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SECOND-ORDER ELASTICITY WITH AXIAL SYMMETRY–I GENERAL THEORY

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Abstract–The equations of second-order elasticity are developed for axially symmetric deformations of incompressible isotropic elastic materials. It is shown that by introducing a 'displacement function' the second-order problem can be reduced to the solution of an equation of the form $E^4\psi = f(R, Z)$ where E^2 is Stokes' differential operator, R, Z are cylindrical polar coordinates, and $f(R, Z)$ depends only on the first-order solution. The problem is formulated in terms of both cylindrical polar and spherical polar coordinates.

1. INTRODUCTION

BECAUSE of the non-linear character of the equations, in the solution of boundary value problems in the theory of finite elasticity recourse must often be made to approximate methods. One such method which has been employed quite extensively is the method of successive approximations, in which the first approximation is taken to be the solution of a problem in linear elasticity. The second approximation includes terms which are quadratic in the displacement gradients, and so on. Among the first to formulate such theories were Rivlin[1] and Green and Spratt[2]. The most comprehensive account, which gives additional references up to 1965, is given in Green and Adkins[3]. This uses the convected curvilinear coordinate formulation of finite elasticity, with its associated general tensor calculus. A description using cartesian tensor notation is given by Spencer[4]; this also contains some references to more recent work. Reference should also be made to further developments in a recent paper by Chan and Carlson[5].

Second-order elasticity is obtained by terminating the successive approximation procedure after the second approximation. In fact because of the algebraic complexity of the higher-order approximations it is only in rare cases practicable to go beyond the second approximation. The second-order theory is regarded as adequate to describe the mechanical behaviour of rubber-like materials at moderate, but not at very large, strains.

Applications of second-order elasticity theory have been to problems involving a high degree of symmetry, such as bending, torsion, plane strain and plane stress. In this and subsequent papers we consider problems of axially symmetric deformations, which as far as we are aware have not been considered before except in cases of plane strain and plane stress. We deal only with incompressible isotropic elastic materials. The formulation is in terms of physical components of the tensors involved, and quantities are referred to cylindrical or spherical polar coordinates, as convenient, of particles of a body in its undeformed and deformed configurations.

Section 2 summarises the theory of incompressible finite elasticity expressed in terms of cylindrical polar coordinates. In section 3 the equations of second-order elasticity, referred to these coordinates, are developed. In section 4 we show how the solution of these equations is facilitated by introduction of a 'displacement function'.

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By introducing this function we make direct use of the assumed incompressibility of the material and reduce the solution of the second-order problem to the solution of a single equation of the form $E^4\psi = f(R, Z)$, where E^2 is Stokes operator [see equation (4.4)] and f depends only on the first-order solution. This 'displacement function' technique, which is particularly convenient when displacement boundary conditions are specified, has also been applied to plane strain problems of incompressible second-order elasticity[6] where it provides a useful alternative to the Airy stress function approach. In Section 5 the same theory is developed in terms of spherical polar coordinates.

This paper deals with the basic theory only. Applications to special problems will be made in subsequent papers.

2. FINITE ELASTICITY WITH AXIAL SYMMETRY. CYLINDRICAL POLAR COORDINATES

We initially refer the deformation to a set of rectangular cartesian coordinates, and denote by X_A $(A = 1, 2, 3)$ the coordinates in this frame of a generic particle in its reference configuration and by $x_i(X_A)$ $(i = 1, 2, 3)$ the coordinates of the same particle in a deformed configuration. Cylindrical polar coordinates (R, Φ, Z) of particles in the reference state and (r, ϕ, z) of particles in the deformed state are given by

$$\begin{aligned} X_1 &= R\cos\Phi, \quad X_2 = R\sin\Phi, \quad X_3 = Z, \\ x_1 &= r\cos\phi, \quad x_2 = r\sin\phi, \quad x_3 = z. \end{aligned} \tag{2.1}$$

For axially symmetric deformations we have

$$r = r(R, Z), \quad \phi = \Phi, \quad z = z(R, Z). \tag{2.2}$$

With (2.2), the matrix $\mathbf{F}$ of deformation gradients in the directions of R, Φ, Z is

$$\mathbf{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & 0 & \dfrac{\partial r}{\partial Z} \\ 0 & \dfrac{r}{R} & 0 \\ \dfrac{\partial z}{\partial R} & 0 & \dfrac{\partial z}{\partial Z} \end{bmatrix}. \tag{2.3}$$

We consider incompressible materials for which

$$\det \mathbf{F} = \frac{r}{R}\left\{\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right\} = 1. \tag{2.4}$$

The matrix of the physical components of the Cauchy–Green strain tensor, referred to (r, ϕ, z) coordinates, is

$$\mathbf{B}=\mathbf{F}\mathbf{F}^T=\begin{bmatrix}\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial r}{\partial Z}\right)^2 & 0 & \frac{\partial r}{\partial R}\frac{\partial z}{\partial R}+\frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z}\\ 0 & \frac{r^2}{R^2} & 0\\ \frac{\partial r}{\partial R}\frac{\partial z}{\partial R}+\frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z} & 0 & \left(\frac{\partial z}{\partial R}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2\end{bmatrix} \tag{2.5}$$

and, with (2.4), the inverse of **B** is

$$\mathbf{B}^{-1}=\begin{bmatrix}\frac{r^2}{R^2}\left[\left(\frac{\partial z}{\partial R}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2\right] & & -\frac{r^2}{R^2}\left[\frac{\partial r}{\partial R}\frac{\partial z}{\partial R}+\frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z}\right]\\ 0 & \left[\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z}-\frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right]^2 & 0\\ -\frac{r^2}{R^2}\left[\frac{\partial r}{\partial R}\frac{\partial z}{\partial R}+\frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z}\right] & 0 & \frac{r^2}{R^2}\left[\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial r}{\partial Z}\right)^2\right]\end{bmatrix}. \tag{2.6}$$

The strain invariants of **B** are

$$\begin{aligned}I_1&=\operatorname{tr}\mathbf{B}=\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial r}{\partial Z}\right)^2+\frac{r^2}{R^2}+\left(\frac{\partial z}{\partial R}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2,\\ I_2&=\operatorname{tr}\mathbf{B}^{-1}=\frac{r^2}{R^2}\left[I_1+\frac{R^4}{r^4}-\frac{r^2}{R^2}\right],\quad I_3=\det\mathbf{B}=1.\end{aligned} \tag{2.7}$$

The matrix **S** of the physical components of the symmetrical Cauchy stress tensor (given by resolving the force per unit area in the deformed body in the directions of the axes in the deformed body), referred to (r, ϕ, z) coordinates, is

$$\mathbf{S}=\begin{bmatrix}s_{rr} & 0 & s_{rz}\\ 0 & s_{\phi\phi} & 0\\ s_{rz} & 0 & s_{zz}\end{bmatrix} \tag{2.8}$$

where the zero components are due to the assumed axial symmetry. It is more convenient to work in terms of the transposed nonsymmetrical Piola–Kirchoff stress tensor, of which the matrix **T** of components is, for incompressible material and for the axially symmetric deformations considered here

$$\mathbf{T}=\mathbf{F}^{-1}\mathbf{S}=\begin{bmatrix}t_{Rr} & 0 & t_{Rz}\\ 0 & t_{\Phi\phi} & 0\\ t_{Zr} & 0 & t_{Zz}\end{bmatrix}. \tag{2.9}$$

Here t_{Rr}, for example, denotes the component in the r direction of the force per unit area of an element which is normal to the R axis in the reference configuration. Note that in general $t_{Rz} \neq t_{Zr}$.

The constitutive equation for an incompressible, isotropic elastic material, expressed in cylindrical polar coordinates with axial symmetry, can be stated in the form

$$\mathbf{S} = -p\mathbf{I} + \chi_1 \mathbf{B} + \chi_{-1}\mathbf{B}^{-1}, \tag{2.10}$$

where p is an arbitrary scalar function, and the response functions χ_1 and χ_{-1} are functions of the strain invariants I_1 and I_2. For the present purpose it is immaterial whether or not we regard χ_1 and χ_{-1} as being derived from a strain-energy function. In terms of $\mathbf{T}$, (2.10) takes the form

$$\mathbf{T} = -p\mathbf{F}^{-1} + \chi_1 \mathbf{F}^T + \chi_{-1}\mathbf{F}^{-1}\mathbf{B}^{-1}. \tag{2.11}$$

The equations of equilibrium, in the absence of body forces, are

$$\begin{aligned} \frac{\partial t_{Rr}}{\partial R} + \frac{\partial t_{Zr}}{\partial Z} + \frac{t_{Rr} - t_{\Phi\phi}}{R} = 0, \\ \frac{\partial t_{Rz}}{\partial R} + \frac{\partial t_{Zz}}{\partial Z} + \frac{t_{Rz}}{R} = 0. \end{aligned} \tag{2.12}$$

The components, in the r and z directions, of the traction on a surface $F(R, Z) = 0$ in the undeformed body, are

$$F_r = n_R t_{Rr} + n_Z t_{Zr}, \quad F_z = n_R t_{Rz} + n_Z t_{Zz}, \tag{2.13}$$

where

$$n_R^2 + n_Z^2 = 1, \quad \frac{n_R}{n_Z} = \frac{\partial F/\partial R}{\partial F/\partial Z}. \tag{2.14}$$

3. SECOND-ORDER ELASTICITY

It is now assumed that the coordinates (r, z) of the particle which initially has coordinates R, Z can be expressed as asymptotic power series in a dimensionless parameter ϵ as $\epsilon \to 0$, as follows

$$r = R + u = R + \sum_{n=1}^{\infty} \epsilon^n u_n(R, Z), \quad z = Z + w = Z + \sum_{n=1}^{\infty} \epsilon^n w_n(R, Z). \tag{3.1}$$

The parameter ϵ is at this stage unspecified; usually a natural choice of ϵ arises in the course of discussion of specific problems. Retention of the first terms only in the series in (3.1) gives rise to classical linear elasticity, in this case specialised to incompressible material and axially symmetric deformations. Second-order elasticity results from retaining the terms corresponding to $n = 1$ and $n = 2$ and consistently neglecting terms involving ϵ^3 and higher powers of ϵ. Higher order theories are readily formulated in principle but their algebraic complexity increases rapidly as higher order terms are included. For second-order elasticity we take

$$r = R + \epsilon u_1(R,Z) + \epsilon^2 u_2(R,Z),$$
$$z = Z + \epsilon w_1(R,Z) + \epsilon^2 w_2(R,Z), \tag{3.2}$$

and we will neglect, usually without further comment, terms which contain as factors ϵ^3 and higher powers of ϵ.

By substituting (3.2) into (2.4) and equating coefficients of ϵ and ϵ^2, we obtain the first and second order incompressibility conditions

$$\frac{\partial u_1}{\partial R} + \frac{u_1}{R} + \frac{\partial w_1}{\partial Z} = 0, \tag{3.3}$$

$$\frac{\partial u_2}{\partial R} + \frac{u_2}{R} + \frac{\partial w_2}{\partial Z} = \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial R}\right)^2 + \left(\frac{u_1}{R}\right)^2 + \left(\frac{\partial w_1}{\partial Z}\right)^2 + 2\frac{\partial u_1}{\partial Z}\frac{\partial w_1}{\partial R}\right\}. \tag{3.4}$$

By substituting (3.2) into (2.3) we have

$$\mathbf{F} = \mathbf{I} + \epsilon \mathbf{F}_1 + \epsilon^2 \mathbf{F}_2,$$
$$\mathbf{F}^{-1} = \mathbf{I} - \epsilon \mathbf{F}_1 - \epsilon^2(\mathbf{F}_2 - \mathbf{F}_1^2), \tag{3.5}$$

where

$$\mathbf{F}_1 = \begin{bmatrix} \frac{\partial u_1}{\partial R} & 0 & \frac{\partial u_1}{\partial Z} \\ 0 & \frac{u_1}{R} & 0 \\ \frac{\partial w_1}{\partial R} & 0 & \frac{\partial w_1}{\partial Z} \end{bmatrix}, \quad \mathbf{F}_2 = \begin{bmatrix} \frac{\partial u_2}{\partial R} & 0 & \frac{\partial u_2}{\partial Z} \\ 0 & \frac{u_2}{R} & 0 \\ \frac{\partial w_2}{\partial R} & 0 & \frac{\partial w_2}{\partial Z} \end{bmatrix}. \tag{3.6}$$

From (2.5), (2.6) and (3.5) it then follows that

$$\mathbf{B} = \mathbf{I} + \epsilon \mathbf{B}_1 + \epsilon^2 \mathbf{B}_2,$$
$$\mathbf{B}^{-1} = \mathbf{I} - \epsilon \mathbf{B}_1 - \epsilon^2(\mathbf{B}_2 - \mathbf{B}_1^2) \tag{3.7}$$

where

$$\mathbf{B}_1 = \mathbf{F}_1 + \mathbf{F}_1^T, \quad \mathbf{B}_2 = \mathbf{F}_2 + \mathbf{F}_2^T + \mathbf{F}_1\mathbf{F}_1^T. \tag{3.8}$$

From (3.3), (3.6) and (3.8) it follows that $\operatorname{tr}\mathbf{B}_1 = 0$. Hence, from (2.7) and (3.7), I_1 and I_2 take the forms

$$I_1 = 3 + \epsilon^2 I_1^{(2)}, \quad I_2 = 3 + \epsilon^2 I_2^{(2)}, \tag{3.9}$$

where

$$I_1^{(2)} = \operatorname{tr}\mathbf{B}_2, \quad I_2^{(2)} = -I_1^{(2)} + \operatorname{tr}\mathbf{B}_1^2. \tag{3.10}$$

We now assume that the response functions χ_1 and χ_{-1} can be expressed as Taylor series in I_1-3 and I_2-3, in the form

$$\begin{aligned} \chi_1 &= 2C_1+(I_1-3)\frac{\partial \chi_1}{\partial I_1}+(I_2-3)\frac{\partial \chi_1}{\partial I_2}+\cdots, \\ \chi_{-1} &= -2C_2+(I_1-3)\frac{\partial \chi_{-1}}{\partial I_1}+(I_2-3)\frac{\partial \chi_{-1}}{\partial I_2}+\cdots, \end{aligned} \tag{3.11}$$

where $2C_1$ and $-2C_2$ denote the values of χ_1 and χ_{-1} at $I_1=3, I_2=3$, and all derivatives of χ_1 and χ_{-1} are evaluated at $I_1=3$, $I_2=3$. This notation is chosen because $2C_1$ and $-2C_2$ are the constant values of χ_1 and χ_{-1} when the elastic material has a strain-energy function of the Mooney–Rivlin form

$$W=C_1(I_1-3)+C_2(I_2-3)$$

but the present work is not dependent on W having this form, or even on the existence of a strain-energy function. From (3.9) and (3.11) it follows that, up to terms in ϵ^2

$$\begin{aligned} \chi_1 &= 2C_1+\epsilon^2\left[I_1^{(2)}\frac{\partial \chi_1}{\partial I_1}+I_2^{(2)}\frac{\partial \chi_1}{\partial I_2}\right], \\ \chi_{-1} &= -2C_2+\epsilon^2\left[I_1^{(2)}\frac{\partial \chi_{-1}}{\partial I_1}+I_2^{(2)}\frac{\partial \chi_{-1}}{\partial I_2}\right]. \end{aligned} \tag{3.12}$$

It is further assumed that $\mathbf{T}$ and p can be expressed as power series in ϵ, in the forms

$$\mathbf{T}=\sum_{n=0}^{\infty}\epsilon^n\mathbf{T}_n, \quad p=\sum_{n=0}^{\infty}\epsilon^n p_n. \tag{3.13}$$

We now substitute (3.5), (3.8), (3.12) and (3.13) into the constitutive equation (2.11) and equate coefficients of powers of ϵ. The terms independent of ϵ give

$$\mathbf{T}_0=-(p_0-2C_1+2C_2)\mathbf{I}. \tag{3.14}$$

If this is to satisfy the equilibrium equations, $\mathbf{T}_0$ must be a uniform hydrostatic stress. Since such a stress is, for incompressible material, independent of the deformation, we may without loss of generality take it to be zero, and if necessary superpose any required constant hydrostatic stress on any stress field obtained under this assumption. Thus we take $p_0=2(C_1-C_2)$. The terms linear in ϵ then give

$$\mathbf{T}_1=-p_1\mathbf{I}+2(C_1+C_2)\mathbf{B}_1. \tag{3.15}$$

This is the constitutive equation of linear incompressible elasticity if we identify $2(C_1+C_2)$ with the elastic shear modulus μ. We thus denote

$$\mu=2(C_1+C_2) \tag{3.16}$$

and (3.15) becomes

$$\mathbf{T}_1=-p_1\mathbf{I}+\mu\mathbf{B}_1. \tag{3.17}$$

Equating coefficients of ϵ^2 gives

$$\mathbf{T}_2 = -p_2\mathbf{I} + p_1\mathbf{F}_1 + \mu(\mathbf{B}_2 - \mathbf{F}_1\mathbf{B}_1) - 2C_2\mathbf{B}_1^2, \tag{3.18}$$

where we have used (3.8) and (3.16), and absorbed some other terms into p_2. Equation (3.18) is the constitutive equation for the second order stress components. We note that to this order the stress depends on χ_1 and χ_{-2} only through their values $2C_1$ and $-2C_2$ at $I_1 = 3, I_2 = 3$.

We further develop (3.18) by introducing new variables u_2' and w_2' by the relations

$$u_2 = u_2' + u_2'', \quad w_2 = w_2' + w_2'' \tag{3.19}$$

where

$$u_2'' = \frac{1}{2}\left(u_1\frac{\partial u_1}{\partial R} + w_1\frac{\partial u_1}{\partial Z}\right), \quad w_2'' = \frac{1}{2}\left(u_1\frac{\partial w_1}{\partial R} + w_1\frac{\partial w_1}{\partial Z}\right), \tag{3.20}$$

or, more briefly

$$u_2'' = \tfrac{1}{2}Du_1, \quad w_2'' = \tfrac{1}{2}Dw_1,$$

where D denotes the operator

$$D \equiv u_1\frac{\partial}{\partial R} + w_1\frac{\partial}{\partial Z}. \tag{3.21}$$

On substituting (3.19) and (3.20) in the second-order incompressibility condition (3.4), this condition takes the simple form

$$\frac{\partial u_2'}{\partial R} + \frac{u_2'}{R} + \frac{\partial w_2'}{\partial Z} = 0. \tag{3.22}$$

The substitutions (3.19) and (3.20) are a special case of a result given by Chan and Carlson[5]. Other expressions besides (3.20) which reduce (3.4) to (3.22) exist; two examples are

$$\text{(i)}\quad u_2'' = \frac{1}{2}u_1\left(\frac{\partial u_1}{\partial R} - \frac{\partial w_1}{\partial Z}\right), \quad w_2'' = u_1\frac{\partial w_1}{\partial R}$$

$$\text{(ii)}\quad u_2'' = w_1\frac{\partial u_1}{\partial Z} - \frac{1}{2}\frac{u_1^2}{R}, \quad w_2'' = w_1\frac{\partial w_1}{\partial Z}$$

but (3.20) is most convenient for our purposes and will be used in this paper.

We now denote

$$\mathbf{F}_2' = \begin{bmatrix} \frac{\partial u_2'}{\partial R} & 0 & \frac{\partial u_2'}{\partial Z} \\ 0 & \frac{u_2'}{R} & 0 \\ \frac{\partial w_2'}{\partial R} & 0 & \frac{\partial w_2'}{\partial Z} \end{bmatrix}, \quad \mathbf{F}_2'' = \begin{bmatrix} \frac{\partial u_2''}{\partial R} & 0 & \frac{\partial u_2''}{\partial Z} \\ 0 & \frac{u_2''}{R} & 0 \\ \frac{\partial w_2''}{\partial R} & 0 & \frac{\partial w_2''}{\partial Z} \end{bmatrix}. \tag{3.23}$$

and it immediately follows from (3.6) and (3.19) that

$$\mathbf{F}_2 = \mathbf{F}_2' + \mathbf{F}_2''. \tag{3.24}$$

Also, using (3.8), we set

$$\mathbf{B}_2 = \mathbf{B}_2' + \mathbf{B}_2'' + \mathbf{B}_2''', \tag{3.25}$$

where

$$\mathbf{B}_2' = \mathbf{F}_2' + \mathbf{F}_2'^T, \quad \mathbf{B}_2'' = \mathbf{F}_2'' + \mathbf{F}_2''^T, \quad \mathbf{B}_2''' = \mathbf{F}_1 \mathbf{F}_1^T. \tag{3.26}$$

From (3.20) and (3.23) it follows, after a little manipulation, that

$$\mathbf{F}_2'' = \tfrac{1}{2} D\mathbf{F}_1 + \tfrac{1}{2}\mathbf{F}_1^2.$$

Hence, using (3.25) and (3.26), we may express (3.18) in the form

$$\mathbf{T}_2 = -p_2 \mathbf{I} + \mu \mathbf{B}_2' + \mathbf{T}_2', \tag{3.27}$$

where

$$\mathbf{T}_2' = p_1 \mathbf{F}_1 + \tfrac{1}{2}\mu[D\mathbf{B}_1 - \mathbf{F}_1^2 + (\mathbf{F}_1^T)^2] - 2C_2 \mathbf{B}_1^2 \tag{3.28}$$

and $\mathbf{T}_2'$ depends only on the first-order solution.

From (2.12) and (3.13) the n-th order equilibrium equations are

$$\begin{aligned} &\frac{\partial t_{Rr}^{(n)}}{\partial R} + \frac{\partial t_{Zr}^{(n)}}{\partial Z} + \frac{t_{Rr}^{(n)} - t_{\Phi\Phi}^{(n)}}{R} = 0, \\ &\frac{\partial t_{Rz}^{(n)}}{\partial R} + \frac{\partial t_{Zz}^{(n)}}{\partial Z} + \frac{t_{Rz}^{(n)}}{R} = 0, \end{aligned} \tag{3.29}$$

for $n = 1, 2, \ldots$, where $t_{Rr}^{(n)}, t_{\Phi\Phi}^{(n)}, t_{Zz}^{(n)}, t_{Rz}^{(n)}, t_{Zr}^{(n)}$ are the nonzero elements of $\mathbf{T}_n$.

Equations (3.3), (3.17) and (3.29) with $n = 1$, are the equations of axially symmetric linear incompressible elasticity. A solution of these determines u_1, w_1, p_1 and $\mathbf{T}_1$, and hence the matrix $\mathbf{T}_2'$. Then (3.22), (3.27) and (3.29) with $n = 2$ are a set of equations for u_2', w_2', p_2 and $\mathbf{T}_2$, which we refer to as the equations of second-order elasticity for axially symmetric deformations of incompressible materials. These equations are of the same form as the first-order equations except that they contain inhomogeneous terms represented by the elements of $\mathbf{T}_2'$, which are determined by the first-order solution.

4. DISPLACEMENT FUNCTIONS

We consider first the linear elastic equations. The first-order incompressibility condition (3.3) may be satisfied identically by the introduction of a displacement function ψ_1 such that

$$u_1 = \frac{1}{R}\frac{\partial \psi_1}{\partial Z}, \quad w_1 = -\frac{1}{R}\frac{\partial \psi_1}{\partial R}. \tag{4.1}$$

In terms of ψ_1, (3.17) becomes

$$t^{(1)}_{Rr} = -p_1 + 2\mu\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi_1}{\partial Z}\right), \quad t^{(1)}_{\Phi\phi} = -p_1 + 2\mu\frac{1}{R^2}\frac{\partial \psi_1}{\partial Z}, \tag{4.2}$$

$$t^{(1)}_{Zz} = -p_1 - 2\mu\frac{1}{R}\frac{\partial^2 \psi_1}{\partial R \partial Z}, \quad t^{(1)}_{Rz} = t^{(1)}_{Zr} = \mu\left\{\frac{1}{R}\frac{\partial^2 \psi_1}{\partial Z^2} - \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi_1}{\partial R}\right)\right\}.$$

By substituting (4.2) in (3.29) ($n = 1$) we obtain

$$\begin{aligned} -\frac{\partial p_1}{\partial R} + \mu\left[\frac{1}{R}\frac{\partial}{\partial Z}(E^2\psi_1)\right] &= 0, \\ -\frac{\partial p_1}{\partial Z} - \mu\left[\frac{1}{R}\frac{\partial}{\partial R}(E^2\psi_1)\right] &= 0, \end{aligned} \tag{4.3}$$

where E^2 is Stokes' operator

$$E^2\psi_1 = \left(\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2}\right)\psi_1. \tag{4.4}$$

By eliminating p_1 from (4.3) we have

$$E^4\psi_1 = 0, \tag{4.5}$$

and by eliminating ψ_1 from (4.3)

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2}\right)p_1 = \nabla^2 p_1 = 0. \tag{4.6}$$

This formulation in terms of ψ_1 has a close formal resemblance to the theory of axially symmetric slow flows of an incompressible Newtonian viscous liquid in terms of Stokes' stream function. It is particularly convenient when displacement boundary conditions are given, but can also be readily applied to many problems which have traction boundary conditions. It does not seem to have been exploited to any great extent in linear elasticity, presumably because the incompressible theory has only limited practical interest in the context of small deformation theory. This is not the case with the second-order theory, which can often be applied to materials such as many natural and artificial rubbers which are virtually incompressible.

We adopt a similar approach in the case of the second-order equations. The second-order incompressibility condition, in the form (3.22), is satisfied identically by expressing u_2' and w_2' in terms of a second-order displacement function ψ_2 by the relations

$$u_2' = \frac{1}{R}\frac{\partial \psi_2}{\partial Z}, \quad w_2' = -\frac{1}{R}\frac{\partial \psi_2}{\partial R}. \tag{4.7}$$

The second-order stress components expressed in terms of ψ_2 are, from (3.27) and (4.7)

$$
\begin{aligned}
t_{Rr}^{(2)} &= -p_2 + 2\mu\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi_2}{\partial Z}\right) + T'_{Rr},\\
t_{\Phi\phi}^{(2)} &= -p_2 + 2\mu\frac{1}{R^2}\frac{\partial\psi_2}{\partial Z} + T'_{\Phi\phi},\\
t_{Zz}^{(2)} &= -p_2 - 2\mu\frac{1}{R}\frac{\partial^2\psi_2}{\partial R\partial Z} + T'_{Zz},\\
t_{Rz}^{(2)} &= \mu\left\{\frac{1}{R}\frac{\partial^2\psi_2}{\partial Z^2} - \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi_2}{\partial R}\right)\right\} + T'_{Rz},\\
t_{Zr}^{(2)} &= \mu\left\{\frac{1}{R}\frac{\partial^2\psi_2}{\partial Z^2} - \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi_2}{\partial R}\right)\right\} + T'_{Zr},
\end{aligned}
\tag{4.8}
$$

where T'_{Rr}, $T'_{\Phi\phi}$, T'_{Zz}, T'_{Rz}, T'_{Zr} are the non-zero elements of $\mathbf{T}'_2$. If we now substitute (4.8) in (3.29) (with $n = 2$) we obtain

$$
\begin{aligned}
-\frac{\partial p_2}{\partial R} + \mu\left[\frac{1}{R}\frac{\partial}{\partial Z}(E^2\psi_2)\right] &= G_1(R, Z),\\
-\frac{\partial p_2}{\partial Z} - \mu\left[\frac{1}{R}\frac{\partial}{\partial R}(E^2\psi_2)\right] &= G_2(R, Z),
\end{aligned}
\tag{4.9}
$$

where

$$
\begin{aligned}
G_1(R, Z) &= -\left(\frac{\partial T'_{Rr}}{\partial R} + \frac{\partial T'_{Zr}}{\partial Z} + \frac{T'_{Rr} - T'_{\Phi\phi}}{R}\right),\\
G_2(R, Z) &= -\left(\frac{\partial T'_{Rz}}{\partial R} + \frac{\partial T'_{Zz}}{\partial Z} + \frac{T'_{Rz}}{R}\right),
\end{aligned}
\tag{4.10}
$$

and $G_1(R, Z)$ and $G_2(R, Z)$ depend only on the first-order solution. By eliminating p_2 and ψ_2 in turn from (4.9) we find

$$
E^4\psi_2 = \frac{R}{\mu}\left(\frac{\partial G_1}{\partial Z} - \frac{\partial G_2}{\partial R}\right), \tag{4.11}
$$

$$
\nabla^2 p_2 = -\left(\frac{\partial G_1}{\partial R} + \frac{G_1}{R} + \frac{\partial G_2}{\partial Z}\right). \tag{4.12}
$$

The procedure for determining a second-order solution when the first-order solution is known is as follows. From the first-order solution we determine u''_2 and w''_2 from (3.20) and the elements of $\mathbf{T}'_2$ from (3.28). The right-hand side of (4.11) can then be found. It is then necessary to solve (4.11); the solution is the sum of a particular integral and a solution of the homogeneous equation $E^4\psi_2 = 0$. It is of course necessary to select that solution of (4.11) which will eventually satisfy the boundary conditions. Having thus determined ψ_2, u_2' and w_2' are given by (4.7) and so the second-order displacement field is obtained by (3.19). To determine the stress field we now integrate (4.9) to obtain p_2. Since u_2' and w_2' are already found we readily determine $\mathbf{B}_2'$ from (3.23) and (3.26), and $\mathbf{T}_2'$ is known from the first-order solution. Hence $\mathbf{T}_2$ is given by (3.27), and the second-order solution is complete. The components of the Cauchy stress tensor, if required, are given in terms of $\mathbf{T}$ by (2.9). To second-order in ϵ we have, from (2.9), (3.5) and (3.13), with $\mathbf{T}_0 = 0$

$$\mathbf{S} = \epsilon\mathbf{T}_1 + \epsilon^2(\mathbf{T}_2 + \mathbf{F}_1\mathbf{T}_1). \tag{4.13}$$

5. SPHERICAL POLAR COORDINATES

The theory of sections 2–4 may equally well be developed in a similar way using polar coordinates. It is, however, simpler to make use of the results of these earlier spherical sections.

We denote by (S, Φ, Θ) and (s, ϕ, θ) the spherical polar coordinates of particles in the reference and deformed configurations, so that

$$\begin{aligned} r &= S\sin\Theta, \quad Z = S\cos\Theta, \\ r &= s\sin\theta, \quad z = s\cos\theta. \end{aligned} \tag{5.1}$$

For axially symmetric deformations

$$s = s(S, \Theta), \quad \phi = \Phi, \quad \theta = \theta(S, \Theta). \tag{5.2}$$

The matrix of deformation gradients in the directions of S, Φ, Θ is denoted $\mathbf{G}_s$. Thus

$$\mathbf{G}_s = \begin{bmatrix} \dfrac{\partial s}{\partial S} & 0 & \dfrac{1}{S}\dfrac{\partial s}{\partial \Theta} \\ 0 & \dfrac{s\sin\theta}{S\sin\Theta} & 0 \\ s\dfrac{\partial\theta}{\partial S} & 0 & \dfrac{s}{S}\dfrac{\partial\theta}{\partial\Theta} \end{bmatrix}, \tag{5.3}$$

and we note that

$$\mathbf{G}_s = \mathbf{rFR} \tag{5.4}$$

where

$$\mathbf{r} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \sin\Theta & 0 & \cos\Theta \\ 0 & 1 & 0 \\ \cos\Theta & 0 & -\sin\Theta \end{bmatrix}, \tag{5.5}$$

and that $\mathbf{r}$, $\mathbf{R}$ are symmetric orthogonal matrices with determinant equal to -1. We further denote

$$\mathbf{C}_s = \mathbf{G}_s\mathbf{G}_s^T. \tag{5.6}$$

It is convenient to introduce a matrix $\mathbf{G}$ defined by

$$\mathbf{G} = \mathbf{RFR}. \tag{5.7}$$

From (5.4), (5.5) and (5.7) it follows that

$$\mathbf{G} = \mathbf{RrG}_s, \quad \mathbf{G}_s = \mathbf{rRG}, \tag{5.8}$$

and

$$\mathbf{Rr} = (\mathbf{rR})^T = \begin{bmatrix} \cos(\theta-\Theta) & 0 & -\sin(\theta-\Theta) \\ 0 & 1 & 0 \\ \sin(\theta-\Theta) & 0 & \cos(\theta-\Theta) \end{bmatrix} \tag{5.9}$$

is a proper orthogonal matrix. We also denote

$$\mathbf{C} = \mathbf{GG}^T = \mathbf{RrG}_s\mathbf{G}_s^T\mathbf{rR} = \mathbf{RFF}^T\mathbf{R}. \tag{5.10}$$

The matrices of the components of the Cauchy and transposed Piola–Kirchoff stress tensors referred to the spherical polar coordinates are denoted by $\mathbf{S}_s$ and $\mathbf{T}_s$ respectively, and are given by

$$\mathbf{S}_s = \mathbf{rSr}, \tag{5.11}$$

$$\mathbf{T}_s = \mathbf{RTr} = \mathbf{G}_s^{-1}\mathbf{S}_s, \quad \mathbf{S}_s = \mathbf{GT}_s. \tag{5.12}$$

It is, however, more convenient to use as the measure of stress a further non-symmetric matrix $\mathbf{P}$ defined by

$$\mathbf{P} = \begin{bmatrix} p_{Ss} & 0 & p_{S\theta} \\ 0 & p_{\Phi\phi} & 0 \\ p_{\Theta s} & 0 & p_{\Theta\theta} \end{bmatrix} = \mathbf{RTR}. \tag{5.13}$$

From (5.11), (5.12) and (5.13)

$$\mathbf{P} = \mathbf{T}_s\mathbf{rR}, \quad \mathbf{T}_s = \mathbf{PRr}, \quad \mathbf{P} = \mathbf{G}^{-1}\mathbf{S}_s\mathbf{rR}, \quad \mathbf{S}_s = \mathbf{GPRr}. \tag{5.14}$$

The point of introducing $\mathbf{P}$ is that in terms of its elements the equilibrium equations take the simple form

$$\begin{aligned} &\frac{\partial p_{Ss}}{\partial S} + \frac{1}{S}\frac{\partial p_{\Theta s}}{\partial \Theta} + \frac{1}{S}(2p_{Ss} - p_{\Phi\phi} - p_{\Theta\theta} + p_{\Theta s}\cot\Theta) = 0, \\ &\frac{\partial p_{S\theta}}{\partial S} + \frac{1}{S}\frac{\partial p_{\Theta\theta}}{\partial \Theta} + \frac{1}{S}[(p_{\Theta\theta} - p_{\Phi\phi})\cot\Theta + 2p_{S\theta} + p_{\Theta s}] = 0. \end{aligned} \tag{5.15}$$

Neither $\mathbf{T}_s$ nor $\mathbf{S}_s$ admit correspondingly simple expressions for the equilibrium equations referred to coordinates S, Φ, Θ. It is not necessary to introduce a matrix analogous to $\mathbf{P}$ in dealing with axially symmetric problems in cylindrical coordinates because when $\phi = \Phi$, the matrix of the transformation from (x_1, x_2, x_3) to (r, ϕ, z) is the same as that from (X_1, X_2, X_3) to (R, Φ, Z). In transforming from cylindrical to spherical coordinates, however, in general $\theta \neq \Theta$, and consequently $\mathbf{r} \neq \mathbf{R}$, and the matrices $\mathbf{P}$ and $\mathbf{T}_s$ are different.

The constitutive equation for $\mathbf{P}$ is most conveniently obtained by multiplying (2.11) on the left and on the right by $\mathbf{R}$ and using (5.7), (5.10) and (5.13). This gives

$$\mathbf{P} = -p\mathbf{G}^{-1} + \chi_1 \mathbf{G}^T + \chi_{-1}\mathbf{G}^{-1}\mathbf{C}^{-1}. \tag{5.16}$$

The components, in the s and θ directions, of the traction on a surface which in the undeformed configuration has normal with components m_S and m_Θ in the S and Θ directions are F_s and F_θ, where

$$\begin{pmatrix} F_s \\ F_\theta \end{pmatrix} = \begin{pmatrix} \cos(\theta - \Theta) & \sin(\theta - \Theta) \\ -\sin(\theta - \Theta) & \cos(\theta - \Theta) \end{pmatrix} \begin{pmatrix} p_{Ss} & p_{\Theta s} \\ p_{S\theta} & p_{\Theta\theta} \end{pmatrix} \begin{pmatrix} m_S \\ m_\Theta \end{pmatrix}. \tag{5.17}$$

The components of displacement in the directions of increasing S and Θ are denoted ξ, η, so that

$$\xi = s\cos(\theta - \Theta) - S, \quad \eta = s\sin(\theta - \Theta). \tag{5.18}$$

We now follow the procedure of section 3 and expand formally in powers of ϵ. Thus we write

$$\xi = \sum_{n=1}^{\infty} \epsilon^n \xi_n, \quad \eta = \sum_{n=1}^{\infty} \epsilon^n \eta_n \tag{5.19}$$

where, from (3.1)

$$\xi_n = u_n \sin\Theta + w_n \cos\Theta, \quad \eta_n = u_n \cos\Theta - w_n \sin\Theta. \tag{5.20}$$

In addition, we denote

$$\xi_2' = u_2' \sin\Theta + w_2' \cos\Theta, \quad \eta_2' = u_2' \cos\Theta - w_2' \sin\Theta, \tag{5.21}$$

$$\begin{aligned} \xi_2'' &= u_2'' \sin\Theta + w_2'' \cos\Theta = \frac{1}{2}D\xi_1 - \frac{1}{2}\frac{\eta_1^2}{S}, \\ \eta_2'' &= u_2'' \cos\Theta - w_2'' \sin\Theta = \frac{1}{2}D\eta_1 + \frac{1}{2}\frac{\xi_1\eta_1}{S}, \end{aligned} \tag{5.22}$$

where the operator D (equation 3.21) can now be expressed in the form

$$D = \xi_1 \frac{\partial}{\partial S} + \frac{\eta_1}{S}\frac{\partial}{\partial \Theta} \tag{5.23}$$

and

$$\xi_2 = \xi_2' + \xi_2'', \quad \eta_2 = \eta_2' + \eta_2''.$$

The first and second order incompressibility conditions (3.3) and (3.22) now take the forms

$$\frac{\partial \xi_1}{\partial S} + \frac{2\xi_1}{S} + \frac{1}{S}\frac{\partial \eta_1}{\partial \Theta} + \cot\Theta \frac{\eta_1}{S} = 0, \tag{5.24}$$

$$\frac{\partial \xi_2'}{\partial S} + \frac{2\xi_2'}{S} + \frac{1}{S}\frac{\partial \eta_2'}{\partial \Theta} + \cot\Theta \frac{\eta_2'}{S} = 0, \tag{5.25}$$

and it follows from (4.1) and (4.7) with (5.20) and (5.21), that ξ_1, η_1, ξ_2' and η_2' can be derived from the first and second order displacement functions ψ_1 and ψ_2 by means of the relations

$$\xi_1 = -\frac{1}{S^2 \sin\Theta}\frac{\partial \psi_1}{\partial \Theta}, \quad \eta_1 = \frac{1}{S\sin\Theta}\frac{\partial \psi_1}{\partial S}, \tag{5.26}$$

$$\xi_2' = -\frac{1}{S^2 \sin\Theta}\frac{\partial \psi_2}{\partial \Theta}, \quad \eta_2' = \frac{1}{S\sin\Theta}\frac{\partial \psi_2}{\partial S}. \tag{5.27}$$

We now adopt the notations

$$\mathbf{G}_1 = \mathbf{R}\mathbf{F}_1\mathbf{R}, \quad \mathbf{G}_2' = \mathbf{R}\mathbf{F}_2'\mathbf{R}, \tag{5.28}$$

$$\mathbf{C}_1 = \mathbf{R}\mathbf{B}_1\mathbf{R} = \mathbf{G}_1 + \mathbf{G}_1^T, \quad \mathbf{C}_2' = \mathbf{R}\mathbf{B}_2'\mathbf{R} = \mathbf{G}_2' + \mathbf{G}_2'^T, \tag{5.29}$$

$$\mathbf{P}_1 = \mathbf{R}\mathbf{T}_1\mathbf{R}, \quad \mathbf{P}_2 = \mathbf{R}\mathbf{T}_2\mathbf{R}, \quad \mathbf{P}_2' = \mathbf{R}\mathbf{T}_2'\mathbf{R}. \tag{5.30}$$

We require explicit expressions for $\mathbf{G}_1$ and $\mathbf{G}_2'$ in terms of the displacement components. These may be obtained in various ways, the most direct being by substituting (3.6), (3.23) and (5.5) in (5.28) and using (5.20) and (5.21). This gives

$$\mathbf{G}_1 = \begin{bmatrix} \dfrac{\partial \xi_1}{\partial S} & 0 & \dfrac{1}{S}\dfrac{\partial \xi_1}{\partial \Theta} - \dfrac{\eta_1}{S} \\ 0 & \dfrac{\xi_1}{S} + \dfrac{\eta_1}{S}\cot\Theta & 0 \\ \dfrac{\partial \eta_1}{\partial S} & 0 & \dfrac{1}{S}\dfrac{\partial \eta_1}{\partial \Theta} + \dfrac{\xi_1}{S} \end{bmatrix}, \tag{5.31}$$

$$\mathbf{G}_2' = \begin{bmatrix} \dfrac{\partial \xi_2'}{\partial S} & 0 & \dfrac{1}{S}\dfrac{\partial \xi_2'}{\partial \Theta} - \dfrac{\eta_2'}{S} \\ 0 & \dfrac{\xi_2'}{S} + \dfrac{\eta_2'}{S}\cot\Theta & 0 \\ \dfrac{\partial \eta_2'}{\partial S} & 0 & \dfrac{1}{S}\dfrac{\partial \eta_2'}{\partial \Theta} + \dfrac{\xi_2'}{S} \end{bmatrix}. \tag{5.32}$$

To order ϵ^2, from (3.13) (recalling $\mathbf{T}_0 = 0$), (5.13) and (5.30)

$$\mathbf{P} = \epsilon\mathbf{P}_1 + \epsilon^2\mathbf{P}_2 \tag{5.33}$$

and, from (3.17), (5.29) and (5.30)

$$\mathbf{P}_1 = -p_1\mathbf{I} + \mu\mathbf{C}_1; \tag{5.34}$$

from (3.27), (5.29) and (5.30)

$$\mathbf{P}_2 = -p_2\mathbf{I} + \mu\mathbf{C}_2' + \mathbf{P}_2'; \tag{5.35}$$

and, from (3.28), (5.28), (5.29) and (5.30)

$$\mathbf{P}_2' = p_1\mathbf{G}_1 + \tfrac{1}{2}\mu[\mathbf{R}(D\mathbf{B}_1)\mathbf{R} - \mathbf{G}_1^2 + (\mathbf{G}_1^T)^2] - 2C_2\mathbf{C}_1^2. \tag{5.36}$$

Now

$$\begin{aligned}\mathbf{R}(D\mathbf{B}_1)R &= D(\mathbf{R}\mathbf{B}_1\mathbf{R}) - (D\mathbf{R})\mathbf{B}_1\mathbf{R} - \mathbf{R}\mathbf{B}_1(D\mathbf{R}) \\ &= D\mathbf{C}_1 - (D\mathbf{R})\mathbf{R}\mathbf{C}_1 - \mathbf{C}_1\mathbf{R}(D\mathbf{R}).\end{aligned}$$

Also, from (5.5) and (5.23)

$$\mathbf{R}(D\mathbf{R}) = -(D\mathbf{R})\mathbf{R} = \frac{\eta_1}{S}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \frac{\eta_1}{S}\mathbf{Q}, \tag{5.37}$$

where $\mathbf{Q}$ denotes the skew-symmetric matrix shown in (5.37). Hence (5.36) may be written

$$\mathbf{P}_2' = p_1\mathbf{G}_1 + \frac{1}{2}\mu\left[D\mathbf{C}_1 - \mathbf{G}_1^2 + (\mathbf{G}_1^T)^2 + \frac{\eta_1}{S}(\mathbf{Q}\mathbf{C}_1 - \mathbf{C}_1\mathbf{Q})\right] - 2C_2\mathbf{C}_1^2. \tag{5.38}$$

From (5.15), the n-th order equilibrium equations are

$$\begin{aligned}&\frac{\partial p_{Ss}^{(n)}}{\partial S} + \frac{1}{S}\frac{\partial p_{\Theta s}^{(n)}}{\partial \Theta} + \frac{1}{S}(2p_{Ss}^{(n)} - p_{\Phi\phi}^{(n)} - p_{\Theta\theta}^{(n)} + p_{\Theta s}^{(n)}\cot\Theta) = 0, \\ &\frac{\partial p_{S\theta}^{(n)}}{\partial S} + \frac{1}{S}\frac{\partial p_{\Theta\theta}^{(n)}}{\partial \Theta} + \frac{1}{S}((p_{\Theta\theta}^{(n)} - p_{\Phi\phi}^{(n)})\cot\Theta + 2p_{S\theta}^{(n)} + p_{\Theta s}^{(n)}) = 0,\end{aligned} \tag{5.39}$$

for $n = 1, 2, \ldots$, where $p_{Ss}^{(n)}, p_{\Phi\phi}^{(n)}, p_{\Theta\theta}^{(n)}, p_{S\theta}^{(n)}, p_{\Theta s}^{(n)}$ are the non-zero elements of $\mathbf{P}_n$.

In terms of the displacement function ψ_1, the elements of the first-order stress matrix $\mathbf{P}_1$ are, from (5.26), (5.31) and (5.34)

$$p_{Ss}^{(1)} = -p_1 + \frac{2\mu}{S^2\sin\Theta}\left[-\frac{\partial^2\psi_1}{\partial S\partial\Theta} + \frac{2}{S}\frac{\partial\psi_1}{\partial\Theta}\right],$$

$$
\begin{aligned}
p^{(1)}_{\Phi\phi} &= -p_1+\frac{2\mu}{S^2\sin\Theta}\left[\cot\Theta\frac{\partial\psi_1}{\partial S}-\frac{1}{S}\frac{\partial\psi_1}{\partial\Theta}\right],\\
p^{(1)}_{\Theta\theta} &= -p_1+\frac{2\mu}{S^2\sin\Theta}\left[\frac{\partial^2\psi_1}{\partial S\partial\Theta}-\cot\Theta\frac{\partial\psi_1}{\partial S}-\frac{1}{S}\frac{\partial\psi_1}{\partial\Theta}\right],\\
p^{(1)}_{S\theta} = p^{(1)}_{\Theta s} &= \frac{\mu}{S\sin\Theta}\left[\frac{\partial^2\psi_1}{\partial S^2}-\frac{2}{S}\frac{\partial\psi_1}{\partial S}-\frac{1}{S^2}\frac{\partial^2\psi_1}{\partial\Theta^2}+\frac{\cot\Theta}{S^2}\frac{\partial\psi_1}{\partial\Theta}\right].
\end{aligned}
\tag{5.40}
$$

By substituting (5.40) into (5.39) with $n = 1$, we obtain

$$
\begin{aligned}
-\frac{\partial p_1}{\partial S}-\frac{\mu}{S^2\sin\Theta}\frac{\partial}{\partial\Theta}(E^2\psi_1) &= 0,\\
-\frac{1}{S}\frac{\partial p_1}{\partial\Theta}+\frac{\mu}{S\sin\Theta}\frac{\partial}{\partial S}(E^2\psi_1) &= 0,
\end{aligned}
\tag{5.41}
$$

where in spherical polar coordinates the operator E^2 has the form

$$
E^2=\frac{\partial^2}{\partial S^2}+\frac{1}{S^2}\frac{\partial^2}{\partial\Theta^2}-\frac{\cot\Theta}{S^2}\frac{\partial}{\partial\Theta}.
\tag{5.42}
$$

By eliminating p_1 and ψ_1 in turn from (5.41), we have

$$
E^4\psi_1 = 0,
\tag{5.43}
$$

$$
\left(\frac{\partial^2}{\partial S^2}+\frac{2}{S}\frac{\partial}{\partial S}+\frac{1}{S^2}\frac{\partial^2}{\partial\Theta^2}+\frac{\cot\Theta}{S}\frac{\partial}{\partial\Theta}\right)p_1=\nabla^2 p_1 = 0.
\tag{5.44}
$$

These results could of course have been obtained directly from (4.5) and (4.6). A solution of (5.43) determines p_1 through (5.41), ξ_1 and η_1 through (5.26), and the first-order stress components through (5.34) or (5.40), and so gives rise to a complete first-order solution. Note that to first-order in ϵ, the stress matrices $\mathbf{S}_s$, $\mathbf{T}_s$ and $\mathbf{P}$ are the same. The first order equations are thus those of linear elasticity for incompressible materials with axial symmetry expressed in spherical polar coordinates.

In a similar manner, the elements of $\mathbf{P}_2$ are given in terms of ψ_2 by (5.27), (5.32) and (5.35) in the forms

$$
\begin{aligned}
p^{(2)}_{Ss} &= -p_2+\frac{2\mu}{S^2\sin\Theta}\left[-\frac{\partial^2\psi_2}{\partial S\partial\Theta}+\frac{2}{S}\frac{\partial\psi_2}{\partial\Theta}\right]+P'_{Ss},\\
p^{(2)}_{\Phi\phi} &= -p_2+\frac{2\mu}{S^2\sin\Theta}\left[\cot\Theta\frac{\partial\psi_2}{\partial S}-\frac{1}{S}\frac{\partial\psi_2}{\partial\Theta}\right]+P'_{\Phi\phi},\\
p^{(2)}_{\Theta\theta} &= -p_2+\frac{2\mu}{S^2\sin\Theta}\left[\frac{\partial^2\psi_2}{\partial S\partial\Theta}-\cot\Theta\frac{\partial\psi_2}{\partial S}-\frac{1}{S}\frac{\partial\psi_2}{\partial\Theta}\right]+P'_{\Theta\theta},\\
\left.\begin{matrix}p^{(2)}_{S\theta}\\ p^{(2)}_{\Theta s}\end{matrix}\right\} &= \frac{\mu}{S\sin\Theta}\left[\frac{\partial^2\psi_2}{\partial S^2}-\frac{2}{S}\frac{\partial\psi_2}{\partial S}-\frac{1}{S^2}\frac{\partial^2\psi_2}{\partial\Theta^2}+\frac{\cot\Theta}{S^2}\frac{\partial\psi_2}{\partial\Theta}\right]+\left\{\begin{matrix}P'_{S\theta}\\ P'_{\Theta s}\end{matrix}\right.,
\end{aligned}
\tag{5.45}
$$

where P'_{Ss} etc. are the non-zero elements of $\mathbf{P}'_2$, and are given by (5.38) when the first-order solution is known.

By substituting (5.45) into (5.39) with $n = 2$, we obtain

$$-\frac{\partial p_2}{\partial S}-\frac{\mu}{S^2\sin\Theta}\frac{\partial}{\partial\Theta}(E^2\psi_2)=H_1(S,\Theta),$$
$$-\frac{1}{S}\frac{\partial p_2}{\partial\Theta}+\frac{\mu}{S\sin\Theta}\frac{\partial}{\partial S}(E^2\psi_2)=H_2(S,\Theta), \tag{5.46}$$

where

$$H_1(S,\Theta)=-\left\{\frac{\partial P'_{Ss}}{\partial S}+\frac{1}{S}\frac{\partial P'_{\Theta s}}{\partial\Theta}+\frac{1}{S}(2P'_{Ss}-P'_{\Phi\phi}-P'_{\Theta\theta}+P'_{\Theta s}\cot\Theta)\right\},$$
$$H_2(S,\Theta)=-\left\{\frac{\partial P'_{S\theta}}{\partial S}+\frac{1}{S}\frac{\partial P'_{\Theta\theta}}{\partial\Theta}+\frac{1}{S}[(P'_{\Theta\theta}-P'_{\Phi\phi})\cot\Theta+2P'_{S\theta}+P'_{\Theta s}]\right\}, \tag{5.47}$$

and H_1 and H_2 depend only on the first-order solution. By eliminating p_2 and ψ_2 in turn from (5.46), we have

$$E^4\psi_2=\frac{\sin\Theta}{\mu}\left\{\frac{\partial(SH_2)}{\partial S}-\frac{\partial H_1}{\partial\Theta}\right\}, \tag{5.48}$$

$$\nabla^2 p_2=-\left\{\frac{\partial H_1}{\partial S}+\frac{2H_1}{S}+\frac{1}{S}\frac{\partial H_2}{\partial\Theta}+\frac{H_2\cot\Theta}{S}\right\}. \tag{5.49}$$

Equations (5.48) and (5.49) are equivalent to (4.11) and (4.12) respectively, but are not easily derived directly from these equations.

The procedure for finding a second-order solution when the first-order solution is known is similar to that described at the end of section 4. The first-order solution determines ξ''_2, η''_2 and $\mathbf{P}''_2$, and hence the right-hand sides of (5.46) and (5.48). A solution of (5.48) for ψ_2 is then sought; this gives ξ'_2 and η'_2 by (5.27) and p_2 by integration of (5.46). The second-order displacement components are then $\xi_2=\xi'_2+\xi''_2$ and $\eta_2=\eta'_2+\eta''_2$. The second-order stress matrix $\mathbf{P}_2$ can then be obtained from (5.35).

To express the stress in terms of the physically more meaningful Piola–Kirchoff or Cauchy stress tensor components we use (5.14). From (5.9), (5.18) and (5.19), to first-order in ϵ

$$\mathbf{Rr}=\mathbf{I}+\frac{\epsilon\eta_1}{S}\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}=\mathbf{I}+\epsilon\frac{\eta_1}{S}\mathbf{Q}, \tag{5.50}$$

where $\mathbf{Q}$ was defined in (5.37). Then from (5.14), (5.28), (5.33) and (5.50), to order ϵ^2

$$\mathbf{T}_s=\epsilon\mathbf{P}_1+\epsilon^2\left(\mathbf{P}_2+\frac{\eta_1}{S}\mathbf{P}_1\mathbf{Q}\right), \tag{5.51}$$

$$\mathbf{S}_s=\epsilon\mathbf{P}_1+\epsilon^2\left(\mathbf{P}_2+\frac{\eta_1}{S}\mathbf{P}_1\mathbf{Q}+\mathbf{G}_1\mathbf{P}_1\right). \tag{5.52}$$

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Résumé—Les équations d'élasticité du second ordre sont développées pour des déformations symétriques axiales de matériaux élastiques, isotropes et incompressibles. Il est montré que par l'introduction d'une "fonction de déplacement" le problème du second ordre peut être ramené à la résolution d'une équation de la forme $E^4\psi = f(R,Z)$ où E^2 est l'opérateur différentiel de Stokes, R, Z sont des coordonné polaires cylindriques, et $f(R,Z)$ dépend seulement de la solution du premier ordre. Le problème est formulé en termes de coordonnées polaires aussi bien cylindriques que sphériques.

Zusammenfassung—Die Gleichungen von Elastiziät zweiter Ordnung werden fur achsensymmetrische Deformationen inkompressibler isotropischer elastischer Stoffe entwickelt. Es wird gezeigt, dass das Problem zweiter Ordnung durch Einführung einer 'Verdrängungsfunktion' auf die Lösung einer Gleichung der Form $E^4\psi = f(R,Z)$ in der E^2 Stokes' Differentialoperator ist, R, Z zylindrische Polarkoordinaten sind, und $f(R,Z)$ nur von der Lösung erster Ordnung abhängt. Das Problem wird in Termen von zylindrischen Polarkoordinaten und auch von spherischen Polarkoordinaten formuliert.

Sommario—Si sviluppano le equazioni di elasticità di seconda grandezza per le deformazioni simmetriche assiali di materiali elastici isotropici incomprimibili. Si dimostra che introducendo una "funzione di spostamento", il problema di seconda grandezza può essere ridotto alla soluzione di un'equazione della forma $E^4\psi = f(R,Z)$ dove E^2 è l'operator differenziale di Stokes, R, Z sono coordinate polari cilindriche e $f(R,Z)$ dipende solo dalla soluzione della prima grandezza. Il problema è formulato in rapporto alle coordinate polari cilindriche e polari sferiche.

Абстракт — Развиты уравнения упругости второго порядка в случае осесимметричных деформаций несжимаемых изотрпических упругих материалов. Показано, что при введении «функции смещения» возможно сведение задачи второго порядка к решению уравнения в виде $E^4\psi = F(R,Z)$, где E^4 — оператор дифференцирования Стокса, R, Z — цилиндрические полярные координаты, и $F(R,Z)$ зависит только от решения первого порядка. Проблема формулируется через как цилиндрические полярные так сферические полярные координаты.