9 Appendix C

Appendix C is devoted to deriving a closed-form expression for the information matrix and to verifying the information matrix equality. Proofs of Lemmas 4 and 5 are also provided.

Proof of Lemma 4. From (31), taking $k = 2$, we get $\Psi'(x) = \sum_{i=0}^{\infty} 1/(x+i)^2 > 0$ for all x, implying that $\Psi(\upsilon)$ is a strictly increasing function. From the above expression for $\Psi'(x)$, we can also see that $\Psi'(x)$ is a strictly decreasing function for $x > 0$, implying $D'(\upsilon) = \Psi'(\frac{\upsilon+1}{2}) - \Psi'(\frac{\upsilon}{2}) < 0$. So $D(\upsilon)$ is strictly decreasing. If the first equality in (32) holds, then the second one is easily verified. Now we proceed to show the first equality. In fact, differentiating both sides of $\Gamma(x+1) = x\Gamma(x)$ leads to $\Gamma'(x+1)/\Gamma(x) = 1 + x\Psi(x)$, and then rewriting it does yield the first equality in (32). ■

Proof of Lemma 5. The proofs are immediate. For equality (33), taking the log of the expression for $K(\upsilon)$ and then differentiating both sides, it follows that

$$\begin{aligned}
\frac{K'(\upsilon)}{K(\upsilon)} &= \frac{d \ln K(\upsilon)}{d\upsilon} = d\left[\ln \Gamma((\upsilon+1)/2) - \ln \Gamma(\upsilon/2) - \frac{1}{2}\ln \upsilon - \ln \sqrt{\pi}\right]/d\upsilon \\
&= \frac{1}{2}\left[\Psi(\frac{\upsilon+1}{2}) - \Psi(\frac{\upsilon}{2}) - \frac{1}{\upsilon}\right].
\end{aligned}$$

So we have shown that (33) holds. From the definition of $K(\upsilon)$, the left side of equality (34) is expressed as

$$\left(\frac{\upsilon}{\upsilon+2j}\right)^{1/2} \frac{K(\upsilon)}{K(\upsilon+2j)} = \frac{\Gamma((\upsilon+1)/2)}{\Gamma(\upsilon/2)} \frac{\Gamma(\upsilon/2+j)}{\Gamma((\upsilon+1)/2+j)}.$$

Using the fact $\Gamma(\upsilon+1) = \upsilon\Gamma(\upsilon)$, the proof of equality (34) is easily completed. ■

Now suppose that y_t $(t = 1, 2, ...T)$ are i.i.d. observations from the AST with density $f(y;\theta_0)$ defined in (5), where $\theta_0 = (\alpha_0, \upsilon_{01}, \upsilon_{02}, \mu_0, \sigma_0)$. Expectations are always taken with respect to the true underlying distribution $f(y;\theta_0)$. Then the log-density function with parameter θ is $\ln f(y_t;\theta) =$

$$-\ln \sigma - \frac{\upsilon_1+1}{2}\left[\ln L(y_t;\theta)\right] 1(y_t < \mu) - \frac{\upsilon_2+1}{2}\left[\ln R(y_t;\theta)\right] 1(y_t > \mu)$$

and the score vector for observation t, $\frac{\partial}{\partial\theta}\ln f(y_t;\theta)$, is given by

$$\begin{aligned}
\frac{\partial \ln f}{\partial \alpha} &= \frac{\upsilon_1+1}{\alpha}\left[1-\frac{1}{L(y_t;\theta)}\right]1(y_t<\mu) \\
&\quad -\frac{\upsilon_2+1}{1-\alpha}\left[1-\frac{1}{R(y_t;\theta)}\right]1(y_t>\mu), \qquad (56)\\
\frac{\partial \ln f}{\partial \upsilon_1} &= \left\{-\frac{1}{2}\ln L(y_t;\theta)+\frac{\upsilon_1+1}{2}D(\upsilon_1)\frac{L(y_t;\theta)-1}{L(y_t;\theta)}\right\}1(y_t<\mu), \qquad (57)\\
\frac{\partial \ln f}{\partial \upsilon_2} &= \left\{-\frac{1}{2}\ln R(y_t;\theta)+\frac{\upsilon_2+1}{2}D(\upsilon_2)\frac{R(y_t;\theta)-1}{R(y_t;\theta)}\right\}1(y_t>\mu), \qquad (58)\\
\frac{\partial \ln f}{\partial \mu} &= \frac{\upsilon_1+1}{2}\frac{1}{L(y_t;\theta)}\frac{1}{\upsilon_1}\frac{2(y_t-\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}1(y_t<\mu) \\
&\quad +\frac{\upsilon_2+1}{2}\frac{1}{R(y_t;\theta)}\frac{1}{\upsilon_2}\frac{2(y_t-\mu)}{[2(1-\alpha)\sigma K(\upsilon_2)]^2}1(y_t>\mu), \qquad (59)\\
\frac{\partial \ln f}{\partial \sigma} &= -\frac{1}{\sigma}+\frac{\upsilon_1+1}{\sigma}\left[1-\frac{1}{L(y_t;\theta)}\right]1(y_t<\mu) \\
&\quad +\frac{\upsilon_2+1}{\sigma}\left[1-\frac{1}{R(y_t;\theta)}\right]1(y_t>\mu), \qquad (60)
\end{aligned}$$

where we used equality (33) in the expressions for the components $\frac{\partial \ln f}{\partial \upsilon_1}$ and $\frac{\partial \ln f}{\partial \upsilon_2}$. To derive the information matrix $I(\theta_0)\equiv E[\frac{\partial}{\partial\theta}\ln f(y_t,\theta_0)\frac{\partial}{\partial\theta'}\ln f(y_t;\theta_0)]$ and the Hessian $H(\theta_0)\equiv E[\frac{\partial^2}{\partial\theta\partial\theta'}\ln f(y_t;\theta_0)]$ and to verify the information matrix equality $I(\theta_0)=-H(\theta_0)$, the following Lemma is needed.

Lemma 9 *For any $j,m=0,1,2,...$, the following moment equalities hold:*

$$\begin{aligned}
E\left\{\frac{1}{[L(y_t;\theta_0)]^j}1(y_t<\mu_0)\right\} &= \alpha\left(\frac{\upsilon_1}{\upsilon_1+2j}\right)^{1/2}\frac{K(\upsilon_1)}{K(\upsilon_1+2j)} \qquad (61)\\
&= \begin{cases} \alpha, & j=0,\\ \alpha\upsilon_1/(\upsilon_1+1), & j=1,\\ \alpha\frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}, & j=2; \end{cases}
\end{aligned}$$

$$\begin{aligned}
E\left\{\frac{1}{[R(y_t;\theta_0)]^j}1(y_t>\mu_0)\right\} &= (1-\alpha)\left(\frac{\upsilon_2}{\upsilon_2+2j}\right)^{1/2}\frac{K(\upsilon_2)}{K(\upsilon_2+2j)} \qquad (62)\\
&= \begin{cases} (1-\alpha), & j=0,\\ (1-\alpha)\upsilon_2/(\upsilon_2+1), & j=1,\\ (1-\alpha)\frac{\upsilon_2(\upsilon_2+2)}{(\upsilon_2+1)(\upsilon_2+3)}, & j=2; \end{cases}
\end{aligned}$$

$$
\begin{aligned}
E\left\{\frac{\ln L(y_t;\theta_0)}{[L(y_t;\theta_0)]^j}1(y_t<\mu_0)\right\} &= \alpha\left(\frac{\upsilon_1}{\upsilon_1+2j}\right)^{1/2}\frac{K(\upsilon_1)}{K(\upsilon_1+2j)}D(\upsilon_1+2j) \\
&= \begin{cases} \alpha D(\upsilon_1), & j=0, \\ \alpha(\frac{\upsilon_1}{\upsilon_1+1})D(\upsilon_1+2), & j=1; \end{cases} \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
E\left\{\frac{\ln R(y_t;\theta_0)}{[R(y_t;\theta_0)]^j}1(y_t>\mu_0)\right\} &= (1-\alpha)\left(\frac{\upsilon_2}{\upsilon_2+2j}\right)^{1/2}\frac{K(\upsilon_2)D(\upsilon_2+2j)}{K(\upsilon_2+2j)} \\
&= \begin{cases} (1-\alpha)D(\upsilon_2), & j=0, \\ (1-\alpha)(\frac{\upsilon_2}{\upsilon_2+1})D(\upsilon_2+2), & j=1; \end{cases} \qquad (64)
\end{aligned}
$$

$$
E\left\{\frac{(y_t-\mu_0)[\ln L(y_t;\theta_0)]^m}{[L(y_t;\theta_0)]^j}1(y_t<\mu_0)\right\} = -\frac{2^{m+1}m!\upsilon_1[2\alpha\sigma K(\upsilon_1)]^2}{2\sigma(\upsilon_1+2j-1)^{m+1}}, \quad (65)
$$

$$
E\left\{\frac{(y_t-\mu_0)[\ln R(y_t;\theta_0)]^m}{[R(y_t;\theta_0)]^j}1(y_t>\mu_0)\right\} = \frac{2^{m+1}m!\upsilon_2[2(1-\alpha)\sigma K(\upsilon_2)]^2}{2\sigma(\upsilon_2+2j-1)^{m+1}}, \quad (66)
$$

$$
\begin{aligned}
E\left\{[\ln L(y_t;\theta_0)]^2 1(y_t<\mu_0)\right\} &= \alpha[D^2(\upsilon_1)-2D'(\upsilon_1)], \qquad (67) \\
E\left\{[\ln R(y_t;\theta_0)]^2 1(y_t>\mu_0)\right\} &= (1-\alpha)[D^2(\upsilon_1)-2D'(\upsilon_1)], \qquad (68)
\end{aligned}
$$

where the right hand sides of all the equalities from (61) to (68) are evaluated at the true values $(\alpha_0, \upsilon_{01}, \upsilon_{02}, \mu_0, \sigma_0)$.

Proof.[4] We discuss equalities (61), (63), (65) and (67). Other equalities can be proved in the same manner. Note that, for any $j, m = 0, 1, 2, ...$,

$$
\begin{aligned}
EL_1(j,m) &\equiv E\left\{[L(y_t;\theta)]^{-j}[\ln L(y_t;\theta)]^m 1(y_t<\mu)\right\} \\
&= \int_{-\infty}^{\mu}[L(y;\theta)]^{-j}[\ln L(y;\theta)]^m\frac{1}{\sigma}[L(y;\theta)]^{-(\upsilon_1+1)/2}\}dy,
\end{aligned}
$$

and that $L(y;\theta) \equiv 1 + \frac{1}{\upsilon_1}(\frac{y-\mu}{2\alpha\sigma K(\upsilon_1)})^2$. Then using the change of variable $z = -\frac{y-\mu}{2\alpha\sigma\sqrt{\upsilon_1}K(\upsilon_1)}$ yields

$$
EL_1(j,m) = 2\alpha\sqrt{\upsilon_1}K(\upsilon_1)\int_0^{+\infty}(1+z^2)^{-(\upsilon_1+2j+1)/2}[\ln(1+z^2)]^m dz. \quad (69)
$$

Setting $m = 0$, $m = 1$, and $(j, m) = (0, 2)$ respectively, and correspondingly taking into account equality (35) with $\upsilon = \upsilon_1 + 2j$, equality (36) with $\upsilon =$

[4] For simplicity, we omit the subscript on the true parameters θ_0 in all the following proofs.

$\upsilon_1 + 2j$, and equality (37) with $\upsilon = \upsilon_1$, we obtain equalities (61), (63), and (67). These proofs use (34). Now consider equality (65). Denote by $EL_2(j,m)$ the expectation of the left side of equality (65), and note that the change of variable $z = \frac{1}{\upsilon_1}(\frac{y-\mu}{2\alpha\sigma K(\upsilon_1)})^2$ yields

$$EL_2(j,m) = -\frac{\upsilon_1[2\alpha\sigma K(\upsilon_1)]^2}{2\sigma}\int_0^{+\infty}(1+z)^{-(\upsilon_1+2j+1)/2}[\ln(1+z)]^m dz. \quad (70)$$

Subject to $\upsilon_1 + 2j - 1 > 0$, by integration by parts it follows that

$$EL_2(j,m) = m\left(\frac{2}{\upsilon_1+2j-1}\right)EL_2(j,m-1) = m!\left(\frac{2}{\upsilon_1+2j-1}\right)^m EL_2(j,0).$$

A straightforward calculation for (70) gives $EL_2(j,0) = -\frac{\upsilon_1[2\alpha\sigma K(\upsilon_1)]^2}{\sigma(\upsilon_1+2j-1)}$. ■

Lemma 10 *The score vector for observation* t, $\frac{\partial}{\partial\theta}\ln f(y_t;\theta)$, *satisfies the equation*

$$E\left[\frac{\partial}{\partial\theta}\ln f(y_t;\theta_0)\right] = 0. \quad (71)$$

Proof. By using the equalities from (61) to (68), this Lemma is easily verified. In fact,

(i).

$$\begin{aligned}E\left[\frac{\partial \ln f}{\partial\alpha}\right] &= \frac{\upsilon_1+1}{\alpha}E\left\{\left[1-\frac{1}{L(y_t;\theta)}\right]1(y_t<\mu)\right\} \\ &\quad -\frac{\upsilon_2+1}{1-\alpha}E\left\{\left[1-\frac{1}{R(y_t;\theta)}\right]1(y_t>\mu)\right\} \\ &= \frac{\upsilon_1+1}{\alpha}\alpha\left(1-\frac{\upsilon_1}{\upsilon_1+1}\right)-\frac{\upsilon_2+1}{1-\alpha}(1-\alpha)\left(1-\frac{\upsilon_2}{\upsilon_2+1}\right)=0.\end{aligned}$$

(ii).

$$\begin{aligned}E\left[\frac{\partial \ln f}{\partial\upsilon_1}\right] &= E\left\{\left[-\frac{\ln L(y_t;\theta)}{2}+\frac{\upsilon_1+1}{2}D(\upsilon_1)\frac{L(y_t;\theta)-1}{L(y_t;\theta)}\right]1(y_t<\mu)\right\} \\ &= -\frac{1}{2}\alpha D(\upsilon_1)+\frac{\upsilon_1+1}{2}D(\upsilon_1)\alpha\left(1-\frac{\upsilon_1}{\upsilon_1+1}\right)=0.\end{aligned}$$

(iii). Similarly, we have

$$E\left[\frac{\partial \ln f}{\partial\upsilon_2}\right] = E\left\{\left[\frac{\upsilon_2+1}{2}D(\upsilon_2)\frac{R(y_t;\theta)-1}{R(y_t;\theta)}-\frac{\ln R(y_t;\theta)}{2}\right]1(y_t>\mu)\right\}=0.$$

(iv).

$$\begin{aligned} E\left[\frac{\partial \ln f}{\partial \mu}\right] &= \frac{\upsilon_1+1}{2}E\left\{\frac{1}{L(y_t;\theta)}\frac{1}{\upsilon_1}\frac{2(y_t-\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}1(y_t<\mu)\right\} \\ &\quad +\frac{\upsilon_2+1}{2}E\left\{\frac{1}{R(y_t;\theta)}\frac{1}{\upsilon_2}\frac{2(y_t-\mu)}{[2(1-\alpha)\sigma K(\upsilon_2)]^2}1(y_t>\mu)\right\} \\ &= \frac{\upsilon_1+1}{2}\frac{-2}{\sigma(\upsilon_1+1)}+\frac{\upsilon_2+1}{2}\frac{2}{\sigma(\upsilon_2+1)}=0. \end{aligned}$$

(v).

$$\begin{aligned} E\left[\frac{\partial \ln f}{\partial \sigma}\right] &= -\frac{1}{\sigma}+\frac{\upsilon_1+1}{\sigma}E\left\{\left[1-\frac{1}{L(y_t;\theta)}\right]1(y_t<\mu)\right\} \\ &\quad +\frac{\upsilon_2+1}{\sigma}E\left\{\left[1-\frac{1}{R(y_t;\theta)}\right]1(y_t>\mu)\right\} \\ &= -\frac{1}{\sigma}+\frac{\upsilon_1+1}{\sigma}\alpha\left(1-\frac{\upsilon_1}{\upsilon_1+1}\right)+\frac{\upsilon_2+1}{\sigma}\frac{(1-\alpha)}{\upsilon_2+1} \\ &= -\frac{1}{\sigma}+\frac{\alpha}{\sigma}+\frac{1-\alpha}{\sigma}=0.\blacksquare \end{aligned}$$

Proof of Proposition 2. We prove this by computing expectations on the both sides of the following equations and then verifying them,

$$E[\frac{\partial \ln f(y_t;\theta)}{\partial\theta_i}\cdot\frac{\partial \ln f(y_t;\theta)}{\partial\theta_j}]=-E[\frac{\partial^2 \ln f(y_t;\theta)}{\partial\theta_i\partial\theta_j}],\quad i,\ j=1,2,...5.$$

In the proof, the fact that $1(y_t<\mu)1(y_t>\mu)=0$ and the equalities (61)-(68) are used repeatedly. In addition, we use $E[\frac{\partial}{\partial\theta}\ln f(y_t;\theta_0)]=0$ shown in (71) and $D(\upsilon+2)=-\frac{2}{\upsilon(\upsilon+1)}+D(\upsilon)$ given in (32). Note that by the construction of the AST distribution, the left-tail parameter υ_1 and the right-tail parameter υ_2 have a symmetry property. Hence we do not consider the terms of the information matrix equality involved in the right-tail parameter υ_2.

(**a**)

$$
\begin{aligned}
E\left[\frac{\partial \ln f}{\partial \alpha}\right]^2 &= \left(\frac{\upsilon_1+1}{\alpha}\right)^2 E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)^2 1(y_t<\mu)\right\} \\
&\quad +\left(\frac{\upsilon_2+1}{1-\alpha}\right)^2 E\left\{\left(1-\frac{1}{R(y_t;\theta)}\right)^2 1(y_t>\mu)\right\} \\
&= \left(\frac{\upsilon_1+1}{\alpha}\right)^2 \alpha\left[1-2\frac{\upsilon_1}{\upsilon_1+1}+\frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right] \\
&\quad +\left(\frac{\upsilon_2+1}{1-\alpha}\right)^2 (1-\alpha)\left[1-\frac{2\upsilon_2}{\upsilon_2+1}+\frac{\upsilon_2(\upsilon_2+2)}{(\upsilon_2+1)(\upsilon_2+3)}\right] \\
&= 3\left[\frac{\upsilon_1+1}{\alpha(\upsilon_1+3)}+\frac{\upsilon_2+1}{(1-\alpha)(\upsilon_2+3)}\right];
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln f}{\partial \alpha^2}\right] &= -\frac{\upsilon_1+1}{\alpha^2}E\left\{\left[1+\frac{1}{L(y_t;\theta)}-\frac{2}{[L(y_t;\theta)]^2}\right]1(y_t<\mu)\right\} \\
&\quad -\frac{\upsilon_2+1}{(1-\alpha)^2}E\left\{\left[1+\frac{1}{R(y_t;\theta)}-\frac{2}{[R(y_t;\theta)]^2}\right]1(y_t>\mu)\right\} \\
&= -\frac{\upsilon_1+1}{\alpha^2}\alpha\left[1+\frac{\upsilon_1}{\upsilon_1+1}-2\frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right] \\
&\quad -\frac{\upsilon_2+1}{(1-\alpha)^2}(1-\alpha)\left[1+\frac{\upsilon_2}{\upsilon_2+1}-2\frac{\upsilon_2(\upsilon_2+2)}{(\upsilon_2+1)(\upsilon_2+3)}\right] \\
&= -3\left[\frac{\upsilon_1+1}{\alpha(\upsilon_1+3)}+\frac{\upsilon_2+1}{(1-\alpha)(\upsilon_2+3)}\right].
\end{aligned}
$$

(**b**)

$$
\begin{aligned}
& E\left[\frac{\partial \ln f}{\partial \upsilon_1}\right]^2 \\
= \;& E\left\{\left[\frac{[\ln L(y_t;\theta)]^2}{4} + \left(\frac{\upsilon_1+1}{2}\right)^2 D^2(\upsilon_1)\left(1-\frac{1}{L(y_t;\theta)}\right)^2\right] 1(y_t<\mu)\right\} \\
& -\left(\frac{\upsilon_1+1}{2}\right) D(\upsilon_1) E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)\ln L(y_t;\theta) 1(y_t<\mu)\right\} \\
= \;& \alpha\frac{D^2(\upsilon_1)-2D'(\upsilon_1)}{4} + \left(\frac{\upsilon_1+1}{2}\right)^2 D^2(\upsilon_1)\frac{3\alpha}{(\upsilon_1+1)(\upsilon_1+3)} \\
& -\left(\frac{\upsilon_1+1}{2}\right) D(\upsilon_1)\alpha\left[D(\upsilon_1)-\frac{\upsilon_1}{\upsilon_1+1}D(\upsilon_1+2)\right] \\
= \;& \frac{\alpha}{2}\left\{\frac{\upsilon_1}{\upsilon_1+3}D^2(\upsilon_1)-\frac{2}{\upsilon_1+1}D(\upsilon_1)-D'(\upsilon_1)\right\};
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln f}{\partial \upsilon_1^2}\right] = \;& \left[D(\upsilon_1)+\frac{\upsilon_1+1}{2}D'(\upsilon_1)\right] E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right) 1(y_t<\mu)\right\} \\
& -\frac{\upsilon_1+1}{2}D^2(\upsilon_1)E\left\{\left(\frac{1}{L(y_t;\theta)}-\frac{1}{[L(y_t;\theta)]^2}\right) 1(y_t<\mu)\right\} \\
= \;& \left[D(\upsilon_1)+\frac{\upsilon_1+1}{2}D'(\upsilon_1)\right]\alpha\left[1-\frac{\upsilon_1}{\upsilon_1+1}\right] \\
& -\frac{\upsilon_1+1}{2}D^2(\upsilon_1)\alpha\left[\frac{\upsilon_1}{\upsilon_1+1}-\frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right] \\
= \;& -\frac{\alpha}{2}\left\{\frac{\upsilon_1}{\upsilon_1+3}D^2(\upsilon_1)-\frac{2}{\upsilon_1+1}D(\upsilon_1)-D'(\upsilon_1)\right\}.
\end{aligned}
$$

(**c**).

$$
\begin{aligned}
E\left[\frac{\partial \ln f}{\partial \mu}\right]^2 &= \frac{1}{\upsilon_1}\left[\frac{\upsilon_1+1}{2\alpha\sigma K(\upsilon_1)}\right]^2 E\left\{\left(\frac{1}{L(y_t;\theta)}-\frac{1}{[L(y_t;\theta)]^2}\right)1(y_t<\mu)\right\} \\
&\quad +\frac{1}{\upsilon_2}\left[\frac{\upsilon_2+1}{2(1-\alpha)\sigma K(\upsilon_2)}\right]^2 E\left\{\left(\frac{1(y_t>\mu)}{R(y_t;\theta)}-\frac{1(y_t>\mu)}{[R(y_t;\theta)]^2}\right)\right\} \\
&= \left[\frac{\upsilon_1+1}{2\alpha\sigma K(\upsilon_1)}\right]^2\frac{\alpha}{\upsilon_1}\left[\frac{\upsilon_1}{\upsilon_1+1}-\frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right]+ \\
&\quad +\left[\frac{\upsilon_2+1}{2(1-\alpha)\sigma K(\upsilon_2)}\right]^2\frac{1-\alpha}{\upsilon_2}\left[\frac{\upsilon_2}{\upsilon_2+1}-\frac{\upsilon_2(\upsilon_2+2)}{(\upsilon_2+1)(\upsilon_2+3)}\right] \\
&= \frac{1}{4\sigma^2}\left[\frac{\upsilon_1+1}{\alpha(\upsilon_1+3)}\frac{1}{K^2(\upsilon_1)}+\frac{\upsilon_2+1}{(1-\alpha)(\upsilon_2+3)}\frac{1}{K^2(\upsilon_2)}\right].
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln f}{\partial \mu^2}\right] &= \frac{1}{\upsilon_1}\frac{\upsilon_1+1}{[2\alpha\sigma K(\upsilon_1)]^2}E\left\{\left(\frac{1}{L(y_t;\theta)}-\frac{2}{[L(y_t;\theta)]^2}\right)1(y_t<\mu)\right\} \\
&\quad +\frac{1}{\upsilon_2}\frac{\upsilon_2+1}{[2(1-\alpha)\sigma K(\upsilon_2)]^2}E\left\{\frac{1(y_t>\mu)}{R(y_t;\theta)}-2\frac{1(y_t>\mu)}{[R(y_t;\theta)]^2}\right\} \\
&= \frac{\upsilon_1+1}{[2\alpha\sigma K(\upsilon_1)]^2}\frac{\alpha}{\upsilon_1}\left[\frac{\upsilon_1}{\upsilon_1+1}-\frac{2\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right] \\
&\quad +\frac{\upsilon_2+1}{[2(1-\alpha)\sigma K(\upsilon_2)]^2}\frac{1-\alpha}{\upsilon_2}\left[\frac{\upsilon_2}{\upsilon_2+1}-\frac{2\upsilon_2(\upsilon_2+2)}{(\upsilon_2+1)(\upsilon_2+3)}\right] \\
&= -\frac{1}{4\sigma^2}\left[\frac{\upsilon_1+1}{\alpha(\upsilon_1+3)}\frac{1}{K^2(\upsilon_1)}+\frac{\upsilon_2+1}{(1-\alpha)(\upsilon_2+3)}\frac{1}{K^2(\upsilon_2)}\right].
\end{aligned}
$$

(**d**).

$$
\begin{aligned}
E\left[\frac{\partial \ln f}{\partial \sigma}\right]^2 &= 0-\frac{1}{\sigma^2}+\left(\frac{\upsilon_1+1}{\sigma}\right)^2 E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)^2 1(y_t<\mu)\right\} \\
&\quad +\left(\frac{\upsilon_2+1}{\sigma}\right)^2 E\left\{\left(1-\frac{1}{R(y_t;\theta)}\right)^2 1(y_t>\mu)\right\} \\
&= -\frac{1}{\sigma^2}+\left(\frac{\upsilon_1+1}{\sigma}\right)^2\alpha\left[1-2\frac{\upsilon_1}{\upsilon_1+1}+\frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right] \\
&\quad +\left(\frac{\upsilon_2+1}{\sigma}\right)^2(1-\alpha)\left[1-2\frac{\upsilon_2}{\upsilon_2+1}+\frac{\upsilon_2(\upsilon_2+2)}{(\upsilon_2+1)(\upsilon_2+3)}\right] \\
&= \frac{2}{\sigma^2}\left[\alpha\frac{\upsilon_1}{\upsilon_1+3}+(1-\alpha)\frac{\upsilon_2}{\upsilon_2+3}\right];
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln f}{\partial \sigma^2}\right] &= \frac{1}{\sigma^2} - \frac{\upsilon_1+1}{\sigma^2} E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)\left(1+\frac{2}{L(y_t;\theta)}\right)1(y_t<\mu)\right\} \\
&\quad - \frac{\upsilon_2+1}{\sigma^2} E\left\{\left(1-\frac{1}{R(y_t;\theta)}\right)\left(1+\frac{2}{R(y_t;\theta)}\right)1(y_t>\mu)\right\} \\
&= \frac{1}{\sigma^2} - \frac{\upsilon_1+1}{\sigma^2}\alpha\left[1+\frac{\upsilon_1}{\upsilon_1+1} - 2\frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right] \\
&\quad - \frac{\upsilon_2+1}{\sigma^2}(1-\alpha)\left[1+\frac{\upsilon_2}{\upsilon_2+1} - 2\frac{\upsilon_2(\upsilon_2+2)}{(\upsilon_2+1)(\upsilon_2+3)}\right] \\
&= -\frac{2}{\sigma^2}\left[\alpha\frac{\upsilon_1}{\upsilon_1+3} + (1-\alpha)\frac{\upsilon_2}{\upsilon_2+3}\right].
\end{aligned}
$$

(**e**).

$$
\begin{aligned}
E\left[\frac{\partial \ln f}{\partial \alpha}\frac{\partial \ln f}{\partial \upsilon_1}\right] &= -\frac{\upsilon_1+1}{2\alpha} E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)\ln L(y_t;\theta) 1(y_t<\mu)\right\} \\
&\quad + \frac{(\upsilon_1+1)^2}{2\alpha} D(\upsilon_1) E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)^2 1(y_t<\mu)\right\} \\
&= -\frac{\upsilon_1+1}{2\alpha}\alpha\left[D(\upsilon_1) - \frac{\upsilon_1}{\upsilon_1+1}D(\upsilon_1+2)\right] \\
&\quad + \frac{(\upsilon_1+1)^2}{2\alpha} D(\upsilon_1)\alpha\left[1-\frac{2\upsilon_1}{\upsilon_1+1} + \frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right] \\
&= -\frac{1}{\upsilon_1+1} + \frac{\upsilon_1}{\upsilon_1+3}D(\upsilon_1);
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln f}{\partial \alpha \partial \upsilon_1}\right] &= \frac{1}{\alpha} E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)1(y_t<\mu)\right\} \\
&\quad - \frac{\upsilon_1+1}{\alpha} D(\upsilon_1) E\left\{\left(\frac{1}{L(y_t;\theta)} - \frac{1}{[L(y_t;\theta)]^2}\right)1(y_t<\mu)\right\} \\
&= \frac{1}{\alpha}\alpha\left[1-\frac{\upsilon_1}{\upsilon_1+1}\right] - \frac{\upsilon_1+1}{\alpha}D(\upsilon_1)\frac{\alpha\upsilon_1}{\upsilon_1+1}\left[1-\frac{\upsilon_1+2}{\upsilon_1+3}\right] \\
&= \frac{1}{\upsilon_1+1} - \frac{\upsilon_1}{\upsilon_1+3}D(\upsilon_1).
\end{aligned}
$$

(**f**).

$$
\begin{aligned}
& E\left[\frac{\partial \ln f}{\partial \alpha}\frac{\partial \ln f}{\partial \mu}\right] \\
= \;& \frac{(\upsilon_1+1)^2}{2\alpha}E\left\{\left(\frac{1}{L(y_t;\theta)}-\frac{1}{[L(y_t;\theta)]^2}\right)\frac{1}{\upsilon_1}\frac{2(y_t-\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}1(y_t<\mu)\right\} \\
& -\frac{(\upsilon_2+1)^2}{2(1-\alpha)}E\left\{\left(\frac{1}{R(y_t;\theta)}-\frac{1}{[R(y_t;\theta)]^2}\right)\frac{1}{\upsilon_2}\frac{2(y_t-\mu)1(y_t>\mu)}{[2(1-\alpha)\sigma K(\upsilon_2)]^2}\right\} \\
= \;& \frac{(\upsilon_1+1)^2}{2\alpha\sigma}\left[-\frac{2}{\upsilon_1+1}+\frac{2}{\upsilon_1+3}\right]-\frac{(\upsilon_2+1)^2}{2(1-\alpha)\sigma}\left[\frac{2}{\upsilon_2+1}-\frac{2}{\upsilon_2+3}\right] \\
= \;& -\frac{2}{\sigma}\left[\frac{\upsilon_1+1}{\alpha(\upsilon_1+3)}+\frac{\upsilon_2+1}{(1-\alpha)(\upsilon_2+3)}\right];
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln f}{\partial \alpha \partial \mu}\right] &= -\frac{\upsilon_1+1}{\alpha}E\left\{\frac{1}{[L(y_t;\theta)]^2}\frac{1}{\upsilon_1}\frac{2(y_t-\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}1(y_t<\mu)\right\} \\
&\quad +\frac{\upsilon_2+1}{1-\alpha}E\left\{\frac{1}{[R(y_t;\theta)]^2}\frac{1}{\upsilon_2}\frac{2(y_t-\mu)}{[2(1-\alpha)\sigma K(\upsilon_2)]^2}1(y_t>\mu)\right\} \\
&= \frac{\upsilon_1+1}{\alpha}\frac{1}{\sigma}\frac{2}{\upsilon_1+3}+\frac{\upsilon_2+1}{1-\alpha}\frac{1}{\sigma}\frac{2}{\upsilon_2+3} \\
&= \frac{2}{\sigma}\left[\frac{\upsilon_1+1}{\alpha(\upsilon_1+3)}+\frac{\upsilon_2+1}{(1-\alpha)(\upsilon_2+3)}\right].
\end{aligned}
$$

(**g**).

$$
\begin{aligned}
E\left[\frac{\partial \ln f}{\partial \alpha}\frac{\partial \ln f}{\partial \sigma}\right] &= -\frac{1}{\sigma}E[\frac{\partial \ln f}{\partial \alpha}]+\frac{(\upsilon_1+1)^2}{\alpha\sigma}E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)^2 1(y_t<\mu)\right\} \\
&\quad -\frac{(\upsilon_2+1)^2}{(1-\alpha)\sigma}E\left\{\left(1-\frac{1}{R(y_t;\theta)}\right)^2 1(y_t>\mu)\right\} \\
&= 0+\frac{(\upsilon_1+1)^2}{\alpha\sigma}\alpha\left[1-\frac{2\upsilon_1}{\upsilon_1+1}+\frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right] \\
&\quad -\frac{(\upsilon_2+1)^2}{(1-\alpha)\sigma}(1-\alpha)\left[1-\frac{2\upsilon_2}{\upsilon_2+1}+\frac{\upsilon_2(\upsilon_2+2)}{(\upsilon_2+1)(\upsilon_2+3)}\right] \\
&= \frac{2}{\sigma}\left[\frac{\upsilon_1}{\upsilon_1+3}-\frac{\upsilon_2}{\upsilon_2+3}\right];
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln f}{\partial\alpha\partial\sigma}\right] &= -2\left(\frac{\upsilon_1+1}{\alpha\sigma}\right)E\left\{\left(\frac{1}{L(y_t;\theta)}-\frac{1}{[L(y_t;\theta)]^2}\right)1(y_t<\mu)\right\} \\
&\quad +2\left[\frac{\upsilon_2+1}{(1-\alpha)\sigma}\right]E\left\{\left(\frac{1}{R(y_t;\theta)}-\frac{1}{[R(y_t;\theta)]^2}\right)1(y_t>\mu)\right\} \\
&= -2\left(\frac{\upsilon_1+1}{\alpha\sigma}\right)\alpha\left[\frac{\upsilon_1}{\upsilon_1+1}-\frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right] \\
&\quad +2\left(\frac{\upsilon_2+1}{(1-\alpha)\sigma}\right)(1-\alpha)\left[\frac{\upsilon_2}{\upsilon_2+1}-\frac{\upsilon_2(\upsilon_2+2)}{(\upsilon_2+1)(\upsilon_2+3)}\right] \\
&= -\frac{2}{\sigma}\left(\frac{\upsilon_1}{\upsilon_1+3}-\frac{\upsilon_2}{\upsilon_2+3}\right).
\end{aligned}
$$

(**h**). Note that $\frac{\partial \ln f}{\partial \upsilon_1}\frac{\partial \ln f}{\partial \upsilon_2}=0$ and $\frac{\partial^2 \ln f}{\partial \upsilon_1\partial\upsilon_2}=0$. Then we have

$$
E\left(\frac{\partial \ln f}{\partial \upsilon_1}\frac{\partial \ln f}{\partial \upsilon_2}\right)=-E\left(\frac{\partial^2 \ln f}{\partial\upsilon_1\partial\upsilon_2}\right)=0.
$$

(**i**).

$$
\begin{aligned}
&E\left[\frac{\partial \ln f}{\partial \upsilon_1}\frac{\partial \ln f}{\partial \mu}\right] \\
&= -\frac{\upsilon_1+1}{4}E\left\{\frac{1}{L(y_t;\theta)}\frac{1}{\upsilon_1}\frac{2(y_t-\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}\ln L(y_t;\theta)1(y_t<\mu)\right\} \\
&\quad +\left(\frac{\upsilon_1+1}{2}\right)^2 D(\upsilon_1)E\left\{\frac{1}{L(y_t;\theta)}\frac{1}{\upsilon_1}\frac{2(y_t-\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}1(y_t<\mu)\right\} \\
&\quad -\left(\frac{\upsilon_1+1}{2}\right)^2 D(\upsilon_1)E\left\{\frac{1}{[L(y_t;\theta)]^2}\frac{1}{\upsilon_1}\frac{2(y_t-\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}1(y_t<\mu)\right\} \\
&= \frac{\upsilon_1+1}{4\sigma}\left(\frac{2}{\upsilon_1+1}\right)^2-\left(\frac{\upsilon_1+1}{2}\right)^2\frac{D(\upsilon_1)}{\sigma}\left[\frac{2}{\upsilon_2+1}-\frac{2}{\upsilon_2+3}\right] \\
&= \frac{1}{\sigma}\left[\frac{1}{\upsilon_1+1}-\frac{\upsilon_1+1}{\upsilon_1+3}D(\upsilon_1)\right];
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln f}{\partial\upsilon_1\partial\mu}\right] &= \frac{1}{2}E\left\{\frac{1}{L(y_t;\theta)}\frac{1}{\upsilon_1}\frac{2(y_t-\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}1(y_t<\mu)\right\} \\
&\quad -\left(\frac{\upsilon_1+1}{2}\right)D(\upsilon_1)E\left\{\frac{1}{[L(y_t;\theta)]^2}\frac{1}{\upsilon_1}\frac{2(y_t-\mu)1(y_t<\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}\right\} \\
&= -\frac{1}{2}\frac{1}{\sigma}\left(\frac{2}{\upsilon_1+1}\right)+\left(\frac{\upsilon_1+1}{2}\right)D(\upsilon_1)\frac{1}{\sigma}\frac{2}{\upsilon_2+3} \\
&= -\frac{1}{\sigma}\left[\frac{1}{\upsilon_1+1}-\frac{\upsilon_1+1}{\upsilon_1+3}D(\upsilon_1)\right].
\end{aligned}
$$

(**j**).

$$
\begin{aligned}
E\left[\frac{\partial \ln f}{\partial \upsilon_1}\frac{\partial \ln f}{\partial \sigma}\right] &= 0-\frac{\upsilon_1+1}{2\sigma}E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)\ln L(y_t;\theta)1(y_t<\mu)\right\} \\
&\quad +\frac{(\upsilon_1+1)^2}{2\sigma}D(\upsilon_1)E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)^2 1(y_t<\mu)\right\} \\
&= -\frac{\upsilon_1+1}{2\sigma}\alpha\left[D(\upsilon_1)-\frac{\upsilon_1}{\upsilon_1+1}D(\upsilon_1+2)\right] \\
&\quad +\frac{(\upsilon_1+1)^2}{2\sigma}D(\upsilon_1)\alpha\left[1-2\frac{\upsilon_1}{\upsilon_1+1}+\frac{\upsilon_1(\upsilon_1+2)}{(\upsilon_1+1)(\upsilon_1+3)}\right] \\
&= \frac{\alpha}{\sigma}\left[-\frac{1}{\upsilon_1+1}+\frac{\upsilon_1}{\upsilon_1+3}D(\upsilon_1)\right];
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln f}{\partial \upsilon_1 \partial \sigma}\right] &= \frac{1}{\sigma}E\left\{\left(1-\frac{1}{L(y_t;\theta)}\right)1(y_t<\mu)\right\} \\
&\quad -\frac{\upsilon_1+1}{\sigma}D(\upsilon_1)E\left\{\left(\frac{1}{L(y_t;\theta)}-\frac{1}{[L(y_t;\theta)]^2}\right)1(y_t<\mu)\right\} \\
&= \frac{1}{\sigma}\alpha\left[1-\frac{\upsilon_1}{\upsilon_1+1}\right]-\frac{\upsilon_1+1}{\sigma}D(\upsilon_1)\alpha\frac{\upsilon_1}{\upsilon_1+1}\left[1-\frac{\upsilon_1+2}{\upsilon_1+3}\right] \\
&= \frac{\alpha}{\sigma}\left[\frac{1}{\upsilon_1+1}-\frac{\upsilon_1}{\upsilon_1+3}D(\upsilon_1)\right].
\end{aligned}
$$

(**k**).

$$
\begin{aligned}
& E\left[\frac{\partial \ln f}{\partial \mu}\frac{\partial \ln f}{\partial \sigma}\right] \\
= {} & \frac{(\upsilon_1+1)^2}{2\sigma}E\left\{\left(\frac{1}{L(y_t;\theta)}-\frac{1}{[L(y_t;\theta)]^2}\right)\frac{1}{\upsilon_1}\frac{2(y_t-\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}1(y_t<\mu)\right\} \\
& +\frac{(\upsilon_2+1)^2}{2\sigma}E\left\{\left(\frac{1}{R(y_t;\theta)}-\frac{1}{[R(y_t;\theta)]^2}\right)\frac{1}{\upsilon_2}\frac{2(y_t-\mu)1(y_t>\mu)}{[2(1-\alpha)\sigma K(\upsilon_1)]^2}\right\} \\
= {} & \frac{(\upsilon_1+1)^2}{2\sigma^2}\left[-\frac{2}{\upsilon_1+1}+\frac{2}{\upsilon_1+3}\right]+\frac{(\upsilon_2+1)^2}{2\sigma^2}\left[\frac{2}{\upsilon_2+1}-\frac{2}{\upsilon_2+3}\right] \\
= {} & \frac{2}{\sigma^2}\left[-\frac{\upsilon_1+1}{\upsilon_1+3}+\frac{\upsilon_2+1}{\upsilon_2+3}\right];
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{\partial^2 \ln f}{\partial\mu\partial\sigma}\right] &= -\frac{\upsilon_1+1}{\sigma}E\left\{\frac{1}{[L(y_t;\theta)]^2}\frac{1}{\upsilon_1}\frac{2(y_t-\mu)}{[2\alpha\sigma K(\upsilon_1)]^2}1(y_t<\mu)\right\} \\
&\quad -\frac{\upsilon_2+1}{\sigma}E\left\{\frac{1}{[R(y_t;\theta)]^2}\frac{1}{\upsilon_2}\frac{2(y_t-\mu)}{[2(1-\alpha)\sigma K(\upsilon_1)]^2}1(y_t>\mu)\right\} \\
&= \left(\frac{\upsilon_1+1}{\sigma}\right)\frac{1}{\sigma}\frac{2}{\upsilon_1+3} - \frac{\upsilon_2+1}{\sigma}\frac{1}{\sigma}\frac{2}{\upsilon_2+3} \\
&= \frac{2}{\sigma^2}\left[\frac{\upsilon_1+1}{\upsilon_1+3} - \frac{\upsilon_2+1}{\upsilon_2+3}\right].
\end{aligned}
$$

(**l**). By the symmetry property of υ_1 and υ_2, similarly, we have

$$
E\left[\frac{\partial \ln f}{\partial\upsilon_2}\right]^2 = -E\left[\frac{\partial^2 \ln f}{\partial\upsilon_2^2}\right] = \frac{1-\alpha}{2}\left\{\frac{\upsilon_2 D^2(\upsilon_2)}{\upsilon_2+3} - \frac{2D(\upsilon_2)}{\upsilon_2+1} - D'(\upsilon_2)\right\};
$$

$$
\begin{aligned}
E\left[\frac{\partial \ln f}{\partial\upsilon_2}\frac{\partial \ln f}{\partial\alpha}\right] &= -E\left[\frac{\partial^2 \ln f}{\partial\upsilon_2\partial\alpha}\right] = \frac{1}{\upsilon_2+1} - \frac{\upsilon_2}{\upsilon_2+3}D(\upsilon_2); \\
E\left[\frac{\partial \ln f}{\partial\upsilon_2}\frac{\partial \ln f}{\partial\mu}\right] &= -E\left[\frac{\partial^2 \ln f}{\partial\upsilon_2\partial\mu}\right] = -\frac{1}{\sigma}\left[\frac{1}{\upsilon_2+1} - \frac{\upsilon_2+1}{\upsilon_2+3}D(\upsilon_2)\right]; \\
E\left[\frac{\partial \ln f}{\partial\upsilon_2}\frac{\partial \ln f}{\partial\sigma}\right] &= -E\left[\frac{\partial^2 \ln f}{\partial\upsilon_2\partial\sigma}\right] = \frac{1-\alpha}{\sigma}\left[-\frac{1}{\upsilon_2+1} + \frac{\upsilon_2 D(\upsilon_2)}{\upsilon_2+3}\right]. \blacksquare
\end{aligned}
$$