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Hertzian contact in the presence of a Mindlin force

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Introduction

The analysis of interaction between a smooth rigid sphere and an elastic halfspace is a fundamental problem in contact mechanics. The solution to this problem was developed by Hertz [1] and comprehensive accounts of this and other classes of contact problems in general, are given by de Pater and Kalker [2], Selvadurai [3], Gladwell [4] and Johnson [5]. In the treatment of such unilateral contact problems involving advancing or receding contact regions, it is explicitly assumed that the interaction between the indenter and the halfspace region is achieved by loads that are applied to the indenting punch. The type of problem where the interaction is achieved by the simultaneous action of external loading and loads applied within the halfspace region has received only limited attention. Selvadurai [6] examined the axisymmetric problem of the interaction between a rigid circular punch and a Mindlin force which is located along the axis of symmetry. These studies were extended by Selvadurai [7] and Fabrikant et al. [8] to include asymmetric interactions between the punch and the internal force. In these studies it is assumed that the relative magnitudes of the force on the rigid punch and the internal Mindlin force are such that there is no separation at the contact region. In general, for this assumption to be satisfied, either the external force on the punch must be significantly greater than the internal Mindlin force or the internal force should be located at a distance which is large in comparison to the geometry of the indenting region [6]. In the general treatment of this class of circular punch indentation problems it is also necessary to consider the possible occurrence of separation at the contact region [4, 9].

In this paper we consider the axisymmetric unilateral contact problem pertaining to the indentation of an isotropic elastic halfspace by a smooth rigid sphere in the presence of an internally applied Mindlin force. Spherical indentation tests have been successfully applied to evaluate the mechanical properties of both elastic and inelastic solids. The Mindlin force can be regarded as the influence of an internal anchoring mechanism which is used to provide a self-stressing loading system. The contact between the rigid sphere and the halfspace is assumed to be classical Hertzian in the sense that, the displacement and traction boundary conditions at the contact region and the tractions external to the contact region are specified in relation to the undeformed surface. A Hankel integral transform technique is used to develop exact closed form results for the indentation of the rigid sphere, the external force required to initiate the indentation and the contact stresses developed at the interface. Numerical results presented in the paper illustrate the extent to which these results are influenced by the relative magnitudes of the external and internal forces and Poisson's ratio of the elastic medium.

Fundamental equations

The axisymmetric problem related to the spherical punch indentation in the presence of a Mindlin force can be formulated by appeal to Hankel transforms and the theory of dual integral equations. The mixed boundary conditions associated with the indentation problem are

$$u_z(r, 0) = -D + f(r); \quad 0 \le r \le a \tag{1}$$

$$\sigma_{zz}(r, 0) = 0; \quad a < r < \infty \tag{2}$$

$$\sigma_{rz}(r, 0) = 0; \quad 0 \le r < \infty \tag{3}$$

where u_z is the axial displacement, σ_{zz} and σ_{rz} are the stress components referred to the cylindrical polar coordinate system and a is the radius of the circle of contact. In (1) the function $f(r)$ is related to the geometry of the indenting region and specified in such a way that $f(0) \equiv 0$. Although the parameter D in (1) is unspecified it is related to D^* the net rigid displacement of the rigid sphere at the axis of symmetry, under the combined action of the external force and the Mindlin force (Figure 1).

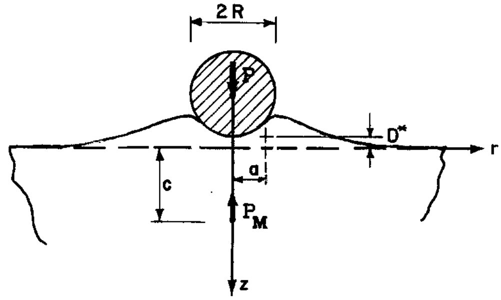

Figure 1
The geometry of the spherical punch indentation problem.

Following Sneddon [10] it can be shown that by making use of the boundary condition (3), the integral expressions for the displacement u_z and the stress components σ_{zz} and σ_{rz} can be reduced to the forms

$$u_z(r, z) = \frac{1}{2(1-\nu)} H_0[-\{2(1-\nu) + \xi z\}\xi^{-1}\psi(\xi)\, e^{-\xi z}; \xi \to r] \tag{4}$$

$$\sigma_{zz}(r, z) = \frac{G}{(1-\nu)} H_0[(1+\xi z)\psi(\xi)\, e^{-\xi z}; \xi \to r] \tag{5}$$

$$\sigma_{rz}(r, z) = \frac{Gz}{(1-\nu)} H_1[\xi\psi(\xi)\, e^{-\xi z}; \xi \to r] \tag{6}$$

where G and ν are the shear modulus and Poisson's ratio for the elastic material, $\psi(\xi)$ is an unknown function and the notation

$$H_n[g(\xi, z); \xi \to r] = \int_0^\infty \xi g(\xi, z) J_n(\xi r)\, d\xi \tag{7}$$

is used to denote the Hankel transform of order n ($n = 0, 1$) with respect to the variable ξ. Using (4)–(6) it is evident that the mixed boundary conditions (1) and (2) yield the following set of dual integral equations

$$H_0[\xi^{-1}\psi(\xi); \xi \to r] = D - f(r); \quad 0 \le r < a \tag{8}$$

$$H_0[\psi(\xi); \xi \to r] = 0; \quad a < r < \infty \tag{9}$$

The solution of the dual integral equations (8) and (9) is given by Sneddon [10]; by representing $\psi(\xi)$ by a result of the form

$$\psi(\xi) = \int_0^a \chi(t) \cos(\xi t)\, dt \tag{10}$$

the boundary condition (9) is identically satisified and the equation (8) reduces to Abel's integral equation. In the ensuing we shall record the results of primary interest to the examination of the unilateral contact problem. It can be shown that the function $\chi(t)$ is related to the displacement D and $f(r)$ by the relationship

$$\chi(t) = -\frac{2D}{\pi} + \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{r f(r)\, dr}{(t^2 - r^2)^{1/2}} \tag{11}$$

Since the contact between the rigid sphere and the elastic halfspace is assumed to be smooth and unilateral the contact stress σ_{zz} at the interface must reduce to zero at the boundary of the contact region $r = a$. This condition which is equivalent to $\chi(a) = 0$, can be used to determine an integral expression for the displacement D, i.e.,

$$D = \int_0^a \left(\frac{df}{dr}\right) \frac{dr}{(a^2 - r^2)^{1/2}} \tag{12}$$

Using the results (5), (10) and (11) we can derive an expression for the contact stresses σ_{zz} at the spherical rigid punch-elastic medium interface in the following form

$$\sigma_{zz}(r) = \frac{G}{r(1-\nu)} \frac{d}{dr} \int_r^a \frac{t\chi(t)\, dt}{(t^2 - r^2)^{1/2}} \tag{13}$$

Finally, the total load at the contact region between the sphere and the halfspace is given by

$$P = \frac{4G}{(1-\nu)} \int_0^a \left(\frac{df}{dr}\right) \frac{r^2\, dr}{(a^2 - r^2)^{1/2}} \tag{14}$$

The indentation problem

We examine the axisymmetric problem of the indentation of the halfspace by a smooth spherical rigid punch, in the presence of an axially directed Mindlin force. Considering the surface displacements of the elastic halfspace region purely due to the action of the Mindlin force P_M located at a distance c from the surface of the halfspace (Figure 1), we have

$$u_z^M(r) = \frac{P_M(1-\nu)}{2\pi G} \left[\frac{1}{(r^2 + c^2)^{1/2}} + \frac{c^2}{2(1-\nu)(r^2 + c^2)^{3/2}}\right] \tag{15}$$

The displacement field $u_z(r)$ associated with the indentation as defined by (1) can be written as

$$u_z(r) = -D + f(r) \tag{16}$$

where

$$f(r) = \{u_z^M(r) - u_z^M(0)\} - \{R - \sqrt{R^2 - r^2}\} \tag{17}$$

such that $f(0) = 0$. Also in (16)

$$D = D^* - \frac{P_M(3 - 2\nu)}{4\pi G c} \tag{18}$$

where D^* is the net rigid displacement of the spherical punch under the combined action of the Mindlin force P_M and the indenting force P. From (11) and (17) we obtain the following result for $\chi(t)$:

$$\chi(t) = -\frac{2D}{\pi} - \frac{2}{\pi}\left\{\frac{P_M(1-\nu)}{2\pi G} \frac{t^2}{c(t^2 + c^2)}\left[1 + \frac{(t^2 + 3c^2)}{2(1-\nu)(t^2 + c^2)}\right] + \frac{t}{2}\ln\left[\left|\frac{R+t}{R-t}\right|\right]\right\} \tag{19}$$

Using the condition $\chi(a) = 0$ and the result (18) we obtain the following result for the indentation D^*:

$$\frac{D^*}{P_M(3-2\nu)/4\pi Gc} = 1 - \frac{2(1-\nu)}{(3-2\nu)}\frac{a^2}{(a^2+c^2)}\left[1 + \frac{(a^2+3c^2)}{2(1-\nu)(a^2+c^2)}\right] - \frac{2\pi Gac}{P_M(3-2\nu)}\ln\left[\left|\frac{R+a}{R-a}\right|\right] \tag{20}$$

By using (14) and (17), the force P required to initiate a contact region of radius a can be evaluated in the following form

$$\frac{P}{P_M} = \frac{2}{\pi}\left\{\frac{ac}{(a^2+c^2)} - \tan^{-1}\left(\frac{a}{c}\right) - \frac{a^3c}{(1-\nu)(a^2+c^2)^2}\right\} + \frac{2G(R^2+a^2)}{(1-\nu)P_M}\left\{\frac{Ra}{(R^2+a^2)} - \sinh^{-1}\left[\frac{a}{(R^2-a^2)^{1/2}}\right]\right\} \tag{21}$$

The contact stress distribution at the spherical punch-elastic halfspace interface can be obtained from the expression (13); we have

$$\begin{aligned}\frac{\sigma_{zz}}{P/\pi a^2} = &-\left(\frac{P_M}{P}\right)\frac{1}{\pi(1-\nu)}\left\{\left[\frac{(1-2\nu)a^2c}{2(r^2+c^2)^{3/2}} + \frac{3a^2c^3}{2(r^2+c^2)^{5/2}}\right]\right.\\ &\times\left[\frac{[(a^2-r^2)(r^2+c^2)]^{1/2}}{(a^2+c^2)} + \tan^{-1}\left[\left(\frac{a^2-r^2}{r^2+c^2}\right)^{1/2}\right]\right]\\ &\left.+\frac{a^2c^3(a^2-r^2)^{1/2}}{(r^2+c^2)(a^2+c^2)^2}\right\}\\ &-\left(\frac{P_M}{P}\right)\frac{2Ga^2}{P_M(1-\nu)}\int_r^a\left[\frac{Rt}{(R^2-t^2)} + \frac{1}{2}\ln\left|\frac{R+t}{R-t}\right|\right]\frac{dt}{(t^2-r^2)^{1/2}}\end{aligned} \tag{22}$$

Numerical results

Owing to the closed form nature of the final results given in the preceding section it is sufficient to present some typical numerical results which are of engineering interest. There are a number of non-dimensional parameters which will be used in the presentation of the results; these include (i) the internal Mindlin force to external load ratio (P_M/P); (ii) the radius of the rigid sphere in relation to the location of the Mindlin force (R/c) and (iii) the magnitude of the Mindlin force in relation to GR^2. Figures 2–4 illustrate the manner in which the displacement of the rigid sphere is influenced by the Poisson's ratio of the elastic medium and the non-dimensional loading and geometrical parameters P_M/P and R/c respectively. It is evident that as P_M/P becomes large, the contact region dimin-

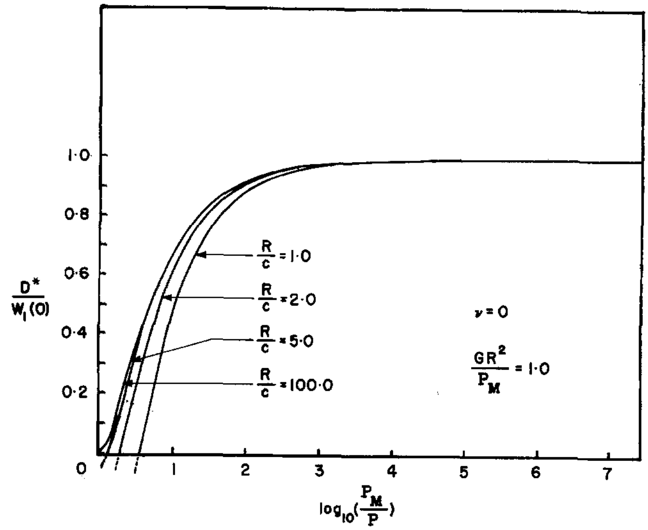

Figure 2
Resultant displacement of the rigid spherical punch; $w_1(0) = P_M(3-2\nu)/4\pi Gc$.

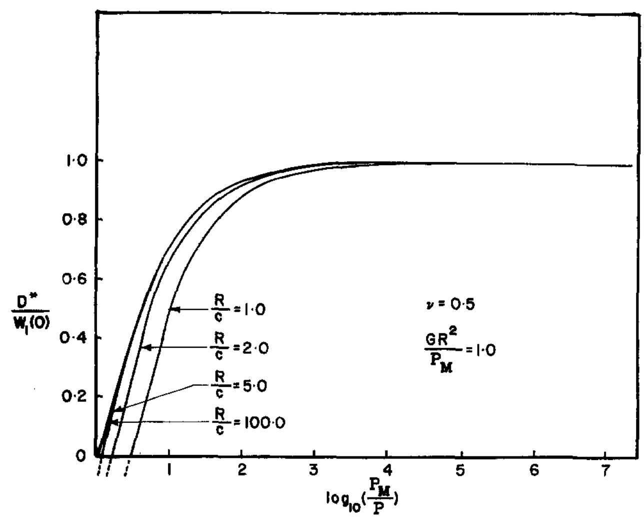

Figure 3
Resultant displacement of the rigid spherical punch.

ishes and the rigid displacement of the sphere approaches the limiting value $P_M(3-2\nu)/4\pi Gc$. Figure 4 also illustrates the fact that when $P_M/P \ll 1$ the indentation D^* occurs in the direction of the force P. Figures 5 and 6 illustrate the extent to which the normalized radius of the contact zone is influenced by the variables P_M/P, R/c and Poisson's ratio ν. Again it is evident that as P_M increases, point contact is established between the rigid

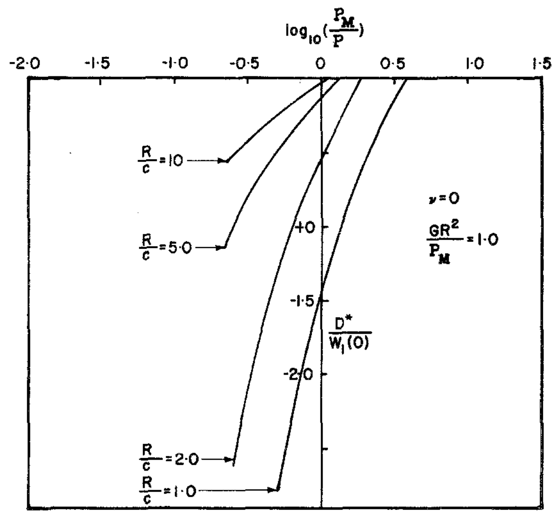

Figure 4
Resultant displacement of the rigid spherical punch.

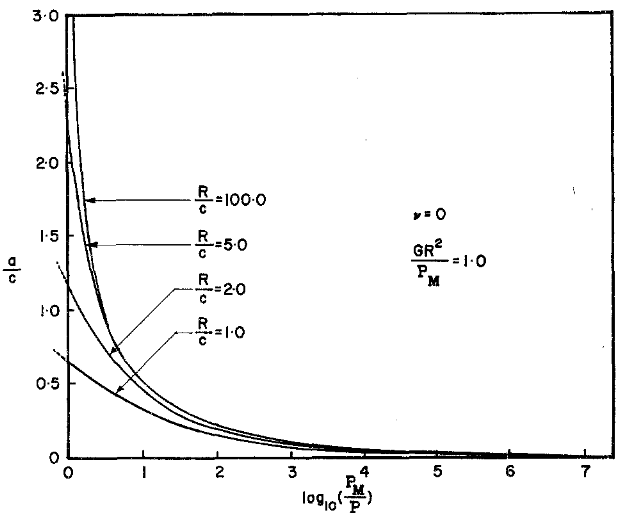

Figure 5
The variation in the radius of the contact region.

sphere and the elastic halfspace. Finally, Figures 7 to 10 illustrate the distribution of normal stress at the contact region between the rigid sphere and the halfspace. It may be observed that as the point of application of the Mindlin force is remote from the surface of the halfspace (i.e., $a/c \rightarrow 0$), the contact stress distribution exhibits the conventional parabolic shape associated with the Hertzian contact between a sphere and a halfspace.

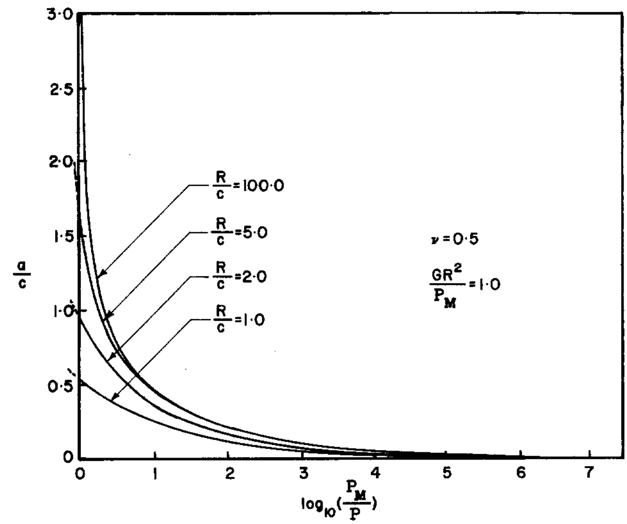

Figure 6
The variation of radius of the contact region.

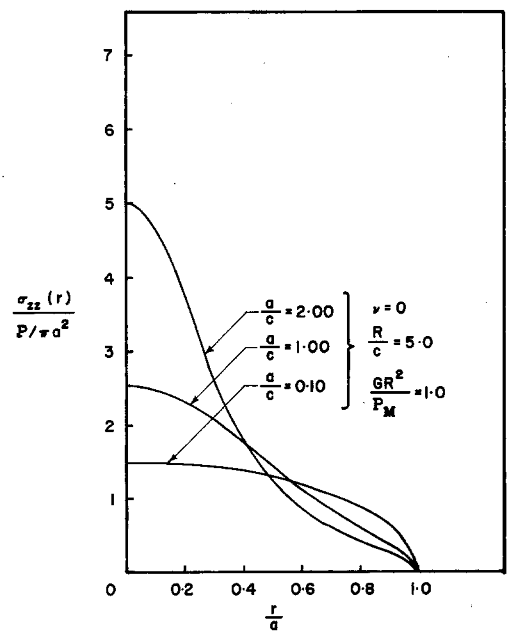

Figure 7
Contact stress distribution at the spherical punch-elastic medium interface.

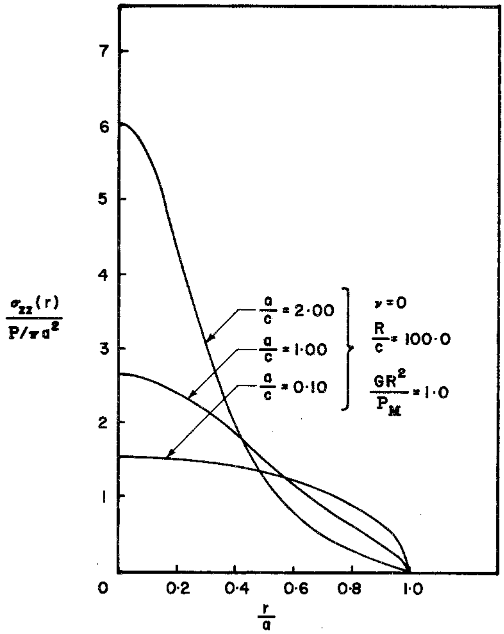

Figure 8
Contact stress distribution at the spherical punch-elastic medium interface.

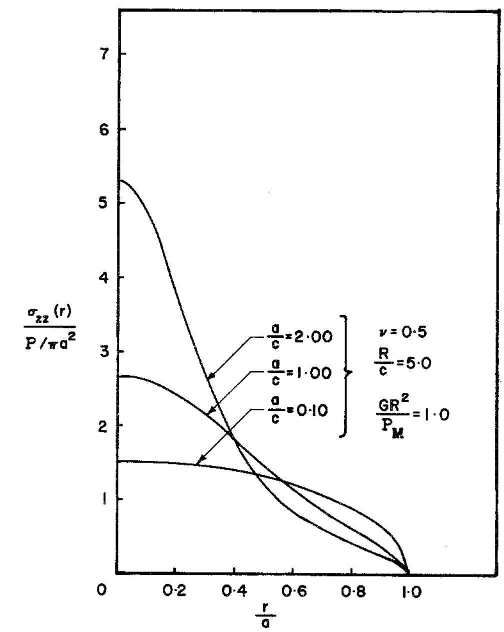

Figure 9
Contact stress distribution at the spherical punch-elastic medium interface.

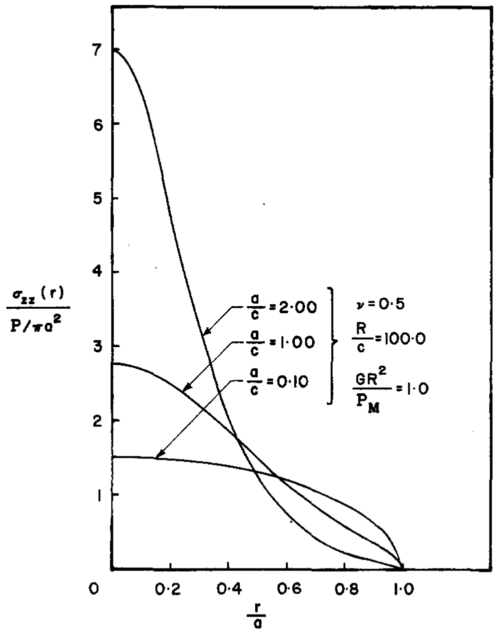

Figure 10
Contact stress distribution at the spherical punch-elastic medium interface.

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Abstract

The smooth axisymmetric contact between a rigid spherical punch and an isotropic elastic halfspace is perturbed by an internally applied Mindlin force which is located along the axis of symmetry. The paper presents certain closed form solutions to this unilateral contact problem.

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