Robust Collusion in Auctions*

(Preliminary and incomplete)

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Abstract

In a symmetric independent private value auction model where bidders are constrained by their privately known budgets, bidders may collude via fund-pooling consortiums and a consortium may coordinate actions, in an incentive-feasible manner, to penalize a nonparticipant of the consortium. A collusive scheme is for all the bidders to form a single consortium that buys the good at the minimally admissible price and then randomly selects one of its members as the winner. It is proved that this collusive scheme, despite its inefficiency, cannot be preempted by a principal even though the principal can choose any grand mechanism from a class that ranges from standard auctions to those that resemble Che and Kim's (2006) collusion-proof mechanisms.

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1 Introduction

Collusion has been an important issue in the study of auctions. A surprising finding in the existing auction-design literature is that collusion does not at all undermine a principal's objective if the principal is free to choose any mechanism before the bidders consider collusion. This result was first observed by Laffont and Martimort [10] and significantly generalized by Che and Kim [6]. It was further extended by Pavlov [16] to some cases where bidders may collude before they participate in the principal's mechanism. These authors have constructed mechanisms for the principal that are robust against collusion. Two assumptions, however, are critical to their constructions. One is that bidders are not budget-constrained, as Che and Kim have noticed. The other is that the outside option for a collusion participant is the noncollisive equilibrium intended by the principal.

In this paper, we remove these two assumptions in an auction-collusion model that follows the same timing as in Laffont and Martimort and Che and Kim. We shall see that a highly inefficient collusive scheme, where everyone bids minimally and gets to win the good with equal probability regardless of the valuations of the good, survives a class of mechanisms. This class of mechanisms includes the standard auctions as well as mechanisms in the spirit of Che and Kim that have bidders bear all the risks via side bets among bidders. Yet none of these mechanisms can preempt the inefficient collusive scheme.

Let me provide the motivation for the removal of the aforementioned assumptions. The outside-option assumption is restrictive to the kind of collusive schemes that a cartel could support, thereby possibly weakening Che and Kim's notion of collusion-proofness. For example, suppose the principal is selling the good via a second-price auction with zero reserve price to n bidders with independent private values drawn from a distribution F supported by $[0, 1]$. Assume that $1 < n \int_0^1 F(x)^{n-1}dx$. Then the collusive scheme "everyone bids zero and the bidding ring randomly picks the winner with equal probability" is not incentive feasible if we maintain the outside-option assumption. That is because any bidder with value sufficiently close to one prefers the noncollusive efficient equilibrium to the collusive scheme: the former gives him an expected payoff approximately $\int_0^1 F(x)^{n-1}dx$, while the latter only approximately $1/n$. By contrast, if we remove the outside-option assumption, we can pick a posterior belief to penalize any bidder who unilaterally deviates from the collusive scheme. For example, if any bidder i rejects the scheme, the other bidders believe that i's value equals one and i will bid one in the second-price auction; they each best-reply by bidding $1 - \frac{1}{2n}$ in the second-price auction. Expecting this response, bidder i cannot profit from rejecting the collusive scheme. (The restrictiveness of the outside-option assumption has been pointed out by Celik and Peters [5] in the context of oligopoly competition.)

Merely removing this assumption, however, does not suffice to make the above collusive scheme robust to various mechanisms. For example, if the principal replaces the second-price auction with an all-pay auction, then the off-path response described above is not a best response any more. To support the collusive scheme across various mechanisms, we also need to remove the no-budget-constraint assumption. This assumption is the basis for Che and Kim's construction, as their construction is based on the idea of having the bidders shoulder all the uncertainty via side bets against one another's types. If bidders are financially constrained, however, such side bets may be infeasible.

In the real world, bidders' financial constraints provide a reason to legitimize coalitional bidding behaviors. In the context of corporate takeovers, bidding consortiums are legal and frequently observed, where bidders pool their funds and submit joint bids together.[1] There a main justification for bidding consortiums to be legal is that they help bidders to pool funds thereby sharing risks.[2] Introducing budget constraints is a simple way to incorporate such risk aversion and fund pooling motives in our model.

The legitimacy of bidding consortiums in corporate takeovers also provides a real-world motivation to remove the outside-option assumption. Being legal, a bidding consortium not only can write binding contracts as assumed in much of the auction-collusion literature (McAfee and McMillan [13], Laffont and Martimort, Che and Kim, etc.), it may also be able to respond to a unilateral defector collectively, thereby making the defector's outside option worse than the noncooperative equilibrium intended by the principal.

Considering the formation of a bidding consortium in auctions, this paper is related to the research on coalition formations with asymmetric information. A main sticking point in this research is how to handle belief-updating conditional on joining a coalition or rejecting to join a coalition. The mechanism-design literature on collusion has been restricting attention to passive updating conditional on the participation decisions of a coalition. This paper

[1] For examples of bidding consortiums in corporate takeovers, see the reports by Alesci and McGracken [1], Fairless [7], Jackson [8], Jones and Hepher [9], Tutt [17], etc.

[2] See for example Assai [2].

by contrast has removed that restriction. There is active updating based on a bidder's nonparticipation in a bidding consortium, and not all types of bidders necessarily participate in a bidding consortium. Biran and Forges [3] have recently proposed a cooperative solution concept to predict the coalition formation in auctions. The solution concept in our paper is noncooperative (though our modeling bidding rings as incentive feasible mechanisms blurs the distinction between the cooperative and noncooperative approaches), which allows us to handle belief-updating and participation decisions.[3]

Considering a symmetric independent private values auction model where each bidder's type consists of the bidder's valuation and wealth (or budget), this paper is also related to the literature on auctions with budget-constrained bidders. Maskin [12] characterized the socially efficient mechanism in the special case where bidders are constrained by the same commonly known wealth. Pai and Vohra [15] characterized the optimal mechanisms, separately for the society and for the seller, in the case of heterogeneous discrete wealth. In our paper, types are drawn from a continuum, so the literature does not have a characterization of the optimal mechanisms, but the kind of mechanisms designed by Maskin and Pai and Vohra are included in the class of mechanisms that we shall show cannot preempt the aforementioned inefficient collusive scheme.

2 The Model

The timing is the same as in Laffont and Martimort [10, 11] and Che and Kim [6]. First, the principal commits to a mechanism, called *grand mechanism*, to offer the good for sale. The grand mechanism does not discriminate bidding consortiums from individual bidders. (Equivalently, the grand mechanism cannot detect precommitted or coordinated actions of the bidders who belong to the same bidding ring.) Second, the bidders independently decide whether to participate in the grand mechanism. Those who participate may form bidding consortiums. Then the bidding consortiums and individual bidders compete in the grand mechanism. Once the grand mechanism has finished and the precommitted intra-consortium

[3] A crucial feature of their method is that when they analyze the interactions among collusive coalitions, the participation decisions of the members of a coalition are not fully considered. In their model, any subset of a coalition has the option of leaving the coalition with the pessimistic expectation that the other players will reconfigure their coalitions in a way that punishes the departing group most among all possible reconfigurations that need not satisfy any participation constraint.

allocations have been carried out, the game ends.

Our model has only two points of departure from the framework of Laffont and Martimort and Che and Kim. First, each bidder is budget-constrained by its wealth. Second, a bidding consortium can devise a plan-B mechanism in response to the nonparticipation of one of its members, thereby manipulating its members' outside options.

2.1 Bidders' Valuations and Budget Constraints

An indivisible object is being pursued by n bidders, with $n \geq 3$. Let I denote the set of bidders. Each bidder i's valuation v_i of the object is independently drawn from a commonly known distribution F, with support $[0, \overline{v}]$ on which F has no atom and no gap. Each bidder i's wealth w_i is independently drawn from a commonly known distribution G, with support $[\underline{w}, \overline{w}]$ on which G has no atom and no gap unless $\underline{w} = \overline{w}$.

Bidder i's valuation v_i, or *value*, and its wealth w_i are the bidder's private information, or type, at the outset. The wealth is interpreted as the quantity of liquid assets available to the bidder such that the bidder cannot individually make any payment that exceeds w_i.

2.2 Grand Mechanisms

A grand mechanism is modeled as a direct revelation mechanism that maps every profile of types (values and wealth) across bidders to a lottery that determines the owner of the good and each bidder's payment. A bidder i's payment consists of two components, the payment delivered to the seller, denoted by $p_i^s((v_j, w_j)_{j\in I}, k)$ if the type profile is $(v_j, w_j)_{j\in I}$ and the winner is k, and the side payment delivered to the other bidders, denoted by $p_i^b((v_j, w_j)_{j\in I}, k)$. We require budget-balance within the bidders, i.e., $\sum_{i\in I} p_i^b = 0$ always. In standard auctions such as the Vickrey and all-pay auctions, $p_i^b \equiv 0$. In mechanisms involving side bets such as the mechanism of Che and Kim, p_i^b is nonzero.

2.3 Bidding Consortiums

Before the grand mechanism starts, the bidders, already privately informed of their types, have the option to form *bidding consortium* such that each consortium submits a joint bid with its total budget pooled from its members. A bidding consortium is a set C of bidders such that every member $i \in C$ delivers a transfer τ_i to other members of C, with $\sum_{i\in C} \tau_i = 0$, and is assigned a share $\alpha_i \in [0,1]$ such that $\sum_{i\in C} \alpha_i = 1$ and the bidders in C decide by unanimity on a joint bid or joint message to submit to the grand mechanism; the share $(\alpha_i)_{i\in C}$, the transfers, and the joint bid may be contingent on the profile of messages the members of C send to the consortium. If a consortium wins the good at a price p, each member $i \in C$ pays the fraction $\alpha_i p$ and exactly one of the members is randomly selected to be the winner of the good, with the winning probability for i being α_i.

Formally, a *consortium plan* consists of a set C of bidders and a mapping M from each profile $(v_i)_{i\in C}$ of values across the members of C to a configuration $((\alpha_i, \tau_i)_{i\in C}, b_C)$ of share vector, internal transfers, and joint bid. Denote $\mathbb{M}(C)$ for the set of the consortium plans for any set C of bidders. Denote a consortium plan by (C, M).

2.4 Modeling the Formation of Bidding Consortiums

We shall model the formation of bidding consortiums as a partition of the set of bidders such that no individual bidder can have a profitable deviation from the partition. The subtlety of this model is to take into account the other bidders' reactions when a bidder ponders about deviation. That differs from the collusion literature, such as McAfee and McMillan [13] and Che and Kim [6], which assume that any unilateral deviation from the partition implies that all consortiums are dissolved.

Formally, a configuration of bidding consortiums corresponds to a partition $\mathbb{P}$ on the set I of all bidders such that any partition cell $C \in \mathbb{P}$ is a consortium unless C is singleton. At equilibrium, each cell $C \in \mathbb{P}$ commits to a consortium plan that is incentive feasible for every possible type of each member of C. This incentive feasibility condition is with respect to an endogenous reaction function $\mathscr{R}$ which responds to unilateral deviation from $\mathbb{P}$.

For any bidder $i \in C \in \mathbb{P}$, a possible deviation d_i from $\mathbb{P}$ is either i, meaning i bids alone instead of joining the consortium C, or $d_i = (C', M_{C'\cup i})$ for some $C' \in \mathbb{P} \setminus \{C\}$ and some consortium plan $M_{C'\cup i} \in \mathbb{M}(C' \cup i)$, meaning that i proposes to another consortium C' to include i with consortium plan $M_{C'\cup i}$.

If bidder $i \in C \in \mathbb{P}$ takes a deviant action d_i, the reaction function $\mathscr{R}(d_i)$ specifies a *plan-B mechanism* $\mathscr{R}(C, d_i) \in \mathbb{M}(C \setminus i)$ for the other members of C and, if $d_i = (C', M_{C'\cup i})$ for some $C' \in \mathbb{P} \setminus \{C\}$, a response $\mathscr{R}(C', d_i)$ from the members of C' to i's proposal.

2.5 The Notion of Equilibrium

An equilibrium corresponds to a list $(\mathbb{P}, (M_C)_{C\in\mathbb{P}}, \mathscr{R})$ such that

a. for each $C \in \mathbb{P}$, the consortium plan M_C is incentive feasible in the sense that

   i. conditional on full participation of C, M_C is incentive compatible for every possible type of every member of C, and

   ii. for any member $i \in C$ and for any possible deviation d_i of i, no possible type of bidder i can profit from the deviation d_i given the reaction $\mathscr{R}(\cdot, d_i)$;

b. the reaction function $\mathscr{R}$ is incentive feasible in the sense that for any $C \in \mathbb{P}$ and any $i \in C$,

   i. the plan-B mechanism $\mathscr{R}(C, d_i)$ for the other members of C is incentive feasible (in the sense of condition a) based on the posterior belief given i's deviation, and

   ii. if $d_i = (C', M_{C'\cup i})$ for some $C' \in \mathbb{P} \setminus \{C\}$, the response $\mathscr{R}(C', d_i)$ instructs each member of C' to accept the deviant proposal if and only if the proposal makes each member strictly better-off than in $\mathbb{P}$.

3 The All-Inclusive Collusive Scheme

We shall show that, given certain parameter values, the following collusive scheme can be supported as an equilibrium of our auction-collusion game: Exactly one bidding consortium is formed and it includes all the bidders. In joining the consortium, each bidder has equal share $1/n$ and pays zero transfer to other bidders. The consortium wins in the grand mechanism at the lowest admissible price and then randomly selects one of its members as the winner of the good. Needless to say, this allocation is devastating to the principal's objective, be it social efficiency or profit maximization.

Supporting this collusive scheme is not easy even when the grand mechanism is the typically collusion-susceptible Vickrey auction. That is because a bidder is budget-constrained, so he need not have enough wealth to submit a sufficiently high bid to penalize the defector in a manner described in the Introduction. (The insight that budget constraints may undermine bidders' ability to collude was observed by Brusco and Lopomo [4].) To penalize the defector, the other members of the bidding ring need to come up with a plan-B mechanism to pool their funds together and share the price of the good in case of outbidding the defector. But then the bidders with low valuations of the good need not want to do that. To give them the ample incentive, the plan-B mechanism needs to devise a configuration of transfers so that the high-value bidders pay the low-bidders to participate in the plan-B mechanism.

At equilibrium, in the off-path event that exactly one bidder, say bidder i, rejects the collusive scheme, the other bidders expect that their expected payoffs will be nonpositive if they bid individually against the defector i. Alternatively, they can participate in the plan-B mechanism. In that mechanism, bidders self-select to one of two groups, a high-value group and a low-value group, according to an endogenous cutoff value v_*. Each high-value member has a higher share $\overline{\alpha}$ than each low-value member. To motivate the low-value members to participate, each high-value member contributes a transfer $\overline{\tau}$ equally distributed among the low-value members. The shares and transfers are contingent to the number of high-value bidders. More precisely, from the viewpoint of any bidder j other than the defector i, the random variables that concern j in his decisions regarding the plan-B mechanism are the numbers of the other bidders whose types are above a threshold say θ, i.e.,

$$N(v_{-i,-j}, \theta) := |\{k \in I \setminus \{i, j\} : v_k \geq \theta\}| \tag{1}$$

with $v_{-i,-j}$ denoting the random vector $(v_k)_{k \in I \setminus \{i,j\}} \in [0, \overline{v}]^{n-2}$. Together with the endogenous cutoff v_*, the contingency shares and transfers are chosen to ensure that the plan-B mechanism satisfies incentive compatibility and participation constraint, as well as each member's budget constraint.

4 The Case of the Vickrey Auction

To illustrate the idea, assume within this section that the support of wealth is degenerated to a commonly known point, w, i.e.,

$$\underline{w} = \overline{w} =: w. \tag{2}$$

Proposition 1 handles the case where the grand mechanism is the Vickrey (second-price) auction.

Proposition 1 *If the grand mechanism is the Vickrey auction and if (2) and*

$$F(w)^{n-2}\left(F(w)+(1-F(w))(n-1)\right)\leq\frac{1}{n} \tag{3}$$

are satisfied, then there exists an equilibrium where all the bidders for sure join a bidding consortium and have equal probability of being the winner.

Proof As outlined in §3, the collusive scheme is for the grand coalition to become a bidding consortium with share vector $(1/n,\ldots,1/n)$ and zero entrance fee. We shall show that there is no profitable unilateral deviation form participating in this bidding consortium.

Suppose that a bidder i deviates by rejecting the above proposal. Seeing bidder i's deviation, the other bidders update that $v_i > w$ and expect that bidder i will bid independently in the Vickrey auction, so i's bid will be w. Then the following plan-B consortium is formed with a large enough probability to render bidder i's deviation unprofitable:

a. for each $j \neq i$, bidder j participates in the plan-B consortium if and only if $v_j \geq w$;

b. if m is the number of participants in the plan-B consortium,

   i. if $m = 1$, the consortium is dissolved (so every bidder has to bid independently),

   ii. if $m \geq 2$ then the consortium submits its total wealth as its joint bid and each participant has the share $1/m$.

If this plan-B consortium is formed (the case $m \geq 2$), then it outbids all individual bidders (as each individual bidder can bid at most w) and pays the price w (since the defector bidder i is expected to bid w). Thus, each participant of the plan-B consortium gets an expected payoff $(v_j - w)/m$. It follows that any bidder $j \neq i$ with value $v_j < w$ does not participate in the plan-B consortium, since the consortium, if formed, gives j a negative payoff, while bidder j get guarantee a zero payoff by bidding individually.

For any bidder $j \neq i$ with value $v_j \geq w$, at any possible value-profile $v_{-i,-j}$ across the bidders other than i and j, bidding individually yields a payoff $(v_j - w)/(N(v_{-i,-j}, w) + 2)$, as there are $N(v_{-i,-j}, w)$ many bidders, in addition to bidders i and j, who will bid w and tie in the Vickrey auction. By contrast, participating in the plan-B consortium in the event that it is formed (if it is not formed then j's decision makes no difference to himself) yields the payoff $(v_j - w)/(N(v_{-i,-j}, w) + 1)$, as bidder i is outbid and bidder j shares the winning probability with only the other $N(v_{-i,-j}, w)$ participants of the plan-B consortium. Thus, bidder j with $v_j \geq w$ prefers to participate in the plan-B consortium.

Since the plan-B consortium is formed if $m \geq 2$, i.e., if there are at least two bidders $j \neq i$ with $v_j \geq w$, the probability with which bidder i's deviation yields a nonzero payoff is equal to the left-hand side of (3). Then Ineq. (3) implies that bidder i's expected payoff from deviation from the grand collusive scheme is less than his expected payoff v_i/n from abiding by the scheme. ■

5 The General Case

We now consider the general environment where

$$\underline{w} < \overline{w}. \tag{4}$$

The extension to (4) is necessary because in the case where $\underline{w} = \overline{w}$, the principal could easily preempt the collusive scheme constructed in the previous subsection by setting a bid ceiling equal to the commonly known constant budget constraint w. With wealth heterogenous across bidders, the principal needs to allow bids at least up to $\overline{w}$.

We shall establish the claim that the above collusive scheme is robust within a broad class of mechanisms satisfying the following assumptions.

Assumption 1 (constrained efficiency) *There exists a noncooperative equilibrium (without bidding consortium) of the grand mechanism; the equilibrium induces an allocation that is efficient unless either the good is not sold or the winner's budget constraint is binding; the seller's expected revenue at this equilibrium is maximized subject to this allocation.*

Assumption 2 (budget constraint) *If i denotes a bidder or bidding consortium and w_i the total wealth of i, then the payment delivered by i to the grand mechanism is always less than or equal to w_i.*

Assumption 3 (side bets) *For every bidder i, the expected value of bidder i's side payment p_i^b in the grand mechanism, based on the prior belief conditional on bidder i's type, is equal to zero.*

Assumption 4 (competitiveness) *For any two bidders or consortiums, i and j, if their actions in the grand mechanism amount to announcing their types being $(\hat{v}_i, \hat{w}_i)$ and $(\hat{v}_j, \hat{w}_j)$ such that $\min\{\hat{v}_i, \hat{w}_i\} > \min\{\hat{v}_j, \hat{w}_j\}$, then i outbids j.*

Assumption 5 (reserve price) *The reserve price of the grand mechanism is announced to be some $r \in [0, \overline{w}]$ when the mechanism is announced.*

These assumptions are satisfied by standard auctions as well as the optimal collusion-proof mechanism in Che and Kim [6, Theorem 1].

The on-path collusive scheme in the general case is the same as in the Vickrey auction. But the plan-B mechanism in the off-path event of having a defector is more complicated.

The first lemma extends the envelope-theorem characterization of incentive compatibility to allow for a bidder's budget as the second dimension of the bidder's type.

Lemma 1 *For any grand mechanism, at the associated noncooperative equilibrium, for any bidder i with any type (v_i, w_i), if $q_i(v_i, w_i)$ denotes the bidder's expected probability of winning, $\overline{p}_i(v_i, w_i)$ denote the expected payment delivered by this bidder, and $U_i(v_i, w_i)$ this bidder's equilibrium surplus, then, for any $w_i \in [\underline{w}, \overline{w}]$, $U_i(\cdot, w_i)$ is absolutely continuous and, for any $v_i \in [0, \overline{v}]$,*

$$U_i(v_i, w_i) = U_i(0, w_i) + \int_0^{v_i} q_i(z, w_i) dz, \tag{5}$$

and $\overline{p}_i(v_i, w_i)$ is weakly increasing in v_i.

Proof For any i and any $w_i \in [\underline{w}, \overline{w}]$, let $\Gamma_i(w_i)$ denote the set of all payment functions that are feasible with respect to budget w_i. For each $p_i \in \Gamma_i(w_i)$, let $\overline{p}_i(v_i, w_i)$ denote the expected payment delivered by bidder i of type (v_i, w_i) at the associated equilibrium. Incentive compatibility implies that, for any w_i,

$$\begin{aligned} U_i(v_i, w_i) \;\; = \;\; & \max\nolimits_{\hat{v}_i \in [0, \overline{v}]} \;\; v_i q_i(\hat{v}_i, w_i) - \overline{p}_i(\hat{v}_i, w_i) \\ & \text{subject to} \;\; p_i(\hat{v}_i, w_i, \cdot) \in \Gamma_i(w_i). \end{aligned}$$

In this optimization problem, since the report about i's budget is fixed at the true budget w_i, the budget constraint $p_i(\hat{v}_i, w_i, \cdot) \in \Gamma_i(w_i)$ is automatically satisfied when bidder i switches his valuation report from v_i to v_i' thereby switching his payment function from $p_i(v_i, w_i, \cdot)$ to $p_i(v_i', w_i, \cdot)$. Thus,

$$U_i(v_i, w_i) = \max_{\hat{v}_i \in [0, \overline{v}]} v_i q_i(\hat{v}_i, w_i) - \overline{p}_i(\hat{v}_i, w_i).$$

Thus, by the standard revealed-preference argument, $q_i(v_i, w_i)$ is weakly increasing in v_i, and Eq. (5) follows from the proof of Milgrom and Segal [14]. Then

$$\overline{p}_i(v_i, w_i) = v_i q_i(v_i, w_i) - \int_0^{v_i} q_i(z, w_i) dz - U_i(0, w_i)$$

and the monotonicity of $\overline{p}_i(\cdot, w_i)$ follows from the monotonicity of $q_i(\cdot, w_i)$. ■

The next lemma says that for any possible budget level a bidder's budget constraint at the noncooperative equilibrium of the grand mechanism binds if the bidder's valuation is sufficiently high. Thus, the side bets constructed by Che and Kim [6] need not be feasible in our model with budget constraints.

Lemma 2 *Suppose that*

$$\int_0^{\overline{v}} \nu F(\nu)^{n-1} d\nu > \overline{w}. \tag{6}$$

For any grand mechanism that satisfies Assumptions 1–3, for any $w \in (\underline{w}, \overline{w})$ *there exists* $\nu(w) \in [\overline{w}, \overline{v}]$ *such that if a bidder or consortium's alleged type is* (v, w) *with* $v \geq \nu(w)$ *and if it wins, then its payment to the losers is equal to zero.*

Proof It suffices to show, at the noncooperative equilibrium of the grand mechanism, for any bidder i, with $\overline{p}_i$ denoting the expected value of his payment in the mechanism,

$$\forall w_i \in [\underline{w}, \overline{w}] \, \exists \nu(w_i) \in [\overline{w}, \overline{v}] \, \forall v_i \in [\nu(w_i), \overline{v}] : \overline{p}_i(v_i, w_i) = w_i. \tag{7}$$

If (7) is true, then any bidder-type (v_i, w_i) with $v_i \geq \nu(w_i)$ cannot make any payment above the expected payment $\overline{p}_i(v_i, w_i)$ (Assumption 2). Recall that this expected payment is equal to the expected payment to the seller plus the expected side payment to other bidders, and the expected side payments is equal to zero (Assumption 3). It follows that the side payment delivered by this bidder-type is equal to zero, desired by this lemma.

To prove (7), suppose it does not hold. Then for any $w_i \in [\underline{w}, \overline{w}]$ there is an infinite sequence $(v_i^m)_{m=1}^{\infty} \to \overline{v}$ such that the expected payment $\overline{p}_i(v_i^m, w_i) < w_i$ for all m. With $\overline{p}_i(\cdot, w_i)$ weakly increasing (Lemma 1), we have $\overline{p}_i(v_i, w_i) < w_i$ for all $v_i \in [\overline{w}, \overline{v}]$. Therefore, it follows from the efficiency of this noncooperative equilibrium (Assumption 1) that any such a bidder-type (v_i, w_i) wins if and only if v_i is greater than the value of all other rivals. It follows from Eq. (5), as well as the fact $U_i(0, w_i) = 0$ due to Assumption 1, that the bidder's expected payment to the seller is equal to $\int_0^{v_i} zF(z)^{n-1} dz < w_i$. Taking the limit of (v_i, w_i) to the highest type $(\overline{v}, \overline{w})$, we get $\int_0^{\overline{v}} zF(z)^{n-1} dz \leq \overline{w}$, contradicting Ineq. (6). ■

In the event of a unilateral defection from the collusive scheme, the plan-B mechanism is a little bit more complicated here than in the case of the Vickrey auction. If it were the Vickrey auction, the high-value bidders join the plan-B consortium because otherwise they will each tie with the defector so the consortium helps them to increase the probability of winning. But such an incentive depends on the uniform tie-breaking rule in the Vickrey auction. In a general mechanism, the tie-breaking rule may be different. So we shall construct the plan-B consortium with a slightly different idea. Here, in the event that a bidder defects from the collusive scheme, the other bidders infer from the defection that the defector's value is atomlessly distributed on a small interval close to $\overline{w}$. Based on this posterior, bidders with either values or budgets below that interval will for sure lose to the defector unless they form a consortium. Some of these bidders would contribute more and others less to the consortium, depending on their valuations of the good. The plan-B mechanism will be constructed to provide the ample incentive for them to self-select in the consortium.

For any $x \in \mathbb{R}$ and any $m \in \{1, \ldots, n\}$, define

$$\begin{aligned} \lceil x \rceil &:= \text{the smallest integer greater than or equal to } x \\ F^{(m)} &:= \text{the cdf of the } m\text{th highest statistic of } n \text{ random variables,} \\ &\quad\ \text{each independently drawn from the cdf } F. \end{aligned}$$

Proposition 2 *For any grand mechanism that satisfies Assumptions 1–5, if*

$$F^{(\lceil \overline{w}/\underline{w} \rceil)}(\overline{w}) < 1 \qquad (8)$$

and Ineq. (6) are satisfied, then there exists an equilibrium such that, whenever $v_k \geq r$ for all bidders k, all the bidders join a bidding consortium and have equal winning probability.

Proof At the proposed equilibrium, a bidder $k \in I$ participates the grand mechanism if and only if $v_k \geq r$ and $w_k \geq r/n$. In the event of full participation, the n bidders form the collusive consortium, which we will construct soon. In the event that the number n' of participants in the grand mechanism is less than n, these n' participants form the collusive consortium if and only if Ineq. (8) holds when n' replaces n. If (8) does not hold when n' replaces n then the consortium is dissolved and every participant bids individually. Note that a bidder k with $v_k < r$ or $w_k < r/n$ cannot profit from participating in the grand mechanism, since each participant's payment conditional on winning is at least r/n and each bidder k's expected payoff, whether the collusive consortium is formed or not, is a fraction of $v_k - p$, with p being a payment greater than or equal to r.

The rest of the proof will demonstrate the collusive scheme in the strictly positive probability event of full participation in the grand mechanism. The other cases where Ineq. (8) holds when n' replaces n are similar. In the full participation case, all the n bidders form a single bidding consortium with the share vector $(1/n, \ldots, 1/n)$ and zero entrance fee, the consortium submits a joint message such that it wins the good at the price r in the grand mechanism (Assumption 5), and every bidder gets to be the winner with probability $1/n$. Conditional on full participation in the collusive scheme, this mechanism satisfies the incentive compatibility condition trivially. The rest of the proof shows that it also satisfies the participation condition, i.e., no bidder can profit from unilaterally not participating in the consortium.

By the strict inequality (8) and the assumption that distribution F is atomless, there exist $\epsilon > 0$ and $\eta > 0$ such that

$$F^{(\lceil \overline{w}/\underline{w} \rceil)}(\overline{w}) + 1 - (1 - F(\eta))^{n-1} + \epsilon \leq 1. \tag{9}$$

By the assumption that distribution G is atomless, there exists a $w \in (\underline{w}, \overline{w})$ such that

$$(n-1)(1 - F(w))(1 - G(w)) < \epsilon. \tag{10}$$

Suppose exactly one bidder, say bidder i, rejects to participate in the grand coalition consortium. Seeing bidder i's nonparticipation, the other bidders' updated belief becomes

$$w_i \sim G\left(\cdot \mid [w, \overline{w}]\right), \quad v_i \sim F\left(\cdot \mid [\nu(w_i), \overline{v}]\right). \tag{11}$$

I.e., they believe that $w \leq w_i \leq \overline{w}$ and $\nu(w_i) \leq v_i$, with $\nu(\cdot)$ specified in Lemma 2. Furthermore, they expect that, if no bidding consortium is formed, bidder i will report in the grand mechanism that i's type is (v_i, w_i), hence by Lemma 2, in case that bidder i wins, i's side payment to other bidders is equal to zero.

Let us describe the plan-B mechanism for $C := I \setminus \{i\}$ conditional on i's nonparticipation. If C forms the plan-B consortium, its joint action will be to announce the type for the consortium C as $(\nu(\overline{w}), \overline{w})$ so that C pays zero to any bidder outside C in the event that C wins (Lemma 2). With $\min\{\nu(\overline{w}), \overline{w}\} = \overline{w}$, Assumption 4 implies that such a joint action will ensure that C defeats bidder i for sure with respect to the posterior belief, Eq. (11). We shall construct such a plan-B mechanism that is budget- and incentive-feasible for a a set of bidder-types whose probability is large enough to deter bidder i's deviation.

Pick any small $\delta \in (0,1)$.

If not all members of $C := I \backslash \{i\}$ participate in the plan-B mechanism, then no bidding consortium is dissolved and the bidders play the noncooperative equilibrium in the grand mechanism. Otherwise, conditional on full participation in C, each bidder $j \neq i$ announces his type as either "high" or "low." Let N_H denote the number of allegedly high-type bidders and N_L the allegedly low-type bidders. If $N_H N_L = 0$ then the consortium is dissolved and everyone bids individually. Otherwise, $N_H N_L > 0$, then the share for each allegedly low-type participant is defined to be

$$\underline{\alpha} := \frac{\delta}{N_L}$$

and the share for each allegedly high-type participant is defined to be

$$\overline{\alpha} := \frac{1-\delta}{N_H}.$$

Upon entry, each allegedly low-type participant receives a transfer

$$\underline{\tau} := -\underline{\alpha}\left(\overline{w} - \eta\right) = -\frac{\delta}{N_L}\left(\overline{w} - \eta\right)$$

and each allegedly high-type participant pays a transfer

$$\overline{\tau} := \frac{\underline{\tau} N_L}{N_H} = \frac{\underline{\alpha}\left(\overline{w} - \eta\right) N_L}{N_H} = \frac{\frac{\delta}{N_L}\left(\overline{w} - \eta\right) N_L}{N_H} = \frac{\delta}{N_H}\left(\overline{w} - \eta\right).$$

Note that the sum of these transfers across all participants is equal to zero.

In the plan-B mechanism, the budget constraint for each allegedly low-type bidder j is satisfied if η is small enough that $w_j \geq \underline{\alpha}\eta$, because

$$w_j \geq \underline{\alpha}\eta = -\underline{\alpha}\left(\overline{w} - \eta\right) + \underline{\alpha}\overline{w} = \underline{\tau} + \underline{\alpha}\overline{w} \geq \underline{\tau} + \underline{\alpha} p(\nu(\overline{w}), \overline{w});$$

for each allegedly high-type participant j, the constraint is satisfied if

$$w_j \geq \overline{\tau} + \overline{\alpha} p\left(\nu(w), \overline{w}\right) = \frac{\delta}{N_H}\left(\overline{w} - \eta\right) + \frac{1-\delta}{N_H} p\left(\nu(w), \overline{w}\right) = \frac{1}{N_H}\left(\delta\left(\overline{w} - \eta\right) + (1-\delta) p\left(\nu(w), \overline{w}\right)\right). \tag{12}$$

If (12) does not hold for some $j \neq i$ then the plan-B consortium is dissolved and everyone bids individually in the grand mechanism. Since $p\left(\nu(w), \overline{w}\right) \leq \overline{w}$ (Assumption 2) and $w_j \geq \underline{w} \geq \max\{\underline{w}, r/n\}$, the above inequality is satisfied if

$$N_H \geq \frac{\overline{w}}{\max\{\underline{w}, r/n\}}.$$

Thus, we shall consider the event where

$$N_H \geq \left\lceil \frac{\overline{w}}{\underline{w}} \right\rceil,$$

so that every bidder $j \neq i$ reports its type to be high or low independently of its wealth w_j. (We shall show that the probability of this event is sufficiently large to deter bidder i's defection.)

Now let us calculate a bidder j's decision regarding the plan-B mechanism. We conjecture that at equilibrium every bidder j satisfying

$$\min\{v_j, w_j\} \leq w \tag{13}$$

and $v_j \geq \eta$ participate in the plan-B consortium and, in the event that all $j \neq i$ participate in it, each bidder j report to be high type if and only if $v_j \geq v_*$ for some cutoff v_* to be determined in the following calculations.

If bidder j does not participate in the plan-B mechanism, then the plan-B mechanism does not have full participation of its members and hence, as specified above, the bidding consortium is dissolved and the bidders will play the noncooperative equilibrium in the grand mechanism (which satisfies the incentive feasibility condition of the plan-B mechanism of the plan-B mechanism). With $\min\{v_i, w_i\} > w \geq \min\{v_k, w_k\}$ for all $k \in C$, bidder i wins for sure in the grand mechanism (Assumption 4) and delivers zero side payment to bidders $k \in C$ (Lemma 2). With each $k \in C$ having zero probability to win at this continuation equilibrium, Eq. (5) implies that each bidder k's expected payoff is equal to his surplus when his type is $(0, w_k)$. This surplus is no more than zero, because if it is equal to some $s > 0$ then the seller could do better, without altering the noncash allocation, by charging bidder k an additional fee $\min\{\underline{w}, s\}$, which is affordable since $\underline{w} \leq w_k$ and positive because $\underline{w} > 0$ by (**??**). But that contradicts the last statement of Assumption 1. Therefore, if bidder $j \in C$ does not participate in the plan-B mechanism, then j's expected payoff is nonpositive.

If bidder j participates and reports to the plan-B mechanism that his type is high, then his expected payoff is equal to

$$\begin{aligned}
\mathbb{E}\left[\overline{\alpha}\left(v_j - p(\nu(w), \overline{w})\right) - \overline{\tau}\right] &= \mathbb{E}\left[\frac{1-\delta}{N_H}\left(v_j - p(\nu(w), \overline{w})\right) - \frac{\delta}{N_H}\left(\overline{w} - \eta\right)\right] \\
&= \mathbb{E}\left[\frac{(1-\delta)\left(v_j - p(\nu(w), \overline{w})\right) - \delta\left(\overline{w} - \eta\right)}{N\left(\mathbf{v}_{-i,-j}, v_*\right) + 1}\right] \\
&=: A(v_j, v_*).
\end{aligned}$$

If bidder j participates and reports to the plan-B mechanism that his type is low, then his expected payoff is equal to

$$\begin{aligned}
\mathbb{E}\left[\underline{\alpha}\left(v_j - p(\nu(w), \overline{w})\right) + \underline{\tau}\right] &= \mathbb{E}\left[\frac{\delta}{N_L}\left(v_j - p(\nu(w), \overline{w})\right) + \frac{\delta}{N_L}\left(\overline{w} - \eta\right)\right] \\
&= \mathbb{E}\left[\frac{\delta\left(v_j - p(\nu(w), \overline{w}) + \overline{w} - \eta\right)}{n - N\left(\mathbf{v}_{-i,-j}, v_*\right)}\right] \\
&=: B(v_j, v_*).
\end{aligned}$$

Note that $B(v_j, v_*) \geq 0$ if $v_j \geq \eta$, as conjectured.

Thus,

$$A(v_j, v_*) - B(v_j, v_*) = \mathbb{E}\left[\frac{(1-\delta)\left(v_j - p(\nu(w), \overline{w})\right) - \delta\left(\overline{w} - \eta\right)}{N\left(\mathbf{v}_{-i,-j}, v_*\right) + 1} - \frac{\delta\left(v_j - p(\nu(w), \overline{w}) + \overline{w} - \eta\right)}{n - N\left(\mathbf{v}_{-i,-j}, v_*\right)}\right].$$

For any δ sufficiently small, $\frac{d}{dv_j}(A(v_j, v_*) - B(v_j, v_*)) > 0$, so if the solution v_* for

$$A(v_*, v_*) = B(v_*, v_*) \tag{14}$$

exists, then bidder j prefers to report his type to be high if $v_j > v_*$ and prefers to report low if $v_j < v_*$, as conjectured for the equilibrium. Such a solution v_* exists because the function $v_* \mapsto A(v_*, v_*) - B(v_*, v_*)$ is continuous and so the intermediate-value theorem applies. Taking limits when $\delta \to 0$ on both sides of (14), we see that

$$v_* \to p\left(\nu(w), \overline{w}\right) \leq \overline{w}$$

as $\delta \to 0$. Thus, we can choose δ sufficiently small so that $v_* \leq \overline{w}$.

In sum, the plan-B consortium is formed if the following conditions are all met:

i. there are at least $\left\lceil \frac{\overline{w}}{\underline{w}} \right\rceil$ bidders with values above $\overline{w}$ (hence, with $v_* \leq \overline{w}$, $N_H \geq \left\lceil \frac{\overline{w}}{\underline{w}} \right\rceil$ and so the budget constraint (12) is satisfied);

ii. every $v_j \geq \eta$ (so that $B(v_j, v_*) \geq 0$); and

iii. Ineq. (13) is satisfied for every $j \neq i$.

If the plan-B consortium is formed, then the deviant bidder i will lose for sure, as pointed out earlier.

Thus, the winning probability of the deviant bidder i is less than

$$\text{Prob}\left\{N_H < \left\lceil \frac{\overline{w}}{\underline{w}} \right\rceil\right\} + 1 - (1 - F(\eta))^{n-1} + (n-1)(1 - F(w))(1 - G(w)),$$

which is less than $1/n$ by (9)–(10). Thus, it follows from Assumption 5 that the bidder's expected payoff from not participating in the bidding consortium is no bigger than $v_i - r$ times the above probability (recall from Lemma 2 that bidder i gets zero payment from the only other rival, C). Thus, bidder i's expected payoff from deviation is less than its expected payoff $(v_i - r)/n$ from participation. ■

6 Discussion

So far we have considered equilibria where only one bidding consortium arises. Can we have multiple bidding consortiums and, if we do, how do these coalitions interact? If there is a tractable prediction of such a coalitional game, what is a socially optimal mechanism that allows bidding consortiums?

In addition to the sharing-rule type of consortium structures studied above, other governance structures may also be relevant. For example, the beneficiary of a consortium might be able to share his value of the good with the other consortium members, e.g., the good for sale may be interpreted as the right to run an enterprise and the beneficiary could share its profits with the other consortium members.

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