and

$$\Omega = \frac{\vartheta F}{\vartheta t_i} E_{lm}^{(e)} \frac{\vartheta \Phi}{\vartheta t_m} - \frac{\vartheta F}{\vartheta W_x^{(p)}} \dot{W}_x^{(p)} \tag{14}$$

where $W_x^{(p)}$ is given by $t_x \dot{R}_x^{(p)}$, which is the plastic work of the tractions tangential to the failure plane. This plastic work is responsible for the degradation of the asperity angle a i.e.

$$a = a_0 \exp(-cW_x^{(p)}) \tag{15}$$

where α_0 is the initial asperity angle and c is a degradation constant. The basic methodologies outlined here can be extended to include interface phenomena which exhibit both frictional and adhesive effects. Such adhesive effects can be characterised by a Mohr-Coulomb model which possesses both cohesive (c) and internal friction (φ; where $\mu = \tan \varphi$).

Stress intensity factors

The study of the influence of interface friction on fracture mechanics primarily concentrates on the evaluation of the stress intensity factors at a crack tip. In these studies, the singular behaviour of the crack tip is modelled by employing the singular traction quarter point elements (Cruse and Wilson [11]) where the displacement and tractions take the following forms

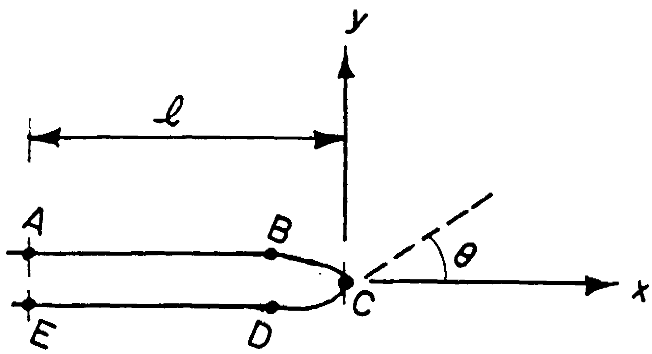
$$[u_i; t_i] = \sum_{m=0}^{2} \left|(b_m r^{m/2}; (c_m r^{(m-1)/2})\right| \tag{16}$$

where b_m and c_m are constants. The stress intensity factors can be obtained by applying the displacement correlation method which utilize the displacement of nodes on either side of the crack. The incremental values of the stress intensity factors are given by

$$\dot{K}_I = \frac{B4}{(k+1)} \sqrt{\frac{B\pi}{l}} \{4[\dot{u}_y(B) - \dot{u}_y(D)] + \dot{u}_y(E) - \dot{u}_y(A)\} \tag{17}$$

$$\dot{K}_{II} = \frac{2G}{(k+1)} \sqrt{\frac{2\pi}{l}} \{4[\dot{u}_x(B) - \dot{u}_x(D)] + \dot{u}_x(E) - \dot{u}_x(A)\} \tag{18}$$

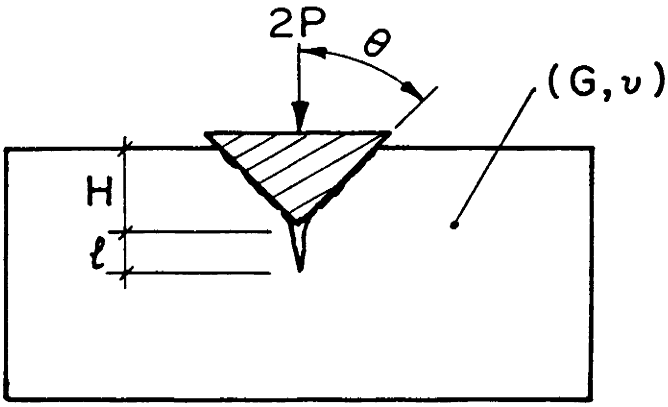
where l is the length of the crack tip element $k = (3-4v)$ for plane strain; $k = (3 - v)/(1 + v)$ for plane stress and the points A, B, C, D, and E are shown in *Fig. 1.* The accuracy of the numerical scheme has been verified by comparison with known exact solutions [12].



1 The crack tip element

A wedge indentation of a cracked region

In this section we apply the basic procedures outlined in the preceding sections to the study of the plane strain frictional indentation of a notch in an elastic medium of finite extent. The base of the notch contains a crack of finite length. The interface between the indenting rigid wedge and the elastic medium is characterized by an interface friction property and cohesion. The rigid wedge is subjected to a central load $2P$. The angle of the wedge and the length of the crack at the root of the notch are variables in the problem. The *Fig. 2* shows the problem under consideration and the *Fig. 3* illustrates the boundary element discretization adopted in the solution of the problem. The *Fig. 4* and *5* illustrate the manner in which the flaw opening mode stress intensity factor at the crack tip is influenced by the magnitude of the load (P), the coefficient of friction (μ) between the indenting wedge and the elastic region and the cohesion (c/G) at the interface. The results clearly indicate the importance of the consideration of friction and adhesion at interfaces in the treatment of problems where interfaces play an active part in fracture mechanics considerations.



2 The wedge indentation of a crack

Acknowledgement

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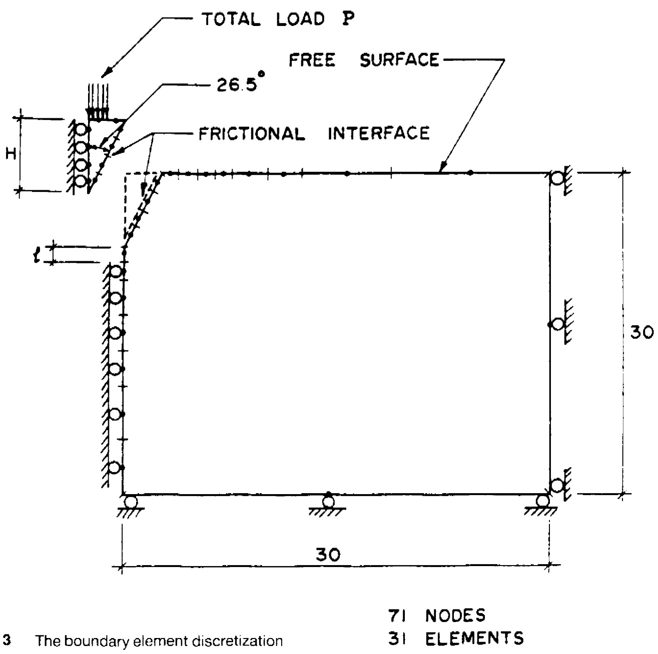

3 The boundary element discretization

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4/5 The stress intensity factor at the crack tip