

A Refined Bootstrap for Heavy Tailed Distributions

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Abstract

It is known that Efron's nonparametric bootstrap of the mean of random variables with common distribution in the domain of attraction of the stable laws is not consistent, in the sense that the limiting distribution of the bootstrap mean is not the same as the limiting distribution of the mean from the real sample. Moreover, the limiting distribution of the bootstrap mean is random and unknown. The remedy for this problem, at least asymptotically, is either the m out of n or the subsampling bootstrap. However, we show that the both these bootstraps can be quite unreliable if the sample is not very large. A refined bootstrap is derived by considering the distribution of the bootstrap P value instead of that of the bootstrap statistic. The quality of inference based on the refined bootstrap is examined in a simulation study, and is found to be satisfactory with heavy-tailed distributions unless the tail index is close to 1 and the distribution is heavily skewed.

Keywords: bootstrap inconsistency, stable distribution, domain of attraction, infinite variance

JEL codes: C12, C15

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1 Introduction

Inference on the parameter δ in the location model

$$Y_i = \delta + U_i, \quad i = 1, \dots, n, \quad (1)$$

requires special attention if the IID random variables U_1, \dots, U_n have a distribution in the domain of attraction of a stable law with a tail index α smaller than 2. The stable laws, introduced by Lévy (1925), are a generalization of the Gaussian distribution that allows for asymmetries and heavy tails, properties frequently encountered with financial data. More important, the stable laws are the only possible limiting laws for suitably centered and normalized sums of IID random variables. A distribution is said to be in the domain of attraction of a stable law if centered and normalized sums of IID variables with that distribution converge in distribution to that stable law.

The stable distributions, denoted $S_\alpha(c, \beta, \delta)$, are characterized by four parameters: the tail index α , which takes values between 0 and 2, the skewness parameter β , which takes values between -1 and 1, the scale parameter c , and the location parameter δ . The most important parameter of the stable distributions is α : the smaller α , the thicker the tails of the distribution and the higher the probability of having extreme observations in a sample. When α is equal to 2, the stable distribution becomes the Gaussian distribution, which is the only stable distribution with light tails and no asymmetries. Inference about the parameter δ from the model (1) is easy to perform in this case. But, when α is smaller than 2, there is no analytical form for the cumulative distribution function (CDF) of the stable laws, except for the Cauchy distribution ($\alpha = 1$) and the Lévy distribution ($\alpha = 0.5$ and $\beta = 1$). Further, asymptotic inference about δ is difficult, since it requires the estimation of the other three parameters, for which there are no general estimation methods that can be used for all distributions in the domain of attraction of the stable laws. As a result, asymptotic inference about δ can be quite unreliable.

The well-known alternative to unreliable asymptotic tests is the nonparametric bootstrap, or naive bootstrap, introduced by Efron (1979). The idea of the naive bootstrap is to treat the sample as if it were the population and estimate the distributions of test statistics by simulation based on resampling with replacement. Bootstrap samples are of the same size as the real sample. For each bootstrap sample a bootstrap statistic is computed in exactly the same way as the statistic of interest was computed from the real sample. The empirical distribution function (EDF) of the bootstrap statistics is an estimate of the CDF of the distribution from which the sample was drawn, and can be used to obtain critical values associated with a given significance level, or to compute P values.

However, when α is smaller than 2, the nonparametric bootstrap is inconsistent – see Athreya (1987) and Knight (1989). This means that the asymptotic distribution of the bootstrap statistic is different from the asymptotic distribution of the original statistic. In fact, the asymptotic distribution of the bootstrap statistic is random. As a consequence, bootstrap critical values or P values remain random even asymptotically, and even if we ignore simulation randomness. The inconsistency is due to the fact that the bootstrap statistic based on δ is greatly influenced by the extreme observations in the sample.

A proposed remedy for the nonparametric bootstrap failure is the m out of n bootstrap – see Arcones and Gine (1989) – which is based on the same principle as the nonparametric

bootstrap, but with the difference that the bootstrap sample size is m , smaller than n . If $m/n \rightarrow 0$ as $n \rightarrow \infty$, this bootstrap is consistent. However, as we will see in simulation experiments, the m out of n bootstrap fails to provide reliable inference if the sample size is not large enough. Like the m out of n bootstrap, the subsampling bootstrap proposed in [Romano and Wolf \(1999\)](#) makes use of samples of size $m < n$. If m is chosen appropriately, this bootstrap too is consistent, and performs somewhat better than the m out of n bootstrap.

In this paper, we introduce a refined bootstrap method that overcomes the failure of both the asymptotic test and the naive bootstrap test for the parameter δ of the model (1). The idea is to consider the distribution of the bootstrap P value instead of the distribution of the bootstrap statistic. It turns out that the limiting distribution of the bootstrap P value exists and is nonrandom.

The paper is organized as follows. In section 2, we introduce the bootstrap P value, and illustrate some of the implications of the nonparametric bootstrap inconsistency for statistical inference about δ . In section 3, our simulation study indicates that the m out of n and subsampling bootstraps, despite their consistency, fail to be reliable in small or medium-sized samples. In section 4, we derive an expression for the limiting nonparametric bootstrap P value and its limiting distribution. In section 5 we introduce the refined bootstrap method making use of the results of the previous section. The refined bootstrap depends on the parameters α and β of the stable distributions, which in practice have to be estimated. In section 6 we investigate some of the best-known methods that can be used for this purpose. In section 7 the quality of the refined bootstrap is investigated, and section 8 concludes.

2 The nonparametric bootstrap test

Given the model (1), suppose that we test the null hypothesis

$$H_0 : \delta = \delta_0 \tag{2}$$

against the alternative:

$$H_1 : \delta \neq \delta_0. \tag{3}$$

In order to test H_0 we can use the nonstudentized statistic τ ,

$$\tau = n^{-1/\alpha} \sum_{i=1}^n (Y_i - \delta_0). \tag{4}$$

which is a function of the data generating process, of the sample size n , and of the tail index α . If the distribution of the U_i is in the domain of attraction of the stable law with tail index α and other parameters c and β , then, by the Generalized Central Limit Theorem, the asymptotic distribution of τ is the stable distribution $S_\alpha(c, \beta, 0)$. If α , c , and β are known, then we can perform asymptotic inference by comparing the realization of the statistic τ with a quantile of the stable distribution $S_\alpha(c, \beta, 0)$. For simplicity, we limit ourselves in what follows to the left tail of $S_\alpha(c, \beta, 0)$ or whatever other distribution we use to approximate the distribution of τ under the null. Rather than choosing a fixed significance level, it is usually preferable to compute the asymptotic P value, the smallest level for which the test rejects in the lower tail of the distribution. The asymptotic P value is

$$p = S_\alpha(c, \beta, 0)(\tau). \tag{5}$$

Even with the improbable assumption that α , c , and β are known, asymptotic inference may be unreliable in finite samples of data drawn from a distribution other than a stable distribution, albeit in the domain of attraction of a stable distribution. In other circumstances, a remedy for this is provided by the nonparametric bootstrap first proposed by [Efron \(1979\)](#). But [Athreya \(1987\)](#) and [Knight \(1989\)](#) demonstrate that this nonparametric bootstrap is not consistent in the case of a simple statistic like τ based on random variables with a distribution in the domain of attraction of a stable law with $1 < \alpha < 2$, for which the variance does not exist. This implies that its sample counterpart varies and increases unboundedly with the sample size. The large variation of the sample variance is due to the presence of extreme observations, and leads to bootstrap inconsistency.

Suppose that we obtain a large number, say B , of bootstrap samples by resampling with replacement from the empirical distribution function of Y_1, \dots, Y_n . For each bootstrap sample Y_1^*, \dots, Y_n^* , a bootstrap statistic is computed in the same way as τ was computed from the real sample, but with δ_0 replaced by $\bar{Y} = \sum_{i=1}^n Y_i$, the sample mean. This replacement is necessary, because we wish to use the bootstrap to estimate the distribution of the statistic *under the null*, and the sample mean, not δ_0 , is the true mean of the bootstrap distribution. Thus the bootstrap statistic is

$$\tau^* = n^{-1/\alpha} \sum_{i=1}^n (Y_i^* - \bar{Y}). \quad (6)$$

The bootstrap P value is the fraction of the bootstrap statistics smaller than the original statistic,

$$p_B^* = \frac{1}{B} \sum_{j=1}^B \mathbf{I}(\tau_j^* < \tau). \quad (7)$$

Here, \mathbf{I} is the indicator function, whose value is 1 when its Boolean argument is true, and 0 when it is false. As B , the number of bootstrap replications, tends at infinity, then, by the strong law of large numbers, the bootstrap P value converges almost surely to the random variable

$$p^* = \mathbf{E}^*(\mathbf{I}(\tau^* < \tau)) = \mathbf{E}(\mathbf{I}(\tau^* < \tau) | Y_1, \dots, Y_n) = F^*(\tau), \quad (8)$$

where \mathbf{E}^* denotes the expectation under the bootstrap data generating process (DGP) and F^* is the CDF of τ^* under that DGP. Since p^* is a deterministic function of the data Y_1, \dots, Y_n , it is a well-defined random variable, with a distribution determined by that of the Y_i . We will see in section 4 that, as the sample size tends to infinity, this distribution tends to a nonrandom limit.

When the asymptotic distribution of τ^* is the same as the asymptotic distribution of τ , the bootstrap is consistent, and the asymptotic distribution of p^* is uniform on $[0, 1]$. Therefore, consistency guarantees that the bootstrap P value can be interpreted as the marginal significance level, so that, if we fix a significance level, the probability of falsely rejecting a true null hypothesis is precisely that level asymptotically.

[Bickel and Freedman \(1981\)](#) prove that the nonparametric bootstrap of τ is consistent when the distribution of the random variable Y_1 is in the domain of attraction of the normal distribution. [Mammen \(1992\)](#) shows that for the consistency property to hold, it is necessary and sufficient that the maximal summand $\max_{1 \leq i \leq n} |Y_i|$ is of smaller order than the sum τ . This condition implies asymptotic normality of τ^* and τ , and that p^* has the uniform $U(0, 1)$ distribution asymptotically.

Athreya (1987) and Knight (1989) demonstrate that the nonparametric bootstrap of τ is not consistent if the Y_i have a distribution in the domain of attraction of the stable law with $\alpha \in (1, 2)$. That is, the asymptotic distribution F^* of the bootstrap statistic τ^* is not the same as the asymptotic distribution F of the statistic τ , and so the bootstrap P value p^* does not have the uniform $U(0, 1)$ distribution.

The nonparametric bootstrap is inconsistent because τ^* is dominated by the extreme values from the sample. Knight (1989) proves that the distribution of τ^* converges in probability to a random distribution whose characteristic function can be expressed in terms of weighted Poisson random measures with unknown intensity.

The inconsistency of the nonparametric bootstrap for τ can be seen if we look at figure 1, which displays P value plots when the data are drawn from symmetric and asymmetric stable distributions with $\alpha = 1.1$ and $\alpha = 1.5$, and with $\beta = 0$ and $\beta = 1$. The simulation results are based on 10,000 replications of τ computed from samples of size $n = 100$. For each replication, a bootstrap value is computed as in (7) with $B = 399$.

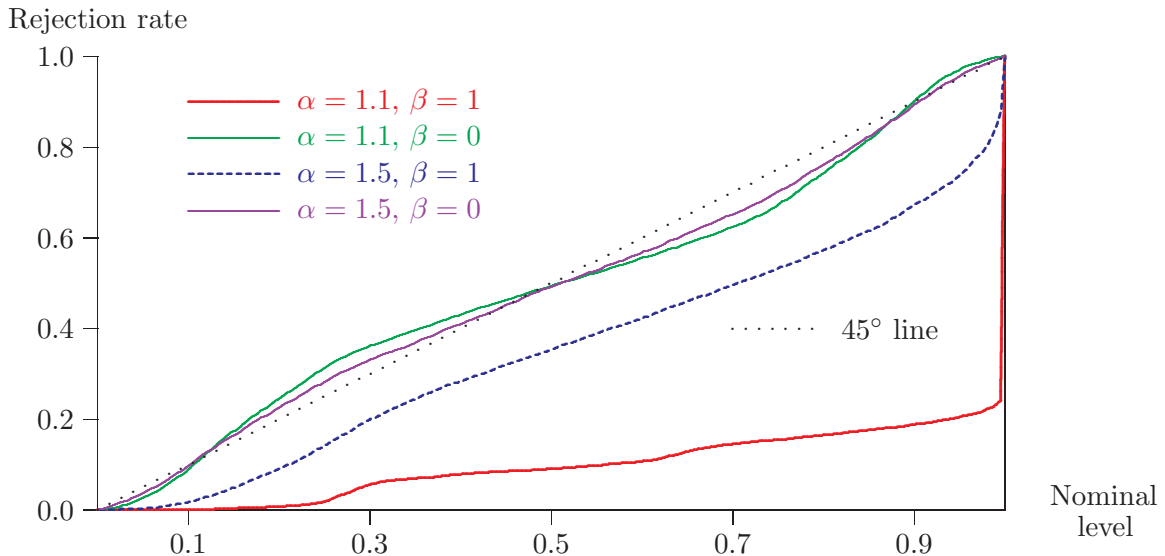


Figure 1: P value plots, naive bootstrap, $\alpha < 2$

As can be seen from figure 1, in all four scenarios the bootstrap P values are not uniformly distributed, since they do not coincide with the 45° line. The difference is more apparent for small values of the tail index α , when the tails of the stable distribution are heavier, and for skewed random variables.

3 The m out of n and subsampling bootstraps

A well-known proposal in the literature to remedy the nonparametric bootstrap inconsistency is the m out of n bootstrap. This bootstrap method is based on the same principle as the nonparametric bootstrap, the only difference being that the bootstrap sample size is equal to m , smaller than n . As a consequence, in the expression (6) of τ^* , n is replaced by m , so that

$$\tau_m^* = m^{-1/\alpha} \sum_{i=1}^m (Y_i^* - \bar{Y}). \quad (9)$$

The m out of n bootstrap for the stable distributions was first studied by Athreya (1985), whose pioneering work was continued by Gine and Zinn (1989), Arcones and Gine (1989), Bickel, Gotze, and van Zwet (1997), and Hall and Jing (1998).

The choice of the bootstrap sample size m is an important matter. It has to be chosen such that the following conditions are satisfied:

$$m \rightarrow \infty, \quad \text{and} \quad m/n \rightarrow 0 \quad \text{or} \quad m(\log \log n)/n \rightarrow 0. \quad (10)$$

The motivation behind the first of these conditions is that it allows us to apply the law of large numbers. In addition, we need the second condition in order for the distribution of τ^* to converge in probability, or the third for almost sure convergence, to the distribution of τ . Proofs of the appropriate large-sample behavior of the m out of n bootstrap can be found in Athreya (1985) and Arcones and Gine (1989). Papers discussing the choice of m include Datta and McCormick (1995) and Bickel and Sakov (2005).

The subsampling bootstrap proposed in Romano and Wolf (1999) is an alternative to the m out of n bootstrap. The main difference between the two is that, in the former, resampling is done without replacement. In this section we show through a simulation study that, despite its consistency, the m out of n bootstrap based on τ does not give reliable inference results in other than very large samples. The subsampling bootstrap does better, but still suffers from serious distortion if m is not chosen with care. Figure 2 displays P value discrepancy plots, as defined in Davidson and MacKinnon (1998). The error on rejection probability (ERP), that is, the difference between the observed rejection rate and the nominal level, is plotted as a function of the nominal level. The plots are based on 10,000 realizations of the statistic τ from samples of size 100 generated by the symmetric stable distribution with the value of $\alpha = 1.5$, supposed known. The bootstrap sample size m took all the values 10, 20, ..., 90, 100. The number of bootstrap replications was $B = 399$.

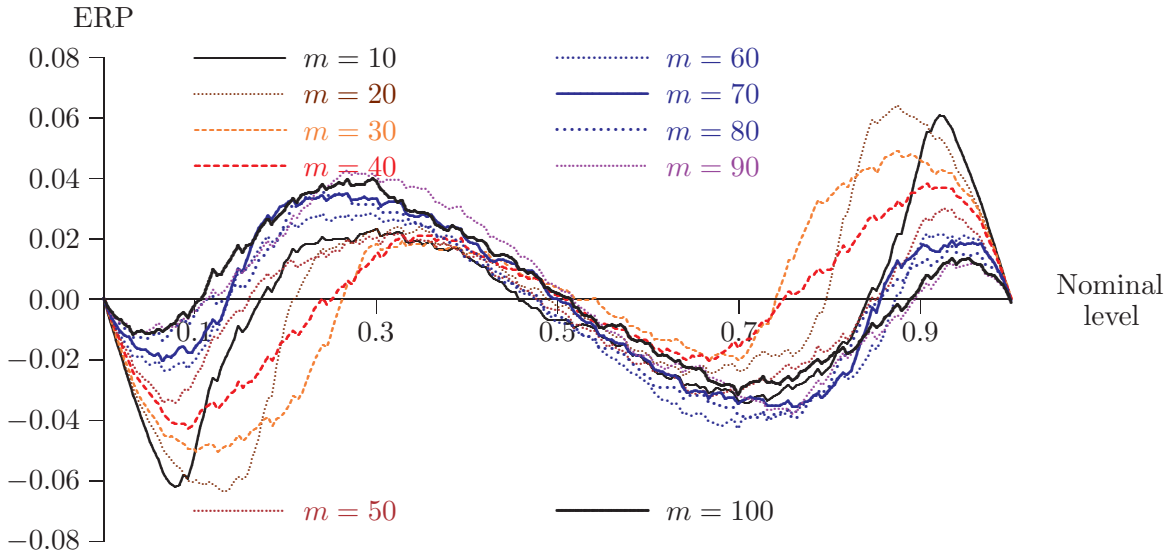


Figure 2: P value discrepancy plots, m out of n bootstrap for τ , $\alpha = 1.5$, $n = 100$

Figure 2 indicates that the m out of n bootstrap suffers from considerable size distortions for all values of m . This means that the rate of convergence of the bootstrap P value to its limit distribution is very slow, and that the m out of n bootstrap is not very reliable in small

samples. Moreover, for the usual significance levels (0.05 and 0.1), the m out of n bootstrap is outperformed by the inconsistent nonparametric bootstrap.

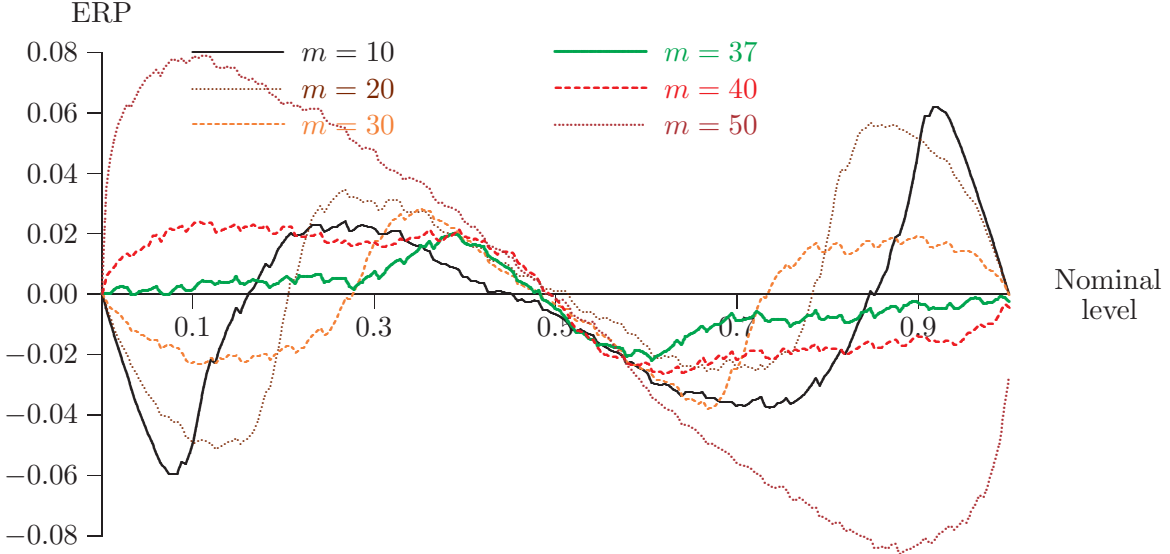


Figure 3: P value discrepancy plots, subsampling bootstrap for τ , $\alpha = 1.5$, $n = 100$

Figure 3 shows comparable results for the subsampling bootstrap. For values of m greater than about 50, the distortions become very large, and so are not shown. Note, though, that for $m = 37$, distortion is quite modest. But even a small difference in m can, as is seen in the figure, give rise to considerable distortion.

The statistic τ depends on the tail index α , which in practice has to be estimated. A natural way to avoid this is to employ the studentized statistic:

$$t = \frac{n^{1/2}(\bar{Y} - \delta_0)}{\sqrt{(n-1)^{-1} \sum_{i=1}^n (Y_i - \bar{Y})^2}}. \quad (11)$$

The statistic for the usual nonparametric bootstrap is

$$t^* = \frac{n^{1/2}(\bar{Y}^* - \bar{Y})}{\sqrt{(n-1)^{-1} \sum_{i=1}^n (Y_i^* - \bar{Y}^*)^2}}, \quad (12)$$

with n replaced by m for the m out of n and subsampling bootstraps. Hall (1990a) shows that the nonparametric bootstrap based on the t statistic is not consistent if the distribution of the Y_i is in the domain of attraction of the stable distributions with tail index $\alpha < 2$. But Hall and LePage (1996) prove that the m out of n bootstrap is justified asymptotically. The same is shown by Romano and Wolf (1999) for the subsampling bootstrap. Moreover, the consistency holds even if the distribution of the Y_i is not in any domain of attraction, with the only requirement that the expectation exists.

Figure 4 displays the P value discrepancy plots for the m out of n bootstrap based on the t statistic, using samples of size $n = 100$ from the stable distribution with $\alpha = 1.5$ and $\beta = 0$, these values again assumed known. As can be observed from the figure, the m out of n bootstrap has large size distortions for any value of m between 10 and 100. Again, the rate of

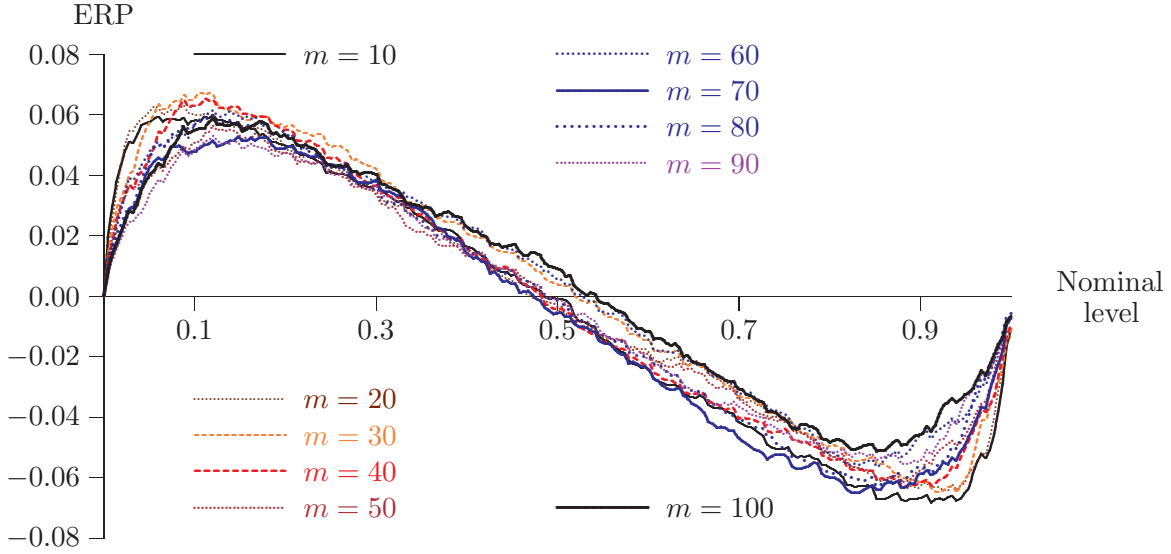


Figure 4: P value discrepancy plots, m out of n bootstrap for t , $\alpha = 1.5$, $n = 100$

convergence to the limit is very slow and we need a larger sample for this bootstrap method to be reliable. In addition, the nonparametric bootstrap performs just as badly as the m out of n bootstrap. Compared with the bootstrap of the nonstudentized statistic τ , which underrejects for conventional significance levels, the bootstrap of the t statistic systematically overrejects.

Figure 5 shows similar results for the subsampling bootstrap. Distortions are smallest for very small values of m , but increase very quickly as m gets larger.

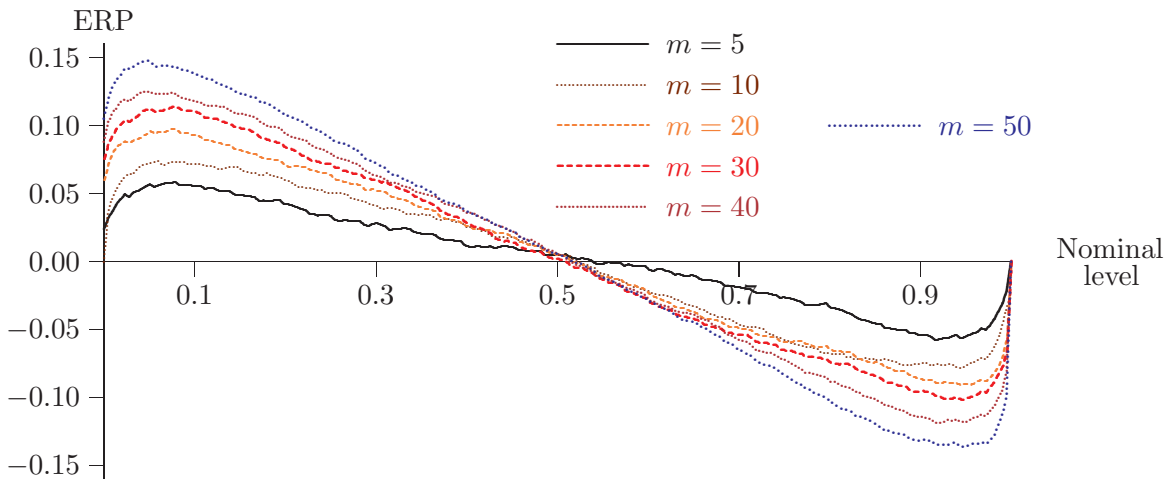


Figure 5: P value discrepancy plots, subsampling bootstrap for t , $\alpha = 1.5$, $n = 100$

The results of this section indicate that, although consistency is necessary to avoid bootstrap failure, it does not guarantee reliable inference in samples of moderate size. For the rest of the paper, we concentrate on the statistic τ rather than the studentized t , since it turns out that the bootstrap P value has an asymptotic distribution that does not depend on α .

4 The limiting distribution of the bootstrap P value

[Knight \(1989\)](#) shows that the distribution of the bootstrap statistic τ^* converges in probability to a random law of unknown form. Because of this, the nonparametric bootstrap is inconsistent and fails to provide reliable inference. However, as we will show shortly, the nonparametric bootstrap can be corrected if we make use of the distribution of the bootstrap P value instead of that of τ^* .

We assume throughout this section that the number of bootstrap replications B is infinitely large, and that the null hypothesis $\delta = \delta_0$ is true. Knight shows that, conditionally on the random variables Y_1, \dots, Y_n , the bootstrap statistic τ^* has the same distribution as

$$\tau(W) = n^{-1/\alpha} \sum_{i=1}^n (Y_i W_i - \bar{Y}) = n^{-1/\alpha} \sum_{i=1}^n Y_i (W_i - 1) = n^{-1/\alpha} \sum_{i=1}^n (Y_i - \bar{Y})(W_i - 1), \quad (13)$$

where W_1, \dots, W_n is a multinomial vector with n trials and each cell probability is $1/n$. The last equality follows because $\sum_i (W_i - 1) = 0$ identically. For large n , the multinomial vector has approximately the same distribution as a vector of n independent Poisson random variables M_1, \dots, M_n with expectation one. Thus, if we make the definition

$$\tau(M) = n^{-1/\alpha} \sum_{i=1}^n (Y_i - \bar{Y})(M_i - 1), \quad (14)$$

then $\tau(W) \xrightarrow{d} \tau(M)$ as $n \rightarrow \infty$ conditionally on Y_1, \dots, Y_n , where \xrightarrow{d} denotes convergence in distribution. But $\tau(M)$ converges in distribution to a normal random variable with mean 0 and variance

$$n^{-2/\alpha} \sum_{i=1}^n (Y_i - \bar{Y})^2. \quad (15)$$

The fact that $\tau(M)$ has a proper and well defined limiting distribution conditional on Y_1, \dots, Y_n follows from [Hall \(1990a\)](#). Briefly, his argument runs as follows. Let $|Y_{(1)}| > \dots > |Y_{(r)}|$ be the most extreme order statistics from the sample of absolute values $|Y_1|, \dots, |Y_n|$. Hall shows, using arguments from extreme value theory, that the vector $|Y_{(1)}|, \dots, |Y_{(r)}|$ suitably normalized converges in distribution to a vector X_1, \dots, X_r , where X_j has the marginal distribution function

$$\Pr(X_j \leq x) = \sum_{i=0}^{j-1} (i! x^{i\alpha})^{-1} \exp(-x^{-\alpha}), \quad x > 0. \quad (16)$$

Then, $E(X_i^2) = \Gamma(1 - 2\alpha^{-1})/\Gamma(i)$, where Γ is the gamma function, and so $\sum_{i>\alpha/2} E(X_i^2) < \infty$.

Let Φ denote the standard normal CDF. By analogy with [\(8\)](#), we see that the bootstrap P value of the nonstudentized statistic $\tau(M)$ is

$$\begin{aligned} p^* &= E(\mathbf{I}(\tau(M) < \tau) \mid Y_1, \dots, Y_n) \\ &= E\left(\mathbf{I}\left(\sum_{i=1}^n Y_i (M_i - 1) < \sum_{i=1}^n (Y_i - \delta_0)\right) \mid Y_1, \dots, Y_n\right) \\ &\rightarrow \Phi\left(\frac{\sum_{i=1}^n (Y_i - \delta_0)}{\sqrt{\sum_{i=1}^n (Y_i - \bar{Y})^2}}\right) \text{ as } n \rightarrow \infty \end{aligned} \quad (17)$$

By definition, the asymptotic distribution of the bootstrap P value (17) has CDF

$$\begin{aligned} \Pr(p^* < u) &= \Pr\left(\Phi\left(\frac{\sum_{i=1}^n(Y_i - \delta_0)}{\sqrt{\sum_{i=1}^n(Y_i - \bar{Y})^2}}\right) < u\right) \\ &= \Pr\left(\frac{\sum_{i=1}^n(Y_i - \delta_0)}{\sqrt{\sum_{i=1}^n(Y_i - \bar{Y})^2}} < \Phi^{-1}(u)\right) = G(\Phi^{-1}(u)), \end{aligned} \quad (18)$$

where G is the distribution of the self-normalized sum

$$\frac{\sum_{i=1}^n(Y_i - \delta_0)}{\sqrt{\sum_{i=1}^n(Y_i - \bar{Y})^2}}. \quad (19)$$

Logan, Mallows, Rice, and Shepp (1973) derive the expression of the limiting density of the statistic (19)¹ as $n \rightarrow \infty$. The density can be written in terms of the parabolic cylinder function

$$D_\nu(z) = \frac{\exp(-z^2/4)}{\Gamma(-\nu)} \int_0^\infty \exp(-zt - t^2/2)t^{-\nu-1} dt, \quad (20)$$

where $\nu < 0$. The density of the self-normalized sum for $y \geq 0$ is

$$g(y) = \operatorname{Re} \left(\int_0^{\infty/i^{1/2}} e^{y^2 t^2/2} \mathcal{K}(t) dt \right), \quad (21)$$

where Re denotes the real part and

$$\mathcal{K}(t) = \frac{\sqrt{2}(1-\alpha)}{\pi^{3/2}} \frac{rD_{\alpha-2}(it) + lD_{\alpha-2}(-it)}{rD_\alpha(it) + lD_\alpha(-it)}, \quad (22)$$

where $i = \sqrt{-1}$. For $y \leq 0$, the density is

$$g(y) = \operatorname{Re} \left(\int_0^{\infty/i^{1/2}} e^{y^2 t^2/2} \mathcal{K}(-t) dt \right). \quad (23)$$

The parameters r and l are defined as follows:

$$\lim_{y \rightarrow \infty} y^\alpha \Pr(Y > y) = r, \quad \lim_{y \rightarrow \infty} y^\alpha \Pr(Y < -y) = l. \quad (24)$$

The density (21) can be integrated with respect to its argument y to yield the CDF of the self-normalized sum for $y \geq 0$,

$$\begin{aligned} G(y) &= \frac{1}{2} + \operatorname{Re} \left[\int_0^{\infty/i^{1/2}} \frac{2(1-\alpha)}{i\pi^{3/2}} \right. \\ &\quad \left. \times \left(\int_0^{yzi/\sqrt{2}} e^{-z^2} dz \right) \frac{rD_{\alpha-2}(it) + lD_{\alpha-2}(-it)}{rD_\alpha(it) + lD_\alpha(-it)} dt \right]. \end{aligned} \quad (25)$$

¹The subtraction of \bar{Y} from the denominator is necessary in order for the distribution G to exist, as emphasized by Logan et al. (1973). However, even if the expectation of the random variable Y_1 is zero, this manipulation does not have any effect on G . This can be easily verified by simulations.

In this paper, we do not actually make use of the CDF (25), since, as we discuss in the next section, we can estimate G quite easily by simulation.

5 Refined bootstrap

The distribution (18) derived in the previous section can be used to correct for the failure of the nonparametric bootstrap. Let \hat{p}^* be a realization of the random variable p^* . The usual rule is to reject the null hypothesis about δ if \hat{p}^* is smaller than the chosen significance level. However, this rule depends on the assumption that the asymptotic distribution of p^* is $U(0, 1)$, which it is not in cases of bootstrap inconsistency. But, as shown in the previous section, the asymptotic CDF of p^* is (18). It follows that the asymptotic CDF of $G(\Phi^{-1}(p^*))$ is indeed the $U(0, 1)$ distribution. The correct rule for rejecting the null hypothesis $\delta = \delta_0$ when $\alpha < 2$ is therefore to reject when $G(\Phi^{-1}(\hat{p}^*))$ is smaller than the significance level.

The limit of the distribution $G(\Phi^{-1}(\cdot))$ is expressed in terms of parabolic cylinder functions. But there is a simpler way to estimate $G(\Phi^{-1}(\cdot))$ without going to the trouble of calculating these functions, and without going to the limit as $n \rightarrow \infty$. First, it should be noticed that the limit of G is the asymptotic distribution of the self-normalized sum (19) where the Y_i are generated by *any* distribution in the domain of attraction of the stable laws. Therefore, G can be consistently estimated by the empirical distribution function of the self-normalized sums from samples of stable random variables with α and β consistently estimated from the original sample. Moreover, it should be noticed that all the distributions in the domain of attraction of the stable laws have tails that behave as in (24), and in particular for the tails of the stable laws we have – see [Samorodnitsky and Taqqu \(1994\)](#) – that

$$\lim_{y \rightarrow \infty} y^\alpha \Pr(Y > y) = C_\alpha \frac{1 + \beta}{2} c^\alpha, \quad \lim_{y \rightarrow \infty} y^\alpha \Pr(Y < -y) = C_\alpha \frac{1 - \beta}{2} c^\alpha, \quad (26)$$

where C_α is the constant

$$C_\alpha = \left(\int_0^\infty y^{-\alpha} \sin y \, dy \right)^{-1}. \quad (27)$$

It follows that

$$\frac{r}{l} = \frac{1 + \beta}{1 - \beta}. \quad (28)$$

On account of (28), for any distribution in the domain of attraction of a stable law $S_\alpha(c, \beta, \delta)$, we can estimate the skewness parameter β if we can estimate r and l .

The refined bootstrap is described by the following steps:

1. Given the sample of random variables Y_1, \dots, Y_n , compute the statistic τ of (4).
2. a) Draw B bootstrap samples by resampling with replacement from the original sample. For each bootstrap sample compute the bootstrap statistic τ_j^* of (6), $j = 1, \dots, B$. Then, the bootstrap P value \hat{p}^* is given by (7).
or
b) Compute the approximate P value \hat{p}^* directly from (17).
3. Estimate α and β from the original sample.

4. Draw N samples of size n from the stable distribution with α and β obtained in the previous step. The parameter δ should be equal to δ_0 , the value under the null hypothesis. The parameter c can take any value since the distribution G does not depend on it.
5. For each sample of the stable random variables compute the bootstrap P value, using whichever method was used in step 2.
6. Compute the empirical distribution function of the bootstrap P value from the previous step. This is the estimate of the distribution $G(\Phi^{-1}(\cdot))$.
7. The null hypothesis is rejected if the estimated distribution of the previous step, evaluated at \hat{p}^* from step 2, is smaller than the chosen significance level.

As can be seen, the refined bootstrap follows the same steps as the nonparametric bootstrap, with the difference that we have to estimate the distribution of the bootstrap P value under the null hypothesis. This particular difference allows us to correct for the inconsistency of the nonparametric bootstrap. Our simulations show that there is very little difference between results obtained using 2a and 2b. It is of course very much less computationally intensive to use 2b.

6 The methods for the estimation of the parameters α and β

The refined bootstrap introduced in the previous section depends on the parameters α and β , which need to be estimated. The problem of estimating them is hampered by the fact that the density and CDF of the stable laws have no closed-form expression. Nevertheless, there are estimation procedures for the stable laws, and also more general procedures that use only the information in the tails of the distributions. In the first category we have maximum likelihood (DuMouchel (1973), Nolan (2001)), characteristic function methods (Koutrouvelis (1980), Kogon and Williams (1998)), the quantile method of McCulloch (1986), the indirect inference method of Garcia, Renault, and Veredas (2006), and the continuous generalized method of moments of Carrasco and Florens (2000). In the second category we may cite Pickands's estimator (Pickands (1975)), Hill's estimator (Hill (1975)), and de Haan's estimator (de Haan and Resnick (1980)). We survey some of these in this section.

6.1 Hill's method

The best known estimator that uses only the information in the tails of the distribution is the Hill estimator. It is based on the order statistics $Y_{(1)} > Y_{(2)} > \dots > Y_{(n)}$ of the sample of IID random variables Y_1, \dots, Y_n . It is equal to

$$\hat{\alpha}_{\text{Hill}} = \left(\frac{1}{k-1} \sum_{j=1}^{k-1} \log Y_{(j)} - \log Y_{(k)} \right)^{-1}. \quad (29)$$

If k is the number of order statistics used for the estimation of α in the right tail of the distribution, then r is estimated by

$$\frac{k}{n} Y_{(k)}^{\hat{\alpha}_{\text{Hill}}}. \quad (30)$$

The parameter l is estimated in a similar way, using the information in the left tail of the distribution.

A variant for the estimation of r and l is introduced by [Aban and Meerschaert \(2004\)](#). They define

$$\hat{\mu} = \log Y_{(k)} - \hat{\alpha}_{\text{Hill}}^{-1} \sum_{j=k}^n \frac{1}{j}, \quad (31)$$

and then the estimate of r is $\exp(\hat{\alpha}_{\text{Hill}} \hat{\mu})$. An estimate of l is obtained similarly.

6.2 Quantile method

One of the methods for estimating the parameters α and β was introduced by [Fama and Roll \(1971\)](#) for symmetric stable distributions, and enhanced by [McCulloch \(1986\)](#) for the general asymmetric case. Using the 0.05, 0.25, 0.5, 0.75, and 0.95 quantiles, one computes the following indices:

$$\nu_{\alpha} = \frac{q_{0.95} - q_{0.05}}{q_{0.75} - q_{0.25}}, \quad (32)$$

$$\nu_{\beta} = \frac{q_{0.95} + q_{0.05} - q_{0.5}}{q_{0.95} - q_{0.05}}. \quad (33)$$

The indices are then inverted using the tables from [McCulloch \(1986\)](#) in order to obtain consistent estimators of α and β . [McCulloch](#) suggests that these estimators can be used as a very good starting point for more sophisticated and theoretically superior estimators, such as maximum likelihood.

6.3 Maximum likelihood

The maximum likelihood estimators of the parameters α and β are based on numerical approximations to the density of the stable distributions. With no closed-form representation of the density, this is not a trivial task. The MLE was first obtained by [DuMouchel \(1973\)](#) by using the Fast Fourier Transform and Bergstrom series expansions for the tails. He showed that the standard theory, in terms of root- n asymptotic normality and Cramér-Rao bounds, applies for the maximum likelihood estimators of the parameters of the stable laws.

[Nolan \(2001\)](#) continued the pioneering work of [DuMouchel](#) and optimized the method by employing direct numerical integration of the stable density derived from one of [Zolotarev's](#) parametrizations of the characteristic function ([Zolotarev \(1986\)](#)).

6.4 Characteristic function methods

The one-one correspondence between the density and the characteristic function motivates the characteristic function approaches for the estimation of the parameters of stable distributions. The methods that fall into this category and have proved to have the best performance are those proposed by [Koutrouvelis \(1980\)](#) and [Kogon and Williams \(1998\)](#), by using different expressions of the cumulant generating function of the stable law. Let

$$\hat{\zeta}(t) = \frac{1}{n} \sum_{j=1}^n e^{itY_j} \quad (34)$$

be the empirical characteristic function. Koutrouvelis' method is based on a parametrization of the characteristic function given by [Zolotarev \(1986\)](#) that is not continuous in all parameters. The tail index α is estimated by ordinary least squares as the coefficient of $\log t$ in the following regression:

$$\log(-\log |\hat{\zeta}(t)|) = \alpha \log c + \alpha \log t + u_t, \quad t = 1, \dots, K. \quad (35)$$

where u_t is a disturbance term. The values of t are chosen, following Koutrouvelis, as $t = \pi k/25$, $k = 1, \dots, K$. Some experimentation showed us that, for sample size $n = 100$, the mean squared error of the estimator is minimised near $K = 16$. The parameter c can be estimated using the estimated constant \hat{a} from (35) as $\hat{c} = \exp(\hat{a}/\hat{\alpha})$.

Then β and δ are estimated by running the regression

$$\text{Arg } \hat{\zeta}(t) = \delta t + \beta(ct)^\alpha \tan \frac{\pi\alpha}{2} + u_t, \quad (36)$$

where Arg denotes the principal argument of the complex number $\hat{\zeta}(t)$, that is, the angle θ such that $\hat{\zeta}(t) = |\hat{\zeta}(t)|e^{i\theta}$ and $-\pi < \theta \leq \pi$. In regression (36), $\text{Arg } \hat{\zeta}(t)$ is regressed on t and $t^{\hat{\alpha}}$, where $\hat{\alpha}$ is from (35). Then the estimated coefficient of t is the estimate of δ , and the estimate of β is $\hat{\beta} = \hat{b}\hat{c}^{-\hat{\alpha}} \cot(\pi\hat{\alpha}/2)$, where \hat{b} is the estimated coefficient of $t^{\hat{\alpha}}$ and \hat{c} is from (35). Koutrouvelis recommends setting the values of t in (36) as $t = \pi l/50$, $l = 1, \dots, L$.

The method of Kogon and Williams is based on another of Zolotarev's parametrizations of the characteristic function, one that is continuous in all parameters. The regression for the estimation of α is the same as in Koutrouvelis' method. The parameter β is estimated by ordinary least squares from the following regression

$$\text{Arg } \hat{\zeta}(t) = \left(\delta + \beta c^\alpha \tan \frac{\pi\alpha}{2}\right)t + \beta ct \tan \frac{\pi\alpha}{2} ((ct)^{\alpha-1} - 1) + u_t. \quad (37)$$

6.5 Comments

For the estimation of α we prefer the method of Koutrouvelis in cases in which the underlying distribution is stable. Simulation results, not reported here, indicate that Koutrouvelis' method performs as well as maximum likelihood in terms of the root mean squared error and bias, and is much less time-consuming. The more robust method of Hill is fast and performs well provided the optimal number of order statistics is used, which in practice can be obtained by employing the Hill plot – see [Hill \(1975\)](#) – or the m out of n bootstrap – see [Hall \(1990b\)](#) and [Caers and Dyck \(1999\)](#). For the estimation of the skewness parameter β , provided the underlying distribution is in fact a stable law, the quantiles method is the best, since it does not depend on the estimate of α , unlike Koutrouvelis' method. For laws that are not stable but belong to the domain of attraction of a stable law, the Aban-Meerschaert estimates of r and l seem to give the best estimates of β .

We made no use of the continuous GMM approach of [Carrasco and Florens \(2000\)](#) or the indirect inference method of [Garcia, Renault, and Veredas \(2006\)](#), since they are almost as time-consuming as maximum likelihood. The latter paper shows in a simulation study that the indirect inference method is superior to the continuous GMM and the quantiles methods.

7 Simulation evidence

In this section we investigate the performance of the refined bootstrap in samples of size $n = 100$ in a simulation study. First, we consider the case in which the null hypothesis $\delta = 0$ is true. All

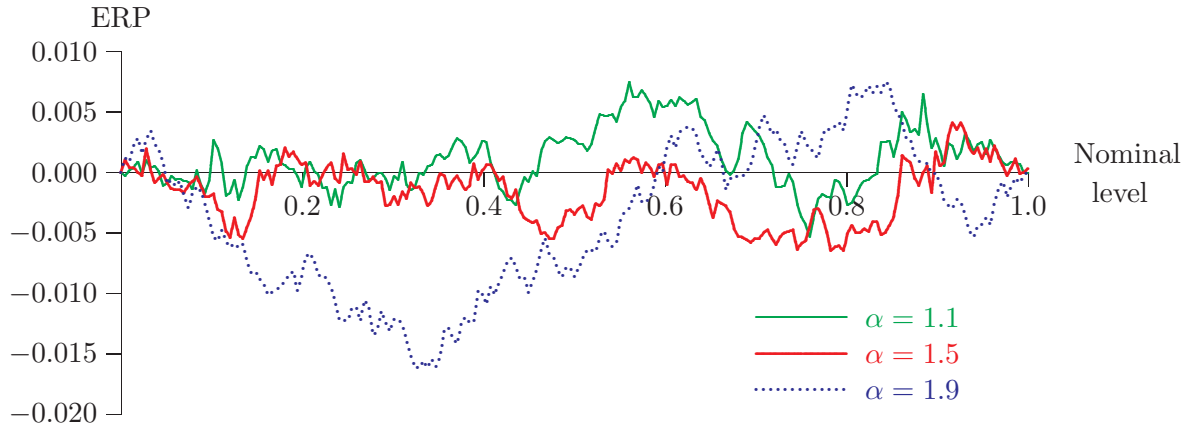


Figure 6: P value discrepancy plots, refined bootstrap, $\hat{\alpha}_{Hill}$, stable law, $\beta = 0$

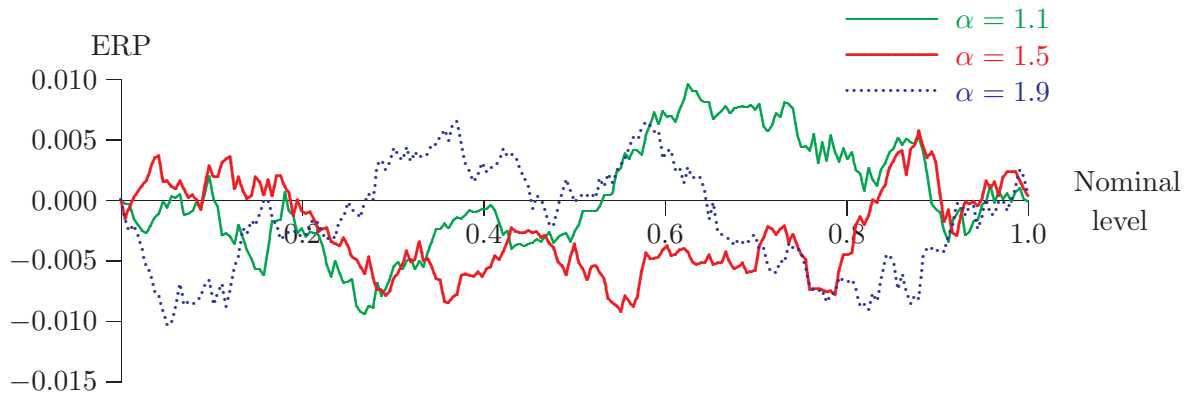


Figure 7: P value discrepancy plots, refined bootstrap, $\hat{\alpha}_{Hill}$, Student's t law

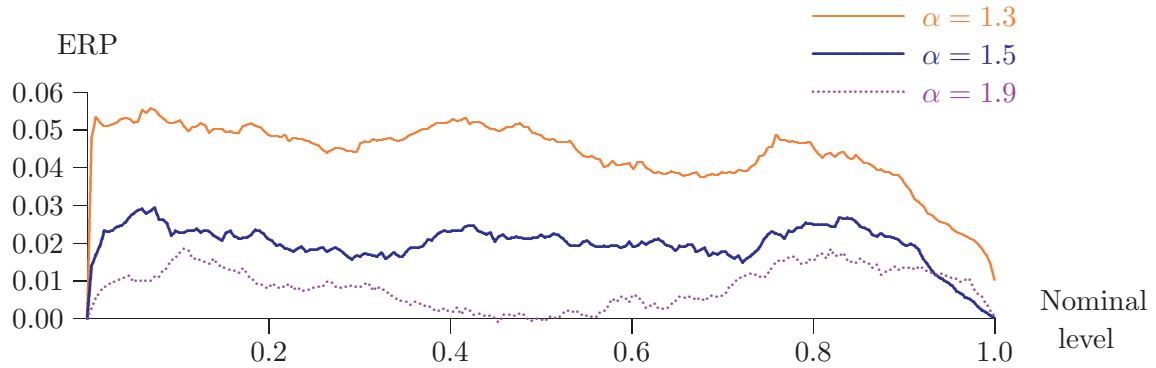


Figure 8: P value discrepancy plots, refined bootstrap, $\hat{\alpha}_{Koutrouvelis}$, $\hat{\beta}_{quantiles}$, stable law $\beta = 1$

results are based on 10,000 replications of the statistic τ of (4) and 399 bootstrap repetitions in step 2a of the refined bootstrap algorithm. They are displayed graphically as P value discrepancy plots. Since the null hypothesis is true, the errors in rejection probability (ERPs), that is, the differences between the actual rejection rates of the refined bootstrap P values and the nominal significance levels, should be close to zero if the refined bootstrap works well.

For figure 6, the data were generated from the stable distribution, with $\alpha = 1.1, 1.5$ and 1.9 , and $\beta = 0$. For bootstrapping, the symmetry of the distribution was assumed to be known, and so the skewness parameter β was set to zero in step 4 of the refined bootstrap algorithm, and was not estimated in step 3. The number N of samples generated in step 4 was 10,000. For figure 7, the data came from Student's t distribution, with α , which is now identified with the degrees-of-freedom parameter, set to 1.1, 1.5, and 1.9. Again the symmetry is assumed known in bootstrapping. In both cases, the parameter α was estimated by Hill's method applied to the absolute values of the data, in order to take advantage of symmetry. The parameter k of (29) was set to 33 for the stable law with $\alpha = 1.1$, to 42 for $\alpha = 1.5$, and to 44 for $\alpha = 1.9$, these values having been determined by some preliminary experimentation to find the best choice. For the Student's t data, we used $k = 30, 22$, and 17 for $\alpha = 1.1, 1.5$, and 1.9 . Although the estimator of Koutrouvelis' method would be more precise when the data are generated by a stable law, it does not perform well with the Student's t data, since it is specific to the stable law. It is in any case clear from figures 6 and 7 that the refined bootstrap performs very well under the true null hypothesis, even when the data come from a distribution which, although in the domain of attraction of a stable law, is not a stable law.

In figure 8, the data are again from the stable law, with $\alpha = 1.3, 1.5$, and 1.9 , and the most extreme case of asymmetry, namely $\beta = 1$. For bootstrapping, α was estimated by Koutrouvelis' method, and β by the quantile method. Hill's method seems to work poorly with such asymmetric data, and Koutrouvelis' method for β is less precise than the quantile method. We see that the performance of the refined bootstrap is much influenced by the value of the skewness parameter β . Compared with the symmetric case, the ERPs are significantly larger, especially for smaller values of α . Indeed, with $\alpha = 1.1$ (not shown in the figure), the ERP is unacceptably large.

Even when the data are generated by a symmetric law, an investigator may not be prepared to assume this. Thus it is of practical importance to see how estimating β affects the performance of the refined bootstrap. The parameters α and β used in the refined bootstrap that give rise to the P value discrepancy plots in figures 9 and 10 were both estimated by Hill's method, although the true value of β is zero. Method 2b was used in the bootstrap algorithm. There is little difference between the results for the stable law in figure 9 and those for Student's t in figure 10, but the size distortions are greater than when it is assumed that $\beta = 0$. They are greater the smaller is α , and for $\alpha = 1.1$, they are again unacceptably large.

Matters can be improved a little by replacing the Hill estimate of β by the variant of Aban and Meerschaert, which we denote by $\hat{\beta}_{AM}$. Results analogous to those of figures 9 and 10 are given in figures 11 and 12. It can be seen that the improvement is slight and that the qualitative features are the same as with Hill's estimator of β .

If β is nonzero, performance becomes still worse. The size distortions that we see in figure 13, where the true β is 0.5, are enormous for small α . In figure 14, we have $\beta = 1$, and the distortions are even greater. For both these figures, we use $\hat{\alpha}_{Hill}$ and $\hat{\beta}_{AM}$. The reason for the large distortions is that, for large β and α close to 1, the distribution G of the self-normalized sum (19) is exceedingly sensitive to the value of α . None of the estimates of α that we have

considered is precise enough to overcome this phenomenon. In this case, the failure of the refined bootstrap is due less to any deficiency of its own than to the imprecision of the estimates of α .

We end this section with some evidence about the behavior of the refined bootstrap when the null hypothesis $\delta = 0$ is false, the true δ being -0.5 . Except for this change, the setups for figures 15 and 16 are the same as for figures 6 and 7. The results are shown as plots of the estimated rejection rates as a function of nominal level. Several interesting facts emerge from the figures. First, power is influenced by the tail index α . The smaller α , the lower the power. Second, compared with the stable distribution, the power is slightly lower when the data come from Student's t . From figure 17, we see that power is also influenced by the skewness parameter: the smaller α and the larger β , the lower the power of the refined bootstrap. For this last figure, since for α close to 1 the size distortion is considerable, we used the *true* value of α for bootstrapping, so as to have a better notion of the theoretical power. The skewness parameter was estimated by the quantile method.

8 Conclusion

In this paper, we have proposed a refined bootstrap for the purposes of inference on the expectation of a heavy-tailed distribution when an IID sample generated by that distribution is available. We show that, unlike the conventional nonparametric bootstrap statistic, the bootstrap P value has a nonrandom limiting distribution when the sample size tends to infinity. By itself, this is no guarantee that inference based on the bootstrap P value is reliable in finite samples, as we see by looking at the finite-sample performance of the m out of n and subsampling bootstraps, which are both consistent but are unreliable with a sample size of 100.

The study of the distribution of the bootstrap P value leads us to an approximate expression for it that can be computed without simulation. Since this expression also has a nonrandom limiting distribution, it is a convenient replacement for the conventional bootstrap P value, and it is independent of an arbitrary choice of the number of bootstrap samples. We have observed very little difference in our simulations between results based on the two statistics, and so recommend the use in practice of the approximate expression, as it is much less computationally intensive.

The distribution of the bootstrap P value or the approximate expression can be estimated consistently if we can estimate the parameters α and β of the stable law to which the centered and normalized sum of the observations converges. This is most conveniently carried out by simulation, rather than by use of the asymptotic distribution, which, although known, is expressed in terms of integrals of functions of parabolic cylinder functions, and is thus awkward to compute. Our results show that, as long as estimation of α and β is reasonably precise, the refined bootstrap gives inference with a sample size of 100 that is reliable by any usual standard. Its performance degrades when the methods we use to estimate these parameters become imprecise, which happens when the expectation is close to nonexistence, and when the distribution is heavily skewed. We conjecture that it is impossible to devise a reliable method of inference for α close to 1, but it may be possible to find better estimators of β .

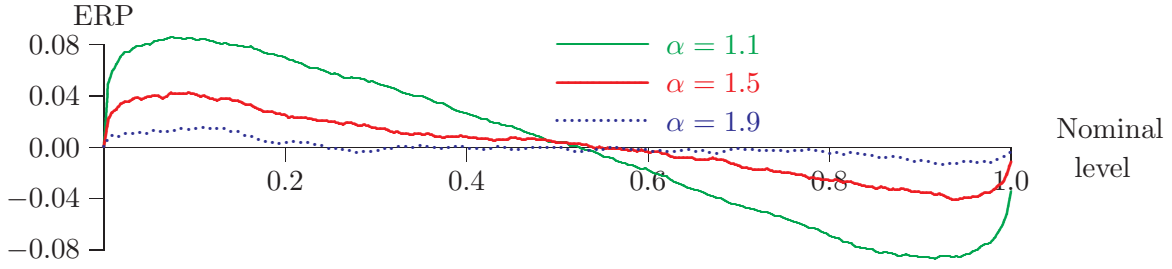


Figure 9: P value discrepancy plots, refined bootstrap, $\hat{\alpha}_{\text{Hill}}$, $\hat{\beta}_{\text{Hill}}$, stable law

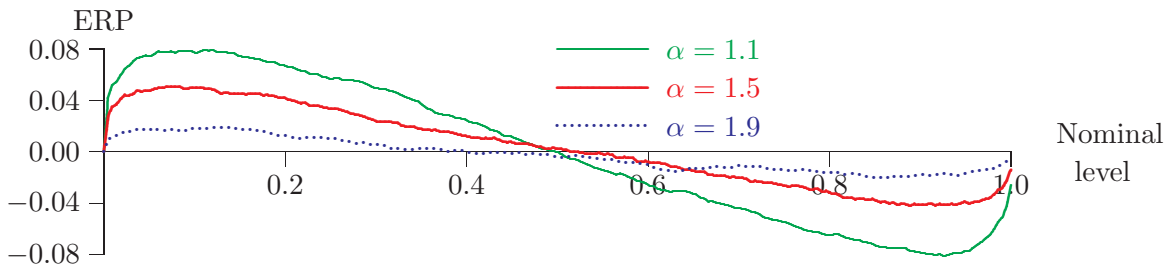


Figure 10: P value discrepancy plots, refined bootstrap, $\hat{\alpha}_{\text{Hill}}$, $\hat{\beta}_{\text{Hill}}$, Student's t law

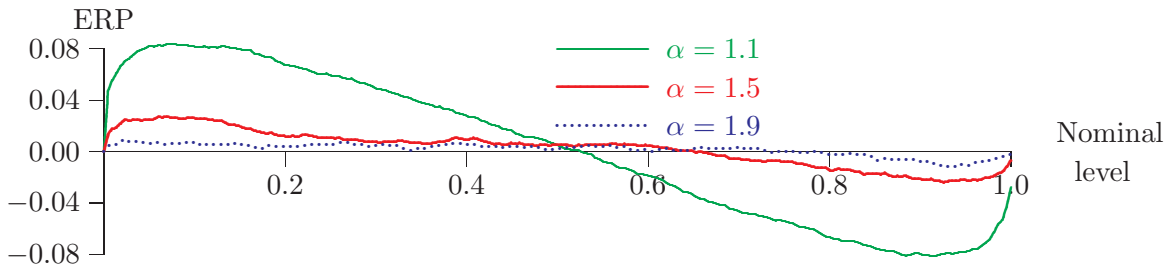


Figure 11: P value discrepancy plots, refined bootstrap, $\hat{\alpha}_{\text{Hill}}$, $\hat{\beta}_{\text{AM}}$, stable law

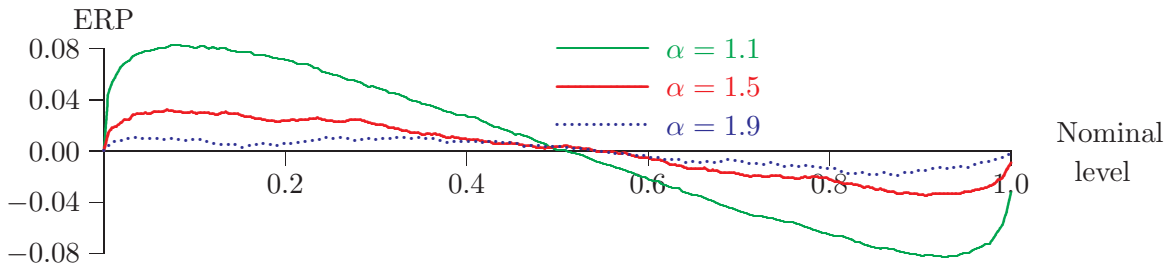


Figure 12: P value discrepancy plots, refined bootstrap, $\hat{\alpha}_{\text{Hill}}$, $\hat{\beta}_{\text{AM}}$, Student's t law

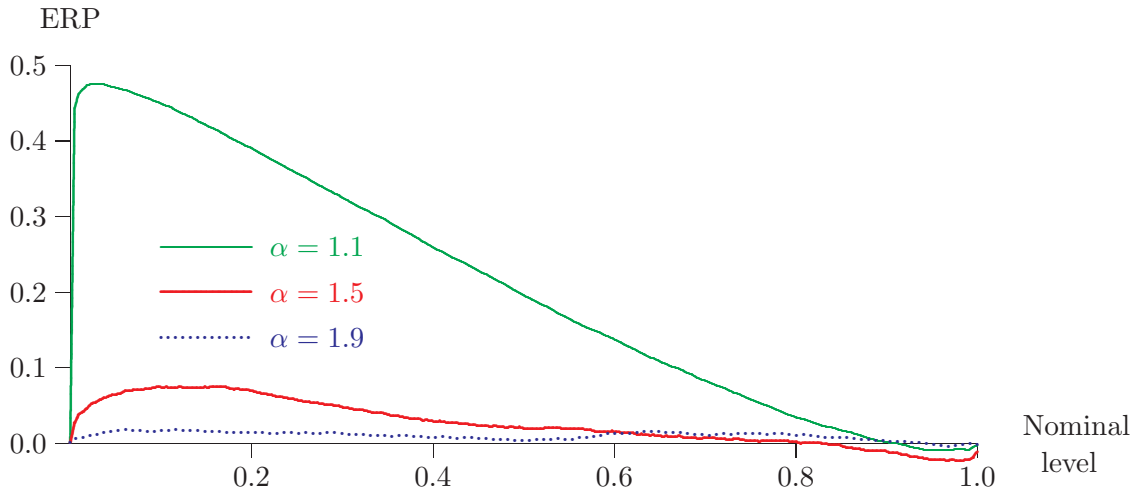


Figure 13: P value discrepancy plots, refined bootstrap, $\hat{\alpha}_{\text{Hill}}$, $\hat{\beta}_{\text{AM}}$, stable law, true $\beta = 0.5$

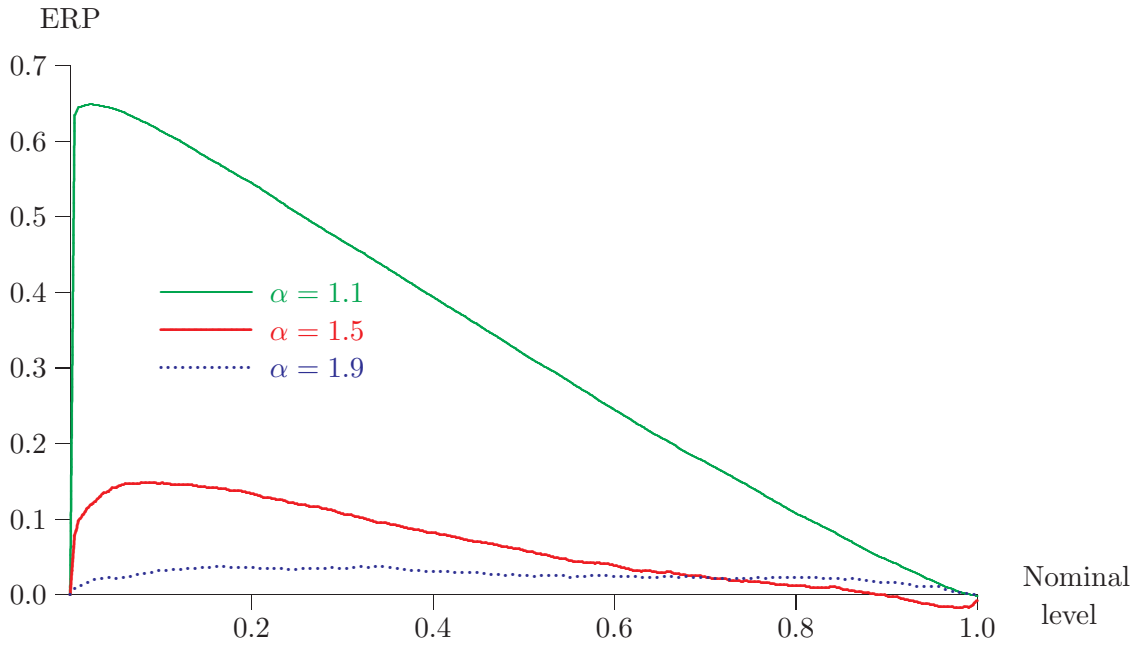


Figure 14: P value discrepancy plots, refined bootstrap, $\hat{\alpha}_{\text{Hill}}$, $\hat{\beta}_{\text{AM}}$, stable law, true $\beta = 1$

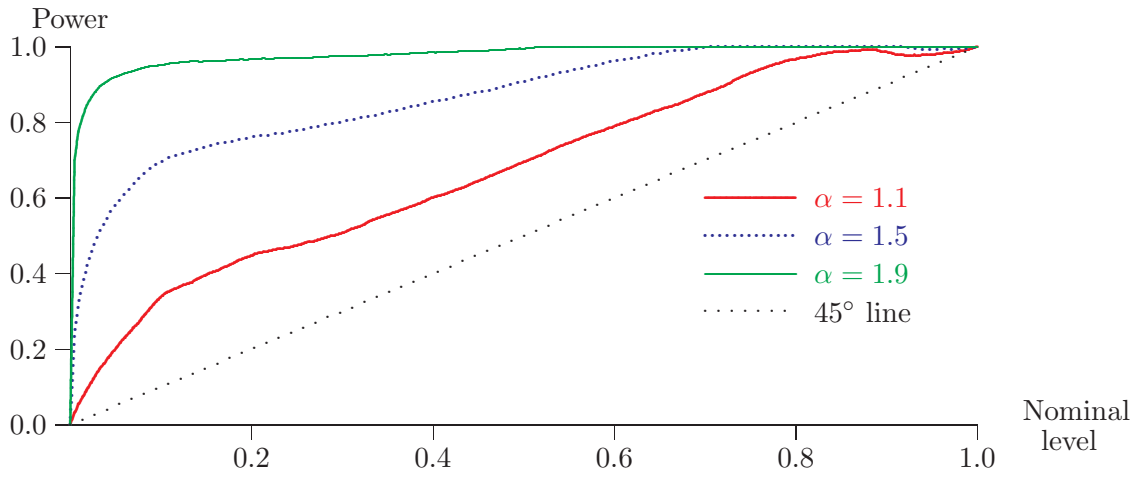


Figure 15: Power of the refined bootstrap, $\hat{\alpha}_{Hill}$, symmetric stable law

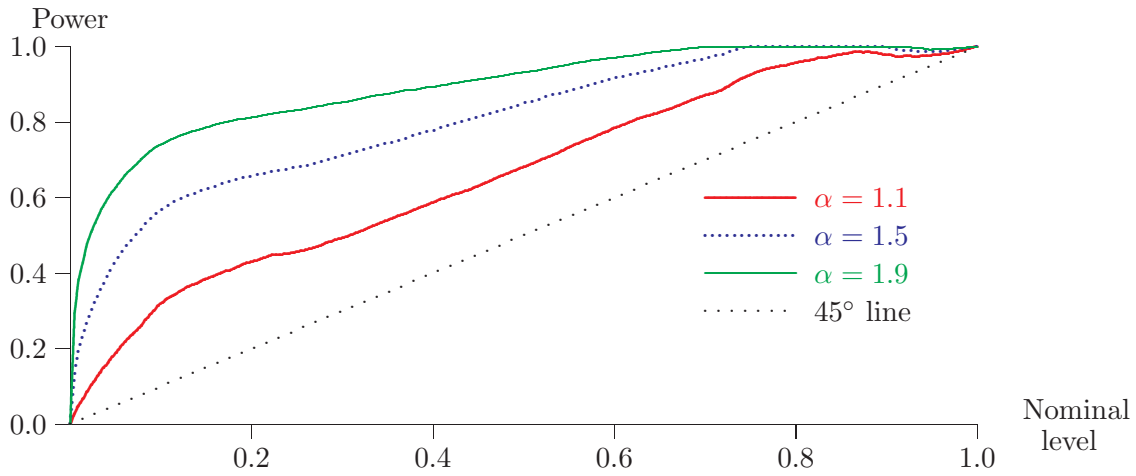


Figure 16: Power of the refined bootstrap, $\hat{\alpha}_{Hill}$, Student's t law

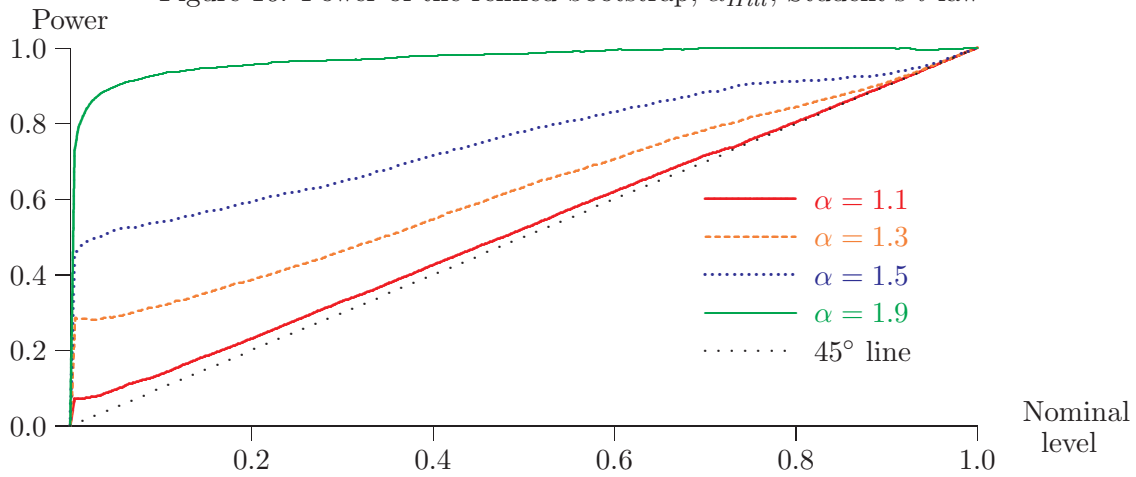


Figure 17: Power of the refined bootstrap, true α , $\hat{\beta}_{quantiles}$, stable law, $\beta = 1$.

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