A Survey on the Sarnak Conjecture

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Introduction

The Sarnak conjecture was raised by Peter Sarnak in 2010 revolving around the randomness of the Möbius function [4]. Understanding the conjecture would help us garner a deeper appreciation of the connections between Number Theory and Ergodic Theory.

Definitions

- A **flow** F is a pair (X, T), where X is a compact metric space equiped with a metric d and $T : X \to X$ a continuous function.
- Note that a more modern and standard terminology for this definition of a flow is a **dynamical system**. Following Furstenberg's style in the 60s, Sarnak chose to use "flow" to refer to a dynamical system.
- A sequence f is **realized in the flow of** F if $\exists x \in X, h \in C(X)$ such that $f(n) = h(T^n x)$ for all n.
- A flow F is **deterministic** if it has zero entropy. A sequence f is **deterministic** if it is realized in a deterministic flow.
- The **Möbius function** $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined to be

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ has a squared prime factor} \\ (-1)^{\omega(n)} & \text{if } n \text{ is square-free} \end{cases},$$

  where $\omega(n)$ denotes the number of distinct prime factors of n.
- A set $E \subset X$ is **(n, ε)-separated** if for all distinct $x, y \in E$, there exists an i with $0 \leqslant i < n$ such that $d(T^i(x), T^i(y)) > \varepsilon$.
- A set $A \subset \mathbb{N}$ is **admissible** if the reduction $A \mod p^2 =: \bar{A} \subsetneq (\mathbb{Z}/p^2\mathbb{Z})^\times$ for all p prime.

Sarnak Conjecture

[4] Let $f : \mathbb{N} \to \mathbb{C}$ be a deterministic bounded sequence. Then

$$\sum_{n \leqslant x} \mu(n) f(n) = o_{x\to\infty}(x)$$

for all $x \in \mathbb{R}$ with $x \geqslant 1$.

Entropy

The entropy of a dynamical system, as its name suggests, measures its temporal complexity and randomness. When unspecified, entropy refers to the topological one.

- [2] The **topological entropy** of T is defined as

$$h_{top}(T) = \lim_{\varepsilon\to 0} \left\{ \limsup_{n\to\infty} \frac{1}{n} \log N(n, \varepsilon) \right\},$$

  where $N(n, \varepsilon)$ denotes the maximal cardinality of all such (n, ε)-separated sets.
- Kolmogorov and Sinai defined in 1959 the **metric entropy** $h_\mu(T)$ with respect to a T—invariant probability measure μ on X borrowing ideas from Information Theory.
- **Variation Principle:** $h_{top}(T) = \sup_\mu h_\mu(T)$.

The Square-Free Flow

We would like to consider a dynamical system obtained from the arithmetic function μ originated in Analytic Number Theory.

- Let T be the left shift: $T(f)(n) := f(n+1)$.
- Let $X_s := \overline{\{T^n \mu^2 : n \in \mathbb{N}\}}$; define a subflow of $(\{0,1\}^{\mathbb{N}}, T)$ the **square-free flow** $S := (X_S, T)$.

Theorem

[4] [3] Let $\mathscr{A}$ be the set of all sequences in $\{0,1\}^{\mathbb{N}}$ whose support is admissible, then $X_s = \mathscr{A}$. Furthermore, $h_{top}(S) = \frac{6}{\pi^2} \log 2$.

Special Instances

1. f being a constant function.
   This statement is equivalent to the Prime Number Theorem, namely
   $$\pi(x) \sim \frac{x}{\log x} \iff \sum_{n\leqslant x} \mu(n) = o(x),$$
   where $\pi(x)$ denotes the number of primes no greater than x.
2. f being a periodic sequence.
   This statement is equivalent to
   $$\sum_{\substack{n \equiv l \bmod k \\ n \leqslant x}} \mu(n) = o(x),$$
   which can be derived from the Prime Number Theorem in arithmetic progressions.

The author provides a complete proof for these two cases lodging purely on number-theoretical manipulations in a more detailed report.

Implication by the Chowla Conjecture

Chowla Conjecture

For any fixed integer m, exponents $a_1, \ldots, a_m \geqslant 0$ with at least one of them being odd, and integers $0 \leqslant b_1 < \ldots < b_m$, we have

$$\sum_{n\leqslant x} \mu(n+b_1)^{a_1} \ldots \mu(n+b_m)^{a_m} = o(x).$$

The Chowla conjecture implies the Sarnak conjecture. But the converse does not hold. Indeed, the Chowla conjecture is significantly more complex than Sarnak's.

The author provides a complete combinatorial proof of this implication following steps outlined in [5]. There is another proof detailed in [1] which relies on more advanced ergodic theory techniques.

Further Inquiry

In a paper by Weiss [6], he proved that a dynamical system has positive entropy if and only if there exists an interpolating set of positive density. Inspired by [4] and [3], the author and her supervisor would like to extend Abromov's formula, which is in terms of metric entropy, to an analogous one in terms of topological entropy: in particular, is there an explicit formula between the entropy of a dynamical system and the density of its interpolating sets?

References

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