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The In-Plane Loading of a Rigid Disk Inclusion Embedded in an Elastic Half-Space

The paper examines the problem of the in-plane loading of a rigid disk inclusion which is embedded in bonded contact with an isotropic elastic half-space region. The governing coupled integral equations, derived via a Hankel transform technique, are evaluated numerically to generate results for the in-plane stiffness of the rigid disk inclusion and the rotation which accompanies the lateral translation.

Introduction

Inclusion problems in classical elasticity theory have a wide range of engineering applications. In the context of mechanics of composite materials, solutions to inclusion problems are used to develop estimates for effective properties of multiphase solids, stress amplification at critical locations of the matrix-inclusion interface, etc. Classical solutions for spheroidal and ellipsoidal inhomogeneities developed by Sadowsky and Sternberg (1949), Dewey (1947), Edwards (1951), Eshelby (1961), Luré (1967), and others have been applied quite extensively in the study of mechanics of composite materials. Extensive accounts of such applications are given by Christensen (1980), Mura (1981), and Hashin and Herakovich (1983). In geomechanical applications, solutions to inclusion problems have been used quite extensively in studies pertaining to ground anchoring mechanisms. Solutions developed for spherical and spheroidal inclusions subjected to axial loads have been used to estimate the stiffness of anchors embedded in isotropic elastic soil media (see, e.g., de Josselin de Jong, 1957; Kanwal and Sharma, 1976; Selvadurai, 1976; Zureick, 1988).

Disk or penny-shaped inclusions are a particular limiting case of the general class of three-dimensional inhomogeneities. Owing to the simplified geometry of the disk inclusion problem it is possible to develop a variety of solutions which are of interest to mechanics of composite materials and geomechanics. Many investigators including Collins (1962), Keer (1965), Kassir and Sih (1968), Selvadurai (1980, 1981, 1982), and Selvadurai and Singh (1982, 1986, 1987) have examined problems related to disk inclusions embedded in elastic media of infinite extent. These studies have been extended to cover disk inclusion problems which deal with the influence of material anisotropy, bimaterial regions, inclusion flexibility, and the interaction of inclusions with cracks. References to these and other studies related to disk inclusion problems are given by Mura (1981, 1988) and Selvadurai (1989, 1992). In a majority of problems dealing with disk inclusions, it is explicitly assumed that the inclusion is contained within a medium of infinite extent. In a generalized treatment of the inclusion problem it is desirable to include the influences of external boundaries. Lawrence (1969) has examined the elastostatic problem of a rigid spherical inclusion which is embedded in an elastic plate. Hunter and Gamblen (1974) have examined the axisymmetric elastostatic problem related to the axial loading of a rigid disk which is embedded in partial bonded contact with an incompressible elastic half-space region. Rajapakse (1988) has examined the problem of the axial loading of a flexible anchor plate embedded in a transversely isotropic elastic half-space by employing a discretization procedure. Wang and Rajapakse (1990) have utilized a discretization procedure to examine a class of inclusion problems related to a transversely isotropic elastic half-space region. References to further work may be found in the works of Mura (1981, 1988).

In this paper we examine the asymmetric elastostatic problem related to the in-plane loading of a rigid disk inclusion which is embedded in bonded contact with an isotropic elastic half-space region. The disk inclusion is embedded at a finite distance from the free surface of the half-space and oriented parallel to the free surface (Fig. 1). The analysis of the problem makes use of a Hankel transform development of the governing equations. The traction-free boundary conditions at the free surface

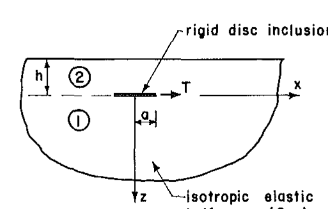

Fig. 1 In-plane loading of a rigid disk inclusion embedded in a half-space

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and the continuity and boundary conditions at the plane of the inclusion yield a system of three coupled integral equations. These integral equations are solved by a quadrature scheme to generate results of engineering interest. Complete numerical results presented in the paper illustrate the manner in which the in-plane stiffness of the disk inclusion and the rotation induced by the lateral force is influenced by the depth of embedment of the inclusion and Poisson's ratio of the elastic medium.

Governing Equations

The asymmetric elastostatic boundary value problem can be analyzed by employing the stress function techniques developed by Muki (1960). The stress functions are governed by the differential equations

$$\nabla^2\nabla^2\Phi(r,\theta,z)=0 \tag{1a}$$

$$\nabla^2\Psi(r,\theta,z)=0 \tag{1b}$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2} \tag{1c}$$

is Laplace's operator referred to the cylindrical polar coordinate system (r, θ, z). The stresses and displacements in the elastic medium can be uniquely expressed in terms of these stress functions (see, e.g., Muki, 1960; Gurtin, 1972). The expressions for the displacement and stress components in terms of $\Phi(r, \theta, z)$ and $\Psi(r, \theta, z)$ take the forms

$$2Gu_r=-\frac{\partial^2\Phi}{\partial r\partial z}+\frac{2}{r}\frac{\partial\Psi}{\partial\theta} \tag{2}$$

$$2Gu_\theta=-\frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\partial z}-2\frac{\partial\Psi}{\partial r} \tag{3}$$

$$2Gu_z=2(1-\nu)\nabla^2\Phi-\frac{\partial^2\Phi}{\partial z^2} \tag{4}$$

and

$$\sigma_{rr}=\frac{\partial}{\partial z}\left(\nu\nabla^2-\frac{\partial^2}{\partial r^2}\right)\Phi+\frac{\partial}{\partial\theta}\left(\frac{2}{r}\frac{\partial}{\partial r}-\frac{2}{r^2}\right)\Psi \tag{5}$$

$$\sigma_{\theta\theta}=\frac{\partial}{\partial z}\left(\nu\nabla^2-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\Phi-\frac{\partial}{\partial\theta}\left(\frac{2}{r}\frac{\partial}{\partial r}-\frac{2}{r^2}\right)\Psi \tag{6}$$

$$\sigma_{zz}=\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\Phi \tag{7}$$

$$\sigma_{\theta z}=\frac{1}{r}\frac{\partial}{\partial\theta}\left[(1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\Phi-\frac{\partial^2\Psi}{\partial r\partial z} \tag{8}$$

$$\sigma_{rz}=\frac{\partial}{\partial r}\left[(1-\nu)\nabla^2-\frac{\partial^2}{\partial z^2}\right]\Phi+\frac{1}{r}\frac{\partial^2\Psi}{\partial\theta\partial z} \tag{9}$$

$$\sigma_{r\theta}=\frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}\left(\frac{1}{r}-\frac{\partial}{\partial r}\right)\Phi-\left(2\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial z^2}\right)\Psi \tag{10}$$

where G and ν are the linear elastic modulus and Poisson's ratio, respectively. It may be noted that for axial symmetry $\Phi=\Phi(r, z)$ and $\Psi=0$, and the results (1)–(10) reduce to those given by Love (1944).

Considering a Hankel transform development of the governing equations (1a) and (1b), we can obtain solutions which are applicable to the regions $0\le z<\infty$ (region 1) and $-h\le z\le 0$ (region 2). For the region 1 $(0\le z<\infty)$ we have

$$\Phi^{(1)}(r,\theta,z)=\cos\theta\int_0^\infty\xi[A(\xi)+zB(\xi)]e^{-\xi z}J_1(\xi r)d\xi \tag{11}$$

$$\Psi^{(1)}(r,\theta,z)=\sin\theta\int_0^\infty\xi C(\xi)e^{-\xi z}J_1(\xi r)d\xi \tag{12}$$

and the corresponding displacement and stress components take the forms

$$2Gu_r^{(1)}(r,\theta,z)=\frac{\cos\theta}{r}\left\{\int_0^\infty[\xi(\xi A(\xi)-(1-\xi z)B(\xi))(\xi rJ_0(\xi r)-J_1(\xi r))+2\xi C(\xi)J_1(\xi r)]e^{-\xi z}d\xi\right\} \tag{13}$$

$$2Gu_\theta^{(1)}(r,\theta,z)=\frac{\sin\theta}{r}\left\{\int_0^\infty[\xi(-\xi A(\xi)+(1-\xi z)B(\xi))J_1(\xi r)-2\xi C(\xi)(r\xi J_0(\xi r)-J_1(\xi r))]e^{-\xi z}d\xi\right\} \tag{14}$$

$$2Gu_z^{(1)}(r,\theta,z)=-\cos\theta\left\{\int_0^\infty\xi^2[\xi A(\xi)+B(\xi)(\xi z+2(1-2\nu))]e^{-\xi z}J_1(\xi r)d\xi\right\} \tag{15}$$

$$\sigma_{zz}^{(1)}(r,\theta,z)=\cos\theta\left\{\int_0^\infty[\xi^4A(\xi)+B(\xi)((1-2\nu)\xi^3+\xi^4z)]e^{-\xi z}J_1(\xi r)d\xi\right\} \tag{16}$$

$$\sigma_{\theta z}^{(1)}(r,\theta,z)=\frac{\sin\theta}{r}\times\left\{\int_0^\infty\xi[\xi^2A(\xi)+B(\xi)(\xi^2z-2\xi\nu)]e^{-\xi z}J_1(\xi r)d\xi+\int_0^\infty\xi^2C(\xi)e^{-\xi z}(\xi rJ_0(\xi r)-J_1(\xi r))d\xi\right\} \tag{17}$$

$$\sigma_{rz}^{(1)}(r,\theta,z)=\frac{\cos\theta}{r}\left\{\int_0^\infty\xi e^{-\xi z}[-\xi^2A(\xi)+(2\xi\nu-\xi^2z)B(\xi)]\cdot(\xi rJ_0(\xi r)-J_1(\xi r))d\xi-\int_0^\infty\xi^2e^{-\xi z}C(\xi)J_1(\xi r)d\xi\right\} \tag{18}$$

where $A(\xi)$, $B(\xi)$, and $C(\xi)$ are arbitrary functions. For the region 2 $(-h\le z\le 0)$,

$$\Phi^{(2)}(r,\theta,z)=\cos\theta\left\{\int_0^\infty\xi[A_1(\xi)+(z+h)B_1(\xi)]\times\frac{\sinh\xi(z+h)}{\sinh(\xi h)}J_1(\xi r)d\xi+\int_0^\infty\xi[C_1(\xi)+(z+h)D_1(\xi)]\frac{\cosh\xi(z+h)}{\sinh(\xi h)}J_1(\xi r)d\xi\right\} \tag{19}$$

$$\Psi^{(2)}(r,\theta,z)=\sin\theta\left\{\int_0^\infty\frac{\xi E_1(\xi)\cosh\xi(z+h)}{\sinh(\xi h)}J_1(\xi r)d\xi\right\} \tag{20}$$

and the corresponding displacement and stress components take the forms

$$2Gu_r^{(2)}=\cos\theta\left\{-\int_0^\infty\xi[\{\xi(A_1(\xi)+(z+h)B_1(\xi))\cosh\xi(z+h)+B_1(\xi)\sinh\xi(z+h)\}\frac{1}{\sinh(\xi h)}+\right.$$
$$+\{\xi(C_1(\xi)+(z+h)D_1(\xi))\sinh\xi(z+h)$$

$$+ D_1(\xi)\cosh\xi(z+h)\}\frac{1}{\sinh(\xi h)}J_1'(\xi r)d\xi + \frac{2}{r}\int_0^\infty \frac{\xi E_1(\xi)\cosh\xi(z+h)J_1(\xi r)d\xi}{\sinh(\xi h)}\Bigg\} \quad (21)$$

$$2Gu_z^{(2)} = \cos\theta\Bigg\{\int_0^\infty [-\xi^3\sinh\xi(z+h)\{A_1(\xi)+(z+h)B_1(\xi)\} + 2(1-2\nu)\xi^2\{\cosh\xi(z+h)B_1(\xi)+\sinh\xi(z+h)D_1(\xi)\} - \xi^3\cosh\xi(z+h)\{C_1(\xi)+(z+h)D_1(\xi)\}]\frac{J_1(\xi r)d\xi}{\sinh(\xi h)}\Bigg\} \quad (22)$$

$$2Gu_\theta^{(2)} = \frac{\sin\theta}{r}\Bigg\{\int_0^\infty \xi[\xi A_1(\xi)\cosh\xi(z+h) + B_1(\xi)\{\sinh\xi(z+h) + \xi(z+h)\cosh\xi(z+h)\} + \xi C_1(\xi)\sinh\xi(z+h) + D_1(\xi)\{\cosh\xi(z+h)+\xi(z+h)\sinh\xi(z+h)]\frac{J_1(\xi r)d\xi}{\sinh(\xi h)} - 2r\int_0^\infty \frac{\xi E_1(\xi)\cosh\xi(z+h)}{\sinh(\xi h)}J_1'(\xi r)d\xi\Bigg\} \quad (23)$$

$$\sigma_{zz}^{(2)} = \cos\theta\Bigg\{\int_0^\infty\Bigg[-\frac{\xi^4 A_1(\xi)\cosh\xi(z+h)}{\sinh(\xi h)} + \frac{\xi^3 B_1(\xi)}{\sinh(\xi h)}\{(1-2\nu)\sinh\xi(z+h) - \xi(z+h)\cosh\xi(z+h)\} - \frac{\xi^4 C_1(\xi)\sinh\xi(z+h)}{\sinh(\xi h)} + \frac{\xi^3 D_1(\xi)}{\sinh(\xi h)}\{(1-2\nu)\cosh\xi(z+h) - \xi(z+h)\sinh\xi(z+h)\}\Bigg]J_1(\xi r)d\xi \quad (24)$$

$$\sigma_{rz}^{(2)} = \cos\theta\Bigg\{-(1-\nu)\int_0^\infty \xi^3[A_1(\xi)+(z+h)B_1(\xi)] \times\frac{\sinh\xi(z+h)}{\sinh(\xi h)}J_1'(\xi r)d\xi - \nu\int_0^\infty\{\xi^2[2\cosh\xi(z+h) + \xi(z+h)\sinh\xi(z+h)]B_1(\xi) + \xi^3 A_1(\xi)\sinh\xi(z+h)\}\frac{J_1'(\xi r)d\xi}{\sinh(\xi h)} - (1-\nu)\int_0^\infty \xi^3[C_1(\xi) + (z+h)D_1(\xi)]\frac{\cosh\xi(z+h)}{\sinh(\xi h)}J_1'(\xi r)d\xi - \nu\int_0^\infty\{\xi^2[2\sinh\xi(z+h)+\xi(z+h)\cosh\xi(z+h)]D_1(\xi) + \xi^3 C_1(\xi)\cosh\xi(z+h)\}\frac{J_1'(\xi r)d\xi}{\sinh(\xi h)}\Bigg\} + \frac{\cos\theta}{r}\int_0^\infty \xi^2 E_1(\xi)\frac{\sinh\xi(z+h)}{\sinh(\xi h)}J_1(\xi r)d\xi \quad (25)$$

$$\sigma_{\theta z}^{(2)} = -\frac{\sin\theta}{r}\Bigg\{-(1-\nu)\int_0^\infty \xi^3[A_1(\xi)+(z+h)B_1(\xi)] \times\frac{\sinh\xi(z+h)}{\sinh(\xi h)}J_1(\xi r)d\xi - \nu\int_0^\infty\{\xi^2[2\cosh\xi(z+h) + \xi(z+h)\sinh\xi(z+h)]B_1(\xi) + \xi^3 A_1(\xi)\sinh\xi(z+h)\}\frac{J_1(\xi r)}{\sinh(\xi h)}d\xi - (1-\nu)\int_0^\infty \xi^3[C_1(\xi)+(z+h)D_1(\xi)]\frac{\cosh\xi(z+h)}{\sinh(\xi h)}J_1(\xi r)d\xi - \nu\int_0^\infty\{\xi^2[2\sinh\xi(z+h)+\xi(z+h)\cosh\xi(z+h)]D_1(\xi) + \xi^3 C_1(\xi)\cosh\xi(z+h)\}\frac{J_1(\xi r)d\xi}{\sinh(\xi h)}\Bigg\} - \sin\theta\int_0^\infty \xi^2 E_1(\xi)\frac{\sinh\xi(z+h)}{\sinh(\xi h)}J_1'(\xi r)d\xi, \quad (26)$$

where $(\ )'$ denotes the derivative with respect to r; i.e., $J_1'(\xi r) = d[J_1(\xi r)]/dr$.

The Inclusion Problem

We consider the problem of a rigid penny-shaped inclusion which is embedded in an isotropic elastic half-space and subjected to an in-plane lateral force T. The boundary and continuity conditions associated with the problem are as follows: In the inclusion region we have

$$u_r^{(1)} = \Delta\cos\theta;\ 0\le r\le a;\ z=0 \quad (27a)$$

$$u_r^{(2)} = \Delta\cos\theta;\ 0\le r\le a;\ z=0 \quad (27b)$$

$$u_\theta^{(1)} = -\Delta\sin\theta;\ 0\le r\le a;\ z=0 \quad (27c)$$

$$u_\theta^{(2)} = -\Delta\sin\theta;\ 0\le r\le a;\ z=0 \quad (27d)$$

$$u_z^{(1)} = \Omega r\cos\theta;\ 0\le r\le a;\ z=0 \quad (27e)$$

$$u_z^{(2)} = \Omega r\cos\theta;\ 0\le r\le a;\ z=0. \quad (27f)$$

At the interface between regions (1) and (2), exterior to the inclusion, we have the following continuity conditions

$$u_z^{(1)} = u_z^{(2)};\ a\le r<\infty;\ z=0 \quad (27g)$$

$$u_r^{(1)} = u_r^{(2)};\ a\le r<\infty;\ z=0 \quad (27h)$$

$$u_\theta^{(1)} = u_\theta^{(2)};\ a\le r<\infty;\ z=0 \quad (27i)$$

$$\sigma_{zz}^{(1)} = \sigma_{zz}^{(2)};\ a<r<\infty;\ z=0 \quad (27j)$$

$$\sigma_{rz}^{(1)} = \sigma_{rz}^{(2)};\ a<r<\infty;\ z=0 \quad (27k)$$

$$\sigma_{\theta z}^{(1)} = \sigma_{\theta z}^{(2)};\ a<r<\infty;\ z=0. \quad (27l)$$

At the surface of the half-space region,

$$\sigma_{zz}^{(2)} = 0;\ 0\le r<\infty;\ z=-h \quad (27m)$$

$$\sigma_{\theta z}^{(2)} = 0;\ 0\le r<\infty;\ z=-h \quad (27n)$$

$$\sigma_{rz}^{(2)} = 0;\ 0\le r<\infty;\ z=-h. \quad (27o)$$

In equations (27a)–(27f) Δ represents the rigid body translation of the disk inclusion due to the in-plane force T and Ω represents the rigid body rotation of the inclusion about the y-axis.

From the boundary conditions (27m), (27n), and (27o) we find that

$$D_1(\xi) = \alpha_1\xi A_1(\xi) \quad (28)$$

$$\xi C_1(\xi) = -\gamma_1 B_1(\xi) \quad (29)$$

where

$$\alpha_1 = \frac{1}{(1-2\nu)};\ \gamma_1 = 2\nu. \quad (30)$$

Considering the continuity conditions (27a) to (27i) it can be shown that

$$A_1(\xi) = a_1 A(\xi) + a_2 B(\xi) \quad (31)$$

$$B_1(\xi) = a_3 A(\xi) + a_4 B(\xi) \quad (32)$$

$$E_1(\xi) = \tanh(\xi h)C(\xi) \quad (33)$$

where

$$a_1 = \frac{(b_1 - b_3)}{(b_0 b_3 + b_1 b_2)} \tag{34}$$

$$a_2 = \frac{1}{\xi}\left\{\frac{b_3 + 2(1-2\nu)b_1}{b_0 b_3 + b_1 b_2}\right\} \tag{35}$$

$$a_3 = \frac{-\xi(b_0 + b_2)}{(b_0 b_3 + b_1 b_2)} \tag{36}$$

$$a_4 = \left\{\frac{b_2 - 2(1-2\nu)b_0}{b_0 b_3 + b_1 b_2}\right\} \tag{37}$$

and

$$b_0 = [\alpha_1(\xi h + \coth(\xi h)) + \coth(\xi h)] \tag{38a}$$

$$b_1 = [1 - \gamma_1 + \xi h \coth(\xi h)] \tag{38b}$$

$$b_2 = [1 - 2\alpha_1(1-2\nu) + \alpha_1 h \xi \coth(\xi h)] \tag{39a}$$

$$b_3 = [-\xi h + \gamma_1 \coth(\xi h) + 2(1-2\nu)\coth(\xi h)]. \tag{39b}$$

From (28), (29), and (33) it is evident that

$$D_1(\xi) = [e_1 A(\xi) + f_1 B(\xi)] \tag{40}$$

$$\xi C_1(\xi) = -\gamma_1[a_3 A(\xi) + a_4 B(\xi)] \tag{41}$$

where

$$e_1 = \alpha_1 \xi a_1 \tag{42}$$

$$f_1 = \alpha_1 \xi a_2. \tag{43}$$

Using the results (31)–(43) it is possible to express the integral expressions for $u_r^{(2)}$, $u_\theta^{(2)}$, $u_z^{(2)}$, $\sigma_{zz}^{(2)}$, $\sigma_{\theta z}^{(2)}$ and $\sigma_{rz}^{(2)}$ in terms of only the functions $A(\xi)$, $B(\xi)$, and $C(\xi)$. On the plane $z=0$, the reduced form of these equations are given by

$$2Gu_r^{(2)} = \cos\theta\left\{-\int_0^\infty \xi[\{\xi(a_1 + ha_3)\coth(\xi h) + a_3 + (\xi h e_1 + \gamma_1 a_3) + e_1\coth(\xi h)\}A(\xi) + \{\xi(a_2 + a_4 h)\coth(\xi h) + a_4 + (\xi h f_1 - \gamma_1 a_4) + f_1 \coth(\xi h)\}B(\xi)]J_1'(\xi r)d\xi + \frac{2}{r}\int_0^\infty \xi J_1(\xi r)C(\xi)d\xi\right\} \tag{44}$$

$$2Gu_z^{(2)} = \cos\theta\left\{\int_0^\infty [\{-\xi^3(a_1 + a_3 h) + 2(1-2\nu)\xi^2[e_1 + a_3\coth(\xi h)] - \xi^2\coth(\xi h)(\xi h e_1 - \gamma_1 a_3)\}A(\xi)J_1(\xi r) + \{-\xi^3(a_2 + a_4 h) + 2(1-2\nu)\xi^2[f_1 + a_4\coth(\xi h)] - \xi^2\coth(\xi h)(\xi h f_1 - a_4\gamma_1)\}B(\xi)J_1(\xi r)]d\xi\right\} \tag{45}$$

$$2Gu_\theta^{(2)} = \frac{\sin\theta}{r}\left\{\int_0^\infty \xi[A(\xi)\{a_3(1 + \xi h\coth(\xi h)) + \xi a_1\coth(\xi h) - \gamma_1 a_3 + e_1(\xi h + \coth(\xi h))\}J_1(\xi r) + B(\xi)\{\xi a_2\coth(\xi h) + a_4(1 + \xi h\coth(\xi h)) - \gamma_1 a_4 + f_1(\xi h + \coth(\xi h))\}J_1(\xi r)]d\xi - 2r\int_0^\infty \xi C(\xi)J_1'(\xi r)d\xi\right\} \tag{46}$$

$$\sigma_{zz}^{(2)} = \cos\theta\left\{\int_0^\infty [A(\xi)\{-a_1\xi^4\coth(\xi h) + a_3\xi^3(1 - 2\nu - \xi h\coth(\xi h)) + \xi^3\gamma_1 a_3 + e_1\xi^3((1-2\nu)\coth(\xi h) - \xi h)\} + B(\xi)\{-a_2\xi^4 + a_4\xi^3((1-2\nu) - \xi h\coth(\xi h)) + a_4\gamma_1\xi^3 + f_1\xi^3((1-2\nu)\coth(\xi h) - \xi h)\}]J_1(\xi r)d\xi\right\} \tag{47}$$

$$\sigma_{rz}^{(2)} = \cos\theta\left\{\int_0^\infty [A(\xi)\{-(1-\nu)\xi^3(a_1 + a_3 h) - \nu\xi^2[(2\coth(\xi h) + \xi h) + a_3 + \xi a_1] - \xi^2(1-\nu)[e_1\xi h - a_3\gamma_1]\coth(\xi h) - \nu\xi^2[(2 + \xi h\coth(\xi h))e_1 - \gamma_1 a_3]\}J_1'(\xi r) + B(\xi)\{-(1-\nu)\xi^3(a_2 + a_4 h) - \nu\xi^2[(2\coth(\xi h) + \xi h)a_4 + \xi a_2] - (1-\nu)\xi^2[f_1\xi h - \gamma_1 a_4]\coth(\xi h) - \nu\xi^2[(2 + \xi h\coth(\xi h))f_1 - \gamma_1 a_4\coth(\xi h)]\}J_1'(\xi r)]d\xi\right\} + \frac{\cos\theta}{r}\int_0^\infty \xi^2\tanh(\xi h)C(\xi)J_1(\xi r)d\xi \tag{48}$$

$$\sigma_{\theta z}^{(2)} = -\frac{\sin\theta}{r}\left\{\int_0^\infty [A(\xi)\{-(1-\nu)\xi^3(a_1 + a_3 h) - \nu\xi^2[(2\coth(\xi h) + \xi h)a_3 + \xi a_1] - \xi^2(1-\nu)[e_1\xi h - \gamma_1 a_3]\coth(\xi h) - \nu\xi^2 \times [(2 + \xi h\coth(\xi h))e_1 - \gamma_1 a_3]\}J_1(\xi r) + B(\xi)\{-(1-\nu)\xi^3(a_2 + a_4 h) - \nu\xi^2[a_4(\xi h + 2\coth(\xi h)) + \xi a_2] - \xi^2(1-\nu)[\xi h f_1 - \gamma_1 a_4]\coth(\xi h) - \nu\xi^2[(2 + \xi h\coth(\xi h))f_1 - \gamma_1 a_4\coth(\xi h)]\}J_1(\xi r)]d\xi\right\} - \sin\theta\int_0^\infty \xi^2 C(\xi)\tanh(\xi h)J_1'(\xi r)d\xi. \tag{49}$$

Using the results (13) to (18) and (44) to (49), the boundary conditions (27*a*), (27*c*), (27*e*), (27*j*), (27*k*) and (27*l*) become

$$\int_0^\infty \{\xi(\xi A(\xi) - B(\xi))(\xi r J_0(\xi r) - J_1(\xi r)) + 2\xi C(\xi)J_1(\xi r)\}d\xi = 2\Delta G r;\ 0 \le r \le a \tag{50a}$$

$$\int_0^\infty \{\xi(-\xi A(\xi) + B(\xi))J_1(\xi r) - 2\xi C(\xi)(\xi r J_0(\xi r) - J_1(\xi r))\}d\xi = -2\Delta G r;\ 0 \le r \le a \tag{50b}$$

$$\int_0^\infty \xi^2\{\xi A(\xi) + 2(1-2\nu)B(\xi)\}J_1(\xi r)d\xi = -2\Omega G r;\ 0 \le r \le a \tag{50c}$$

$$\int_0^\infty \xi^3\{\xi g_0 A(\xi) + g_1 B(\xi)\}J_1(\xi r)d\xi = 0,\ a < r < \infty \tag{51a}$$

$$\int_0^\infty \xi^2\{g_2\xi A(\xi) + g_3 B(\xi)\}J_1(\xi r)d\xi + \int_0^\infty g_4\xi^2\{\xi r J_0(\xi r) - J_1(\xi r)\}C(\xi)d\xi = 0;\ a < r < \infty \tag{51b}$$

$$\int_0^\infty \xi^2\{g_2\xi A(\xi) + g_3 B(\xi)\}\{\xi r J_0(\xi r) - J_1(\xi r)\}d\xi + \int_0^\infty g_4\xi^2 C(\xi)J_1(\xi r)d\xi = 0;\ a < r < \infty, \tag{51c}$$

where the functions $g_i(\xi)$ $(i=0,1,\cdots 4)$ are given by

$$g_0=1+a_1\coth(\xi h)-\frac{a_3}{\xi}\{(1-2\nu)-\xi h\coth(\xi h)\}$$
$$-\frac{\gamma_1 a_3}{\xi}-\frac{e_1}{\xi}\{(1-2\nu)\coth(\xi h)-\xi h\} \quad (52a)$$

$$g_1=(1-2\nu)+a_2\xi\coth(\xi h)-a_4\{(1-2\nu)-\xi h\coth(\xi h)\}$$
$$-a_4\gamma_1-f_1\{(1-2\nu)\coth(\xi h)-\xi h\} \quad (52b)$$

$$g_2=-1+(1-\nu)(a_1+a_3h)+\frac{\nu}{\xi}\{a_3(\xi h+2\coth(\xi h))+a_1\xi\}$$
$$+\frac{(1-\nu)}{\xi}\coth(\xi h)\{\xi he_1-a_3\gamma_1\}$$
$$+\frac{\nu}{\xi}\{e_1(2+\xi h\coth(\xi h))-\gamma_1 a_3\} \quad (52c)$$

$$g_3=\xi\left\{\frac{2\nu}{\xi}+(1-\nu)(a_2+a_4h)+\frac{\nu}{\xi}\{a_4[\xi h+2\coth(\xi h)]\right.$$
$$+\xi a_2\}+\frac{(1-\nu)}{\xi}(\xi hf_1-\gamma_1 a_4)\coth(\xi h)$$
$$\left.+\frac{\nu}{\xi}\{(2+\xi h\coth(\xi h))f_1-\gamma_1 a_4\coth(\xi h)\}\right\} \quad (52d)$$

$$g_4=-\frac{2}{(1+e^{-2\xi h})}. \quad (52e)$$

By introducing the substitutions

$$N(\xi)=\xi^2[\xi g_0 A(\xi)+g_1 B(\xi)] \quad (53)$$
$$L(\xi)=\xi^2[\xi g_2 A(\xi)+g_3 B(\xi)+g_4 C(\xi)] \quad (54)$$
$$M(\xi)=\xi^2[\xi g_2 A(\xi)+g_3 B(\xi)-g_4 C(\xi)], \quad (55)$$

the integral equations (50a)–(50c) and (51a)–(51c) can be further reduced to the following forms:

$$\int_0^\infty L(\xi)J_0(\xi r)d\xi+\int_0^\infty\left\{\frac{g_4(g_0+g_1)+2p(\xi)}{2g_4 p(\xi)}-1\right\}$$
$$\times L(\xi)J_0(\xi r)d\xi+\int_0^\infty\left\{\frac{g_4(g_0+g_1)-2p(\xi)}{2g_4 p(\xi)}\right\}M(\xi)J_0(\xi r)d\xi$$
$$-\int_0^\infty\frac{(g_2+g_3)}{p(\xi)}N(\xi)J_0(\xi r)d\xi=4G\Delta;\ 0\le r\le a \quad (56)$$

$$\int_0^\infty M(\xi)J_2(\xi r)d\xi+\int_0^\infty\left\{\frac{2p(\xi)+g_4(g_0+g_1)}{2p(\xi)g_4}-1\right\}$$
$$\times M(\xi)J_2(\xi r)d\xi+\int_0^\infty L(\xi)\left\{\frac{g_4(g_0+g_1)-2p(\xi)}{2p(\xi)g_4}\right\}J_2(\xi r)d\xi$$
$$-\int_0^\infty N(\xi)\frac{(g_2+g_3)}{p(\xi)}J_2(\xi r)d\xi=0;\ 0\le r\le a \quad (57)$$

$$\int_0^\infty N(\xi)J_1(\xi r)d\xi+\int_0^\infty\left\{\frac{2(1-2\nu)g_2-g_3}{p(\xi)}-1\right\}J_1(\xi r)N(\xi)d\xi$$
$$+\int_0^\infty\left\{\frac{g_1-2(1-2\nu)g_0}{2p(\xi)}\right\}M(\xi)J_1(\xi r)d\xi$$
$$+\int_0^\infty\left\{\frac{g_1-2(1-2\nu)g_0}{2p(\xi)}\right\}L(\xi)J_1(\xi r)d\xi=-2\Omega Gr;$$
$$0\le r\le a \quad (58)$$

$$\int_0^\infty \xi N(\xi)J_1(\xi r)d\xi=0;\ a<r<\infty \quad (59)$$

$$\int_0^\infty \xi L(\xi)J_0(\xi r)d\xi=0;\ a<r<\infty \quad (60)$$

$$\int_0^\infty \xi M(\xi)J_2(\xi r)d\xi=0;\ a<r<\infty \quad (61)$$

where

$$p(\xi)=g_1g_2-g_0g_3. \quad (62)$$

Considering (59), (60), and (61), it it convenient to introduce representations of $L(\xi)$, $N(\xi)$, and $M(\xi)$ in terms of functions $\varphi_1(t)$, $\psi(t)$, and $\varphi_2(t)$ as follows:

$$L(\xi)=\int_0^a\varphi_1(t)\cos(\xi t)dt \quad (63)$$

$$N(\xi)=\int_0^a\psi(t)\sin(\xi t)dt \quad (64)$$

$$M(\xi)=\int_0^a t\varphi_2(t)J_2(\xi t)dt. \quad (65)$$

The representations (63), (64), and (65) identically satisfy (59), (60), and (61), respectively. Using the representations (63) to (65) in (56) to (58), we obtain the following system of coupled integral equations for the functions of $\varphi_1(t)$, $\varphi_2(t)$, and $\psi(t)$; i.e.,

$$\varphi_1(t)+\int_0^a\varphi_1(u)K_1(u,t)du+\int_0^a u\varphi_2(u)K_2(u,t)du$$
$$+\int_0^a\psi(u)K_3(u,t)du=\frac{8G\Delta}{\pi};\ 0\le t\le a \quad (66)$$

$$\varphi_2(t)=-\int_0^a u\varphi_2(u)K_5(u,t)du-\int_0^a\varphi_1(u)K_4(u,t)du$$
$$-\int_0^a\psi(u)K_6(u,t)du;\ 0<t<a \quad (67)$$

$$\psi(t)+\int_0^a\psi(u)K_7(u,t)du+\int_0^a u\varphi_2(u)K_8(u,t)du$$
$$+\int_0^a\varphi_1(u)K_9(u,t)du=-\frac{8\Omega Gt}{\pi};\ 0\le t\le a \quad (68)$$

where, the kernel functions $K_i(u,t)$, $(i=1,2,\cdots 9)$ are given by

$$K_1(u,t)=\frac{2}{\pi}\int_0^\infty\left\{\frac{g_4(g_0+g_1)+2p(\xi)}{2g_4 p(\xi)}-1\right\}$$
$$\times\cos(\xi u)\cos(\xi t)d\xi \quad (69)$$

$$K_2(u,t)=\frac{2}{\pi}\int_0^\infty\left\{\frac{g_4(g_0+g_1)-2p(\xi)}{2g_4 p(\xi)}\right\}\cos(\xi t)J_2(\xi u)d\xi \quad (70)$$

$$K_3(u,t)=-\frac{2}{\pi}\int_0^\infty\left\{\frac{g_2+g_3}{p(\xi)}\right\}\cos(\xi t)\sin(\xi u)d\xi \quad (71)$$

$$K_4(u,t)=-\int_0^\infty\cos(\xi u)I(\xi,t)\left\{\frac{g_4(g_0+g_1)-2p(\xi)}{2p(\xi)g_4}\right\}d\xi \quad (72)$$

$$K_5(u,t)=-\int_0^\infty\left\{\frac{g_4(g_0+g_1)+2p(\xi)}{2p(\xi)g_4}-1\right\}$$
$$\times I(\xi,t)J_2(\xi u)d\xi \quad (73)$$

$$K_6(u,t)=\int_0^\infty\sin(\xi u)\left\{\frac{g_1+g_2}{p(\xi)}\right\}I(\xi,t)d\xi \quad (74)$$

$$K_7(u,\ t)=\frac{2}{\pi}\int_0^\infty\left\{\frac{2(1-2\nu)g_2-g_3}{p(\xi)}-1\right\}\sin(\xi u)\sin(\xi t)d\xi \quad (75)$$

$$K_8(u,\ t)=\frac{2}{\pi}\int_0^\infty\left\{\frac{g_1-2(1-2\nu)g_0}{2p(\xi)}\right\}\sin(\xi t)J_2(\xi u)d\xi \quad (76)$$

$$K_9(u,\ t)=\frac{2}{\pi}\int_0^\infty\left\{\frac{g_1-2(1-2\nu)g_0}{2p(\xi)}\right\}\sin(\xi t)\cos(\xi u)d\xi \quad (77)$$

where

$$I(\xi,\ t)=\sqrt{\frac{2\xi}{\pi}}\,t\frac{d}{dt}\int_t^a\frac{s^{-1/2}J_{3/2}(\xi s)ds}{(s^2-t^2)^{1/2}}. \quad (78)$$

The coupled system of integral equations defined by (66) to (68) can be solved in a numerical fashion to generate results of engineering interest. The results of primary interest relate to the resultant in plane force (T) acting on the inclusion and the resultant moment (M) acting on the inclusion about the y-axis. Considering the resultant in-plane tractions, we have

$$T=\int_0^{2\pi}\int_0^a r\{(\sigma_{rz}^{(1)}-\sigma_{rz}^{(2)})-(\sigma_{\theta z}^{(1)}-\sigma_{\theta z}^{(2)})\}dr\,d\theta. \quad (79)$$

Using the results presented previously, it can be shown that

$$T=\pi\int_0^a rdr\int_0^\infty \xi M(\xi)J_2(\xi r)d\xi=\pi\int_0^a r\varphi_2(r)dr. \quad (80)$$

Considering the axial stresses $\sigma_{zz}^{(1)}$ and $\sigma_{zz}^{(2)}$ acting on the inclusion, we have

$$M=-\int_0^{2\pi}\int_0^a[\sigma_{zz}^{(1)}(r,\ 0,\ \theta)-\sigma_{zz}^{(2)}(r,\ 0,\ \theta)]r^2\cos\theta d\theta dr\,d\theta. \quad (81)$$

Again, using the results presented earlier, we have

$$M=\pi\int_0^a r^2\left\{\frac{d}{dr}\int_r^a\frac{\psi(t)dt}{\sqrt{t^2-r^2}}\right\}dr=2\pi\int_0^a t\psi(t)dt. \quad (82)$$

By setting $M=0$, we can obtain a relationship between the induced rigid rotation Ω and the rigid translation of the disk inclusion Δ.

Numerical Solution of the Integral Equations

Considering the structure of the kernel functions $K_i(i=1,2,\cdots 9)$ given by (69) to (77), it becomes evident that the coupled integral equations (66), (67), and (68) governing $\varphi_1(t)$, $\varphi_2(t)$, and $\psi(t)$ are not amenable to exact solution. Consequently it is necessary to adopt a numerical technique to generate the relevant results. A variety of techniques have been proposed for the numerical solution of integral equations (see, e.g., Baker, 1978; Delves and Mohammed, 1985). In this paper we present a quadrature scheme for the solution of the coupled system of integral equations.

From (66) to (68) we have

$$\begin{bmatrix}\varphi_1(t)\\ \varphi_2(t)\\ \psi(t)\end{bmatrix}+\int_0^a\begin{bmatrix}K_1(u,\ t) & uK_2(u,\ t) & K_3(u,\ t)\\ K_4(u,\ t) & uK_5(u,\ t) & K_6(u,\ t)\\ K_9(u,\ t) & uK_8(u,\ t) & K_7(u,\ t)\end{bmatrix}\times\begin{bmatrix}\varphi_1(u)\\ \varphi_2(u)\\ \psi(u)\end{bmatrix}du=\frac{8G}{\pi}\begin{bmatrix}\Delta\\ 0\\ -\Omega t\end{bmatrix} \quad (83)$$

where the kernel functions are defined by (69) to (77). Considering the moment free condition on the inclusion we have

$$\int_0^a t\psi(t)dt=0. \quad (84)$$

The result (84) is used as a constraint condition on the solution of the system (83). The function $I(t,\ \xi)$ given by (78) can be expressed in the form

$$\sqrt{\frac{\pi}{2\xi}}I(\xi,\ t)=-\frac{t^2J_{3/2}(\xi a)}{a^{3/2}(a^2-t^2)^{1/2}}-\xi t^2\int_t^a\frac{J_{5/2}(\xi s)ds}{s^{3/2}(s^2-t^2)^{1/2}}. \quad (85)$$

Substituting (85) into (72), (73), and (74) we obtain

$$K_i(u,\ t)=-\frac{t^2K_{i1}(u,\ a)}{a^{3/2}(a^2-t^2)^{1/2}}-t^2\int_t^a\frac{K_{i2}(u,\ s)ds}{s^{3/2}(s^2-t^2)^{1/2}};\ (i=4,5,6) \quad (86)$$

where

$$K_{41}(u,\ a)=-\int_0^\infty\left\{\frac{g_4(g_1+g_0)-2p(\xi)}{2p(\xi)g_4}\right\}\sqrt{\frac{2\xi}{\pi}}J_{3/2}(\xi a)\cos(\xi u)d\xi \quad (87)$$

$$K_{42}(u,\ s)=-\int_0^\infty\left\{\frac{g_4(g_0+g_1)-2p(\xi)}{2p(\xi)g_4}\right\}\sqrt{\frac{2\xi}{\pi}}J_{5/2}(\xi a)\cos(\xi u)d\xi \quad (88)$$

$$K_{51}(u,\ a)=-\int_0^\infty\left\{\frac{g_4(g_1+g_0)+2p(\xi)}{2p(\xi)g_4}-1\right\}\times\sqrt{\frac{2\xi}{\pi}}J_2(\xi u)J_{3/2}(\xi a)d\xi \quad (89)$$

$$K_{52}(u,\ s)=-\int_0^\infty\left\{\frac{g_4(g_0+g_1)2p(\xi)}{2p(\xi)g_4}-1\right\}\times\sqrt{\frac{2\xi^3}{\pi}}J_2(\xi u)J_{5/2}(\xi s)d\xi \quad (90)$$

$$K_{61}(u,\ a)=\int_0^\infty\left\{\frac{(g_1+g_2)}{p(\xi)}\right\}\sqrt{\frac{2\xi}{\pi}}J_{3/2}(\xi a)\sin(\xi u)d\xi \quad (91)$$

$$K_{62}(u,\ s)=\int_0^\infty\left\{\frac{(g_1+g_2)}{p(\xi)}\right\}\sqrt{\frac{2\xi^3}{\pi}}J_{5/2}(\xi s)\sin(\xi u)d\xi. \quad (92)$$

For the numerical solution of the problem, the interval [0, a] is divided into R segments with ends defined by $u_i=(i-1)a/R$; $i=1,2,3,\cdots(R+1)$, and the collocation points are $t_i=(u_i+u_{i+1})/2$; $i=1,2,\cdots R$. Therefore, a matrix equation of order $(3R+1)$ can be obtained with $3R$ equations derived from (83) and a single equation derived from (84). This matrix equation can be written in the form

$$[\mathbf{A}_{ij}]\{\mathbf{X}_j\}=\{\mathbf{B}_i\} \quad (93)$$

where $i,j=1,2,\cdots(3R+1)$. Certain coefficients of $\mathbf{A}_{ij}$ are defined by the kernel functions $K_i(u,\ t)$ and other zero coefficients are as follows:

$$A_{(3\ell-2),(3R+1)}=A_{(3\ell-1),(3R+1)}=A_{(3R+1),(3\ell-2)}=A_{(3R+1),(3\ell-1)}=0 \quad (94)$$

and

$$A_{(3\ell),(3R+1)}=A_{(3R+1),(3\ell)}=-t_\ell \quad (95)$$

where $\ell=1,2,\cdots R$. The coefficients of $\mathbf{A}_{ij}$, other than those given by (94) and (95), are obtained by numerical integrations. The right-hand side vector $\{\mathbf{B}_i\}$ in (93) are given by

$$B_{(3\ell-2)}=1.0;\ B_{(3\ell-1)}=B_{(3\ell)}=B_{(3R+1)}=0 \quad (96)$$

for $\ell=1,2,\cdots R$, and the elements of the unknown vector are:

$$\varphi_1(t_\ell)=\frac{8G\Delta}{\pi}X_{(3\ell-2)};\ \varphi_2(t_\ell)=\frac{8G\Delta}{\pi}X_{(3\ell-1)} \quad (97)$$

$$\psi(t_\ell)=\frac{8G\Delta}{\pi}X_{(3\ell)};\ \Omega=\Delta X_{(3R+1)}. \tag{98}$$

Upon solution of the matrix equation (93), the load on the inclusion can be obtained from the discretized version of the integral (80). We obtain

$$\bar{T}=\frac{(7-8\nu)T}{64(1-\nu)G\Delta a}=\frac{(7-8\nu)}{8(1-\nu)}\frac{1}{R}\sum_{\ell=1}^{R}t_\ell X_{(3\ell-1)}. \tag{99}$$

The rotation of the rigid inclusion is given by

$$\bar{\Omega}=\frac{\Omega a}{\Delta}=2aX_{(3R+1)}. \tag{100}$$

Numerical Results

Prior to the presentation of numerical results, it is instructive to record results for certain limiting cases which pertain to inclusions located either within an infinite space region or at the surface of a half-space region.

As the depth of embedment of the inclusion becomes large in comparison with the radius of the inclusion (i.e., $h/a\rightarrow\infty$), the problem reduces to that of the in-plane loading of a rigid disk inclusion which is embedded in an isotropic elastic infinite space. The closed-form analytical solution to this problem was obtained, among others, by Keer (1965), Kassir and Sih (1968), and Selvadurai (1980). The load-displacement relationship for the inclusion takes the form

$$\bar{T}=\frac{T(7-8\nu)}{64G\Delta a(1-\nu)}. \tag{101}$$

By virtue of the symmetry of the problem about the plane $z=0$, the rigid rotation of the inclusion embedded in an infinite space is identically equal to zero.

When the depth of embedment $h\rightarrow 0$, the problem reduces to that of the in-plane loading of a rigid punch which is bonded to the surface of a half-space region. The solution to this problem was developed by Ufliand (1956) and Gladwell (1969, 1980). The Hilbert-problem approach adopted in these investigations takes into consideration the oscillatory nature of the stress singularity at the boundary of the rigid punch. In this case, the closed-form expression for the load-displacement relationship is given by

$$\bar{T}=\frac{T}{16G\Delta a\{2+\beta+3\omega\alpha\}} \tag{102}$$

where

$$\alpha=\frac{1}{2\pi}\ln(3-4\nu);\ \beta=\frac{(1-2\nu)}{\pi\alpha};\ \omega=\frac{\alpha\beta}{(1+\alpha^2)}. \tag{103}$$

The corresponding rigid rotation is given by

$$\bar{\Omega}=\frac{3\omega T}{16G\Delta a}. \tag{104}$$

It may be noted that integral equation technique adopted in the paper does not take into account the influences such oscillatory singularities which can develop as $(H/a)\rightarrow 0$. It has, however, been shown (Selvadurai, 1989) that such oscillatory stress singularities have virtually no influence on the accuracy with which the translational and rotational stiffnesses are computed for a punch bonded to a half-space region. Results developed for the axial loading of an inclusion bonded to a crack (Selvadurai, 1989) indicate that in the extreme case when $\nu=0$, the discrepancy between the Hankel transform-based integral equation approach and the Hilbert problem approach does not exceed 0.5 percent. This is within the computational efficiency of the numerical procedure adopted in the solution of the system of coupled integral equations, which is presented in the previous section.

Figure 2 illustrates the influence of the depth of embedment of the disk inclusion on the in-plane load-displacement relationship. It is evident that the depth of embedment has a significant influence on the behavior of the in-plane stiffness of the inclusion. The results for the exact closed-form solution for the case $(h/a)=0$, derived from (102), are also shown in Fig. 2 for purposes of comparison. It is evident that the numerical procedure provides a very accurate result for the in-plane load-displacement response of the rigid punch bonded to a half-space region. Figure 3 illustrates the influence of the depth of embedment on the rigid rotation experienced by the disk inclusion as a result of the in-plane loading. Again, the numerical results for the rotation induced on the laterally loaded punch bonded to a half-space region agree very closely with the exact result given in (104). It is also evident that for the special case when $\nu=1/2$, the in-plane loading of the inclusion embedded in a half-space region does not induce a corresponding rotation.

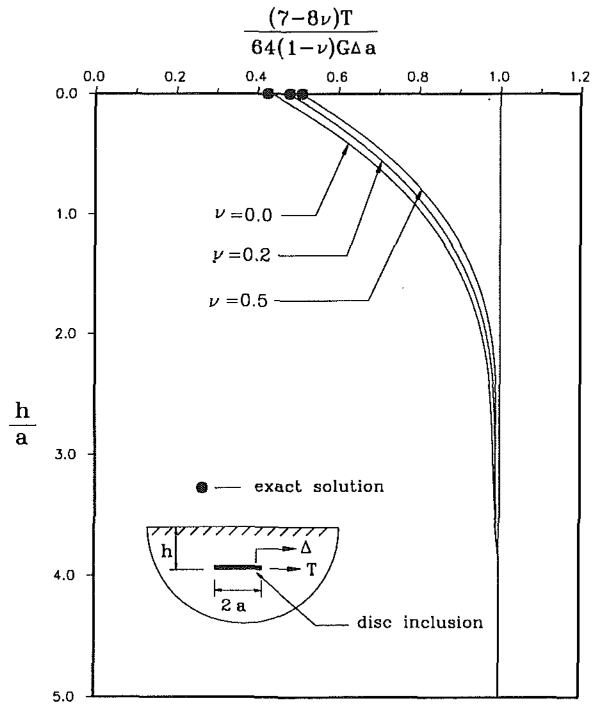

Fig. 2 In-plane stiffness of the rigid disk inclusion embedded in a half-space

Conclusions

An integral equation technique is used to evaluate the load-displacement behavior of a rigid disk inclusion which is embedded in bonded contact with an isotropic elastic half-space region. The mixed boundary value problem associated with the in-plane loading of the rigid inclusion generates a system of three coupled integral equations which are solved numerically by employing a quadrature scheme. The numerical results presented in the paper for the in-plane load-lateral displacement of the inclusion indicate that the influence of the traction-free boundary becomes insignificant as the depth of embedment to inclusion radius ratio exceeds 4. Similar conclusions apply for the results for the rigid rotation of the disk inclusion which is induced as a result of the in-plane loading. As the depth of embedment $h\rightarrow 0$, the problem reduces to that of the in-plane loading of a rigid punch which is bonded to the surface of a half-space region. In existing analytical treatments of this problem, the effects of oscillatory stress singularities at the boundary of the rigid punch are incorporated. The integral equation approach, however, does not allow for such provi-

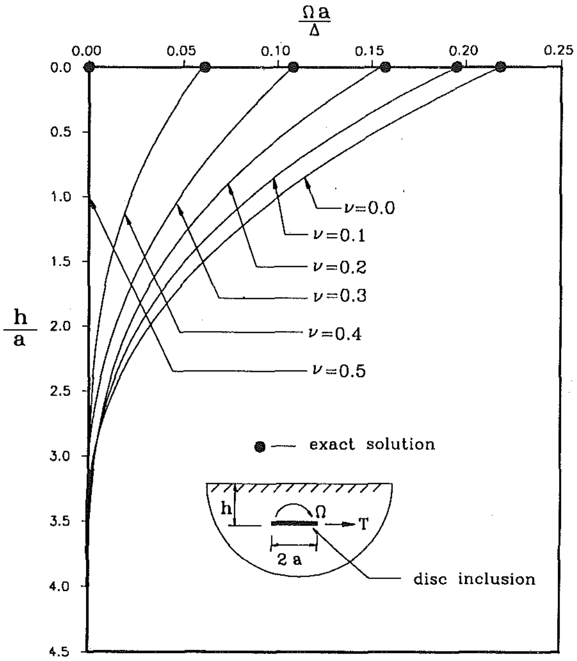

Fig. 3 Rotation of the rigid disk inclusion induced by the in-plane translation

sions. The numerical results derived from the present investigation suggests that such oscillatory phenomena do not influence, to any significant extent, the results for the in-plane displacement and rotation associated with the in-plane loading of the punch.

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