Stability of Price Leadership Cartel with Endogenous Pricing

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Abstract

This paper studies farsighted behavior of firms in an oligopolistic market to form a dominant cartel, which has a power to set and control the price in the market. The von Neumann-Morgenstern stable set is adopted as the solution concept. In contrast to the literature, we do not assume *a priori* the optimal pricing behavior of the cartel; rather, we show that such behavior arises from the result of firms' consideration on the stability. *JEL classification*: C71, D43, L13. *Keywords*: price leadership model, cartel stability, foresight, stable set, endogenous pricing

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1 Introduction

The study of cartel stability is one of the traditional topics in oligopoly theory. In particular, collusive pricing behavior, whether it is a result of overt agreement or not, has been viewed as "the only feasible means of assuring parallel actions among sellers" (Markham [11, p. 901]) and, then, the price leadership cartel has received great concerns for decades. Although there is an extensive literature on the price leadership models with one leader and one follower,[1] the studies of the price leadership cartel with many firms are not still sufficient. In this paper, we study stability of price leadership cartel in an oligopolistic market with many (but, finite) symmetric firms and show that relatively large size of the cartel can prevail in the market when each firm is assumed to have an ability to look ahead and the price leadership cartel can adopt flexible pricing policies to influence the decision by the non-cartel firms.

One of the earliest contributions concerning the stability of the price leadership cartel is d'Aspremont, Jacquemin, Gabszewicz, and Weymark [1], which had been a starting point of subsequent studies by other authors. In their model, it is assumed that there is one and only one cartel in the role of the price leader who announces and sets the price (the size of the cartel in terms of the number of firms in it varies endogenously through entry-exit by firms) and that, taking the price set by the leader as given, the other fringe firms behave in a competitive fashion, that is, they follow the price-equal-marginal-cost principle.[2] Knowing the responses of the fringe firms, the cartel can derive the residual demand function by subtracting the total supply by the fringe firms from the total demand. Taking account of the derived residual demand, the cartel members determine the price to maximize the (joint) profit. In d'Aspremont et al. [1], a certain size of the cartel is considered to be "stable" if (i) no firm in the existing cartel find it

[1]There are several studies exploring the reason that there is a firm in the position of the price leader. Deneckere and Kovenock [2] and Furth and Kovenock [6] have considered a model with the firms' capacity constraints. Pastine and Pastine [14] have used the endogenous timing model of Hamilton and Slutsky [8] to examine endogenous role assignment of leader and follower. van Damme and Hurkens [17] have also used the endogenous timing model with firms' risk consideration by Harsanyi and Selten [9].

[2]Ono [13] regarded this kind of fringe firm's behavior as optimal policy. He argued that because, given the price set by the leader, a fringe firm can set a price infinitesimally lower than the one set by the leader and sell q_f that satisfies the price-equal-marginal-cost condition, then the fringe firm can maximize its profit. However, there is some difficulty in justifying this kind of behavior of fringe firms in a rigorous non-cooperative game model with finite players, because there must be an interaction among fringe firms. Tasnadi [16] shows that such behavior can be justified in a non-atomic model of the fringe firms.

profitable to exit from the cartel and (ii) no fringe firm can be better off by entering the existing cartel. Then, d'Aspremont et al. [1] have shown that there exists a stable size of the cartel.

Although the model in d'Aspremont et al. [1] is simple and their results are clear, there are three inadequacies in their analysis: (a) firm's foresight, (b) cartel identification, and (c) pricing behavior of the cartel. The studies subsequent to d'Aspremont et al. [1] have tried to modify these inadequacies; and our aim in the current paper is also to provide a model that overcomes them.

The first inadequacy concerning firm's foresight has been pointed out by Diamantoudi [3]. She argued that the analysis by d'Aspremont et al. [1] exhibited some inconsistency between an implicit assumption of the firms' brightness embedded in the model and the stability criterion that assumes the firms' myopic view. Consider a firm in the cartel consisting of k firms. When the firm contemplates the deviation (exiting from the cartel), it compares the current profit (the profit of a firm in the size k cartel) with the profit under a new price set by a new cartel established after its deviation (the profit of a firm in the fringe with size $k-1$ cartel). Since the cartel's pricing behavior is restricted to the optimal pricing at the very outset of the model, the deviating firm should expect correctly the response of *readjusting* price by the new cartel against its deviation. In this sense, a firm in their model should have an ability to foresee the reaction of the other firms (in particular, those remaining in the cartel) against its deviation. To the contrary, the stability criterion adopted by d'Aspremont et al. [1] implies that a firm contemplating deviation does not take account of possible subsequent deviations by other firms after its own deviation. That is, the stability criterion assumes firm's myopic view, conflicting the foresight of firms assumed in the model. In view of this inconsistency, Diamantoudi [3] has reconsidered the stability of the price leadership of d'Aspremont et al. [1] adopting a different stability criterion that incorporates firms' farsighted perspective. Then, she has shown that there exists a set of stable sizes of the cartel.

Both d'Aspremont et al. [1] and Diamantoudi [3] share the second inadequacy concerning cartel identification; in their models, cartels are identified by their sizes (in terms of the number of firms) and two distinct cartels with different members are regarded as the same if their sizes are equal. This does not much matter in the case of d'Aspremont et al. [1] because of firms' myopia embedded in their stability criterion. This, however, can become a more serious problem when we fully take account of the farsightedness on the side of firms as in Diamantoudi [3]. Suppose that each firm can foresee a chain reaction of further deviations by other firms after its own deviation. Then, it may be the case that one firm in the cartel finds it profitable to exit from the existing cartel and actually do so, expecting that another fringe firm would enter the cartel after its deviation and that the resulting cartel would be stable. In this argument, there are two cartels of the same size involved: the initial cartel and the resulting cartel. The former is *not* considered to be stable, while the latter is. That is, when firms are farsighted, two distinct cartels of the same size can have different stability properties; when cartels of the equal size are treated as the same, this possibility would be ignored. Therefore, cartels should be identified by their members (not by the numbers of members). Kamijo and Muto [10] have argued as just described and constructed an appropriate model in which cartels are identified by their members. Then, they have shown that any Pareto-efficient and individually rational cartel can always be stable with respect to the stability criterion incorporating the firms' farsighted view.

None of three studies, d'Aspremont et al. [1], Diamantoudi [3], and Kamijo and Muto [10], has dealt with the third inadequacy concerning the cartel's pricing behavior, which we shall take up in this paper. As mentioned earlier, the cartel's pricing policy in the above three studies is restricted to the *optimal pricing* in the sense that the cartel sets the price along the residual demand to maximize the joint profit of the members. Restricting the cartel's pricing to the optimal pricing may seem to be an innocuous assumption, but actually it is not.

From several fields in economics, we can draw many pieces of evidence that some observed outcomes that satisfy certain criteria of rationality, efficiency and/or optimality, can often be sustained through some irrational, inefficient and/or non-optimal behavior. On theoretical ground, take the well-known folk theorem for instance; it states that nearly efficient and cooperative outcomes can be maintained through the "punishment" behavior after one player's deviation, which is irrational (at least, in the one shot game) even if the continuation game satisfies the subgame perfection (see Fudenberg and Tirole [5, Chap. 5]). On empirical ground, among the growing literature on experimental economics, take Fehr and Gachter [4] for example; they have examined a two-stage game composed of a voluntary contribution game in the first stage and a punishment phase in the second and shown that higher contributions by the subjects in the voluntary contribution game are realized by the actual use of the punishment option, which has been designed *not* to constitute the subgame-perfect equilibium. In sum, non-optimal behavior of a player can work as "punishment" and/or "reward" to other players and, therefore, it can induce other players' *optimal* responses. Taking account of the possibility of non-optimal behavior has a significant influence on the final outcomes of the model. In our model, the cartel is allowed to choose not only the optimal price, but also any non-negative price; this flexible pricing policy can be interpreted as punishment and/or reward to the fringe firms and, by this, the cartel can induce the fringe firms to behave *optimally*.

In this paper, we present a model that takes full account of the three points (a) through (c) mentioned above. That is, in our model, the price leadership cartel is identified by its members, each firm has an ability to foresee not only an immediate outcome but also the ultimate outcome after its deviation, and the cartel can choose any nonnegative price, including pricing policy in order to control the behavior of non-cartel members.

One additional difference between our work and Kamijo and Muto [10] is that while sequences of coalitional or simultaneous deviations (of exiting from the cartel or joining the cartel) are allowed in Kamijo and Muto [10], only sequences of individual deviations are allowed in our model. In general, this difference produces a large variation of the results. For example, Suzuki and Muto [15] have analyzed the stability of an n-person prisoners' dilemma game and shown that Pareto-efficient and individually rational outcomes are always stable if coalitional moves are allowed. On the other hand, Nakanishi [12] have considered the same problem with only allowing individual moves and shown that the set of the stable outcomes shows a rather complicated figure, quite different from the result obtained by Suzuki and Muto [15]. Surprisingly, in contrast to the comparison of the prisoners' dilemma cases, we obtain an efficiency result similar to Kamijo and Muto [10] because of the endogeneity of the price set by the cartel. Further, we also show that although we do not restrict our analysis to the optimal pricing of the cartel, the optimal pricing behavior of the stable cartel emerges as the result of stability consideration.

Similar to Diamantoudi [3] and Kamijo and Muto [10], we shall adopt the von Neumann and Morgenstern [18] stable set as the basis of our stability concept in this paper.[3] The stable set is defined for the pair of a set of the outcomes and a dominance relation defined over the set of the outcomes. An outcome of our model, which describes the current market structure, is a pair of a cartel and a quoted price set by the cartel. The dominance relation is extended to two directions: one is to capture firms' farsighted view and the other is to deal with the endogenous pricing by the cartel. The stable set according to the set of the outcomes and this extended dominance relation (called the indirect dominance) constitutes our solution concept called the *farsighted stable set*.

[3]The von Neumann-Morgenstern stable set has been originally defined for characteristic function form games. Greenberg [7]'s "Theory of Social Situations (TOSS)" has opened the way to apply the concept of the von Neumann-Morgenstern stable set to more general game-theoretical settings. Although we do not utilize the framework of TOSS explicitly in this paper, our analysis can be reconstructed within the framework of TOSS.

The rest of this paper is organize as follows. In the next section, we present a price leadership model; we give the definitions of outcomes, the indirect dominance relation, and the farsighted stable set. The endogeneity of the pricing is embedded in the definition of our indirect dominance relation. Main results and the proofs are given in Section 3. Some of the proofs of lemmas are relegated to the Appendices. Section 4 includes some remarks.

2 Model

We consider an industry composed of n ($n \geqq 2$) identical firms, which produce a homogeneous good. The demand for the good is represented by a function $d\colon \mathbb{R}_+ \to \mathbb{R}_+$:

$$Q = d(p), \tag{1}$$

where p is the price and Q is the total amount of demand of the good. We assume that d is continuous and decreasing in p and it satisfies $0 < d(0) < +\infty$ and $d(a) = 0$ for some $a > 0$.

Each firm has an identical cost function $c(q_i)$, where q_i is the output level of a firm (firm i). We assume that c is increasing, twice continuously differentiable in q_i, and it satisfies $c(0) = 0$, $c'(0) = 0$, $c'(q_i) > 0$ for $q_i > 0$, and $c''(q_i) > 0$ for $q_i \geqq 0$.

2.1 Collusive price leadership

Once k firms have decided to combine and form a cartel, the cartel can exercise a power to determine the market price of the good. The remaining $n-k$ firms constitute a competitive fringe, whose members behave competitively. That is, each firm in the fringe takes the price determined by the cartel as given, and choose its output level to maximize its own profit. Given the price p, the supply function of a fringe firm, $q_f(p)$, is determined through the well-known price-equal-marginal-cost condition:

$$p \equiv c'(q_f(p)). \tag{2}$$

Given the responses by the fringe firms, the residual demand for the size k cartel can be written as follows:

$$R(k,p) \equiv \max\left\{d(p) - (n-k)q_f(p),\ 0\right\}. \tag{3}$$

To simplify the exposition, we assume that members in the cartel divide their total quantity of production equally. Thus, the production per firm in the cartel can be written as follows:

$$r(k,p) \equiv \frac{R(k,p)}{k}. \tag{4}$$

With this, the profit of a firm in the cartel can be written as a function of the cartel size k and the price p:

$$\pi_c(k,p) \equiv pr(k,p) - c\left(r(k,p)\right). \tag{5}$$

On the other hand, the profit of a fringe firm can be written as a function of p:

$$\pi_f(p) = pq_f(p) - c\left(q_f(p)\right). \tag{6}$$

The optimal price for the size k cartel is determined by

$$p^*(k) = \arg\max_{p>0} \pi_c(k,p). \tag{7}$$

We simply assume the existence and the uniqueness of $p^*(k)$ for each $k = 1, 2, \ldots, n$. The profits of a cartel firm and a fringe firm evaluated at the optimal price $p^*(k)$ can be written as functions of the cartel size k: For $k = 1, \ldots, n$,

$$\pi_c^*(k) \equiv \pi_c(k, p^*(k)),$$

and for $k = 1, \ldots, n-1$,

$$\pi_f^*(k) \equiv \pi_f(p^*(k)).$$

If $k = 0$, that is, if there is no cartel, then it is assumed that the market structure is competitive. The competitive equilibrium price, denoted by p^{comp}, is determined by $d(p^{\text{comp}}) = nq_f(p^{\text{comp}})$. Then, we have $\pi_f^*(0) = \pi_f(p^{\text{comp}})$.

The following proposition is concerned with the profits of firms in the cartel and in the fringe.

Proposition 1. *π_c and π_f satisfy the following properties:*

(i) *If $p \neq p^{\text{comp}}$ and $\pi_c(k,p) > 0$, then $\pi_c(k,p)$ is strictly increasing in k—[**Size monotonicity of $\boldsymbol{\pi_c}$**]. If $\pi_c(k,p) = 0$, $\pi_c(k+1,p) \geqq \pi_c(k,p)$ holds. Further, if $p = p^{\text{comp}}$, $\pi_c(k,p)$ is invariant against changes in k;*

(ii) *$\pi_f(p)$ is strictly increasing in p;*

(iii) *$\pi_f(p) \geqq \pi_c(k,p)$ for all p and for all $k = 1, \ldots, n-1$ with strict inequality when $p \neq p^{\text{comp}}$.*

Proof. Properties (ii) and (iii) follow immediately from the definition of q_f. Thus, it suffices to show property (i). When $\pi_c(k,p) = 0$, then either $p = 0$ or $d(p) - (n-k)q_f(p) \leqq 0$ holds. In either cases, $\pi_c(k+1,p) \geqq 0$. Next, we consider the case that $\pi_c(k,p) > 0$. Partially differentiating $\pi_c(k,p)$ with respect to k, we obtain

$$\frac{\partial \pi_c}{\partial k} = p r_k(k,p) - c'(r(k,p)) r_k(k,p) = r_k(k,p)\left\{p - c'(r(k,p))\right\}, \qquad (8)$$

where

$$r_k(k,p) \equiv \frac{\partial r(k,p)}{\partial k} = \frac{n q_f(p) - d(p)}{k^2}. \qquad (9)$$

Note that $q_f(p) > r(k,p)$ if and only if $q_f(p) > d(p)/n$. If $r_k(k,p) > 0$ or, equivalently, $q_f(p) > d(p)/n$, then $q_f(p) > r(k,p)$. Because c' is strictly increasing, then this result implies $p = c'(q_f(p)) > c'(r(k,p))$. Hence, we have $\partial\pi_c/\partial k > 0$. In turn, if $r_k(k,p) < 0$ or, equivalently, $q_f(p) < d(p)/n$, then $q_f(p) < r(k,p)$. Since $p = c'(q_f(p)) < c'(r(k,p))$, we have $\partial\pi_c/\partial k > 0$ again. If $r_k(k,p) = 0$, then p satisfies $d(p) - n q_f(p) = 0$. Thus p must be p^{comp}. In this case, $\partial\pi_c/\partial k = 0$. □

The next proposition for the optimal profits is due to d'Aspremont et al. [1] and Kamijo and Muto [10].

Proposition 2. *π_c^* and π_f^* satisfy the following properties:*

(i) *$\pi_c^*(k)$ is increasing in k—***[Size monotonicity of $\boldsymbol{\pi_c^*}$]***;*

(ii) *$\pi_c^*(k) > \pi_f^*(0)$ for all $k = 1, \ldots, n$;*

(iii) *$\pi_f^*(k) > \pi_c^*(k)$ for all $k = 1, \ldots, n-1$.*

2.2 Stability of collusive cartel

Consider an n-vector $x = (x_1, x_2, \ldots, x_n)$ such that for each i, x_i is equal to 0 or 1. Here, $x_i = 1$ means that firm i belongs to the existing cartel, whereas $x_i = 0$ means firm i does not belong to the cartel. That is, an n-vector x represents a cartel structure. Let $X \equiv \{0,1\}^n$ be the set of all possible cartel structures. By definition, $x^f \equiv (0, \ldots, 0)$ represents a situation with no actual cartel and $x^c \equiv (1, \ldots, 1)$ represents a situation with the largest cartel that consists of all the firms. Given $x \in X$, $C(x)$ denotes the set of firms belonging to the cartel at x, that is, $C(x) \equiv \{i \in N |\ x_i = 1\}$. We identify $C(x)$ with the cartel at x. Given $x, y \in X$, $x \wedge y$ denotes a cartel structure z such that $z_i = \min\{x_i, y_i\}$ for $i = 1, \ldots, n$. We can easily verify that $C(x \wedge y) = C(x) \cap C(y)$. For $x \in X$, let us define $|x| \equiv \sum_{i=1}^{n} x_i$, which signifies the cartel size at x in terms of the number of the participating firms.[4]

A pair of a cartel structure $x \in X$ and a quoted price $p \in \mathbb{R}_+$ describes a market structure; it specifies the current price and the firms in the cartel (and, implicitly, the firms in the fringe). Incidentally, what will happen to the market structure if there is no actual cartel (i.e., if $x = x^f$)? In this case, we assume that (x^f, p^{comp}) will be realized. That is, if there is no actual price-leader, only the competitive equilibrium price p^{comp} will prevail in the market. In other words, any market structure such as (x^f, p) with $p \neq p^{\text{comp}}$ is meaningless. Excluding such meaningless market structures, we now define the set A of all possible market structures:

$$A \equiv \left\{ (x, p) \in \{0,1\}^n \times \mathbb{R}_+ \,\middle|\, x \neq x^f \text{ or } (x, p) = (x^f, p^{\text{comp}}) \right\}. \tag{10}$$

We shall call an element in A as an "outcome."[5]

Let g_i be the payoff function of firm i defined on A: For $(x, p) \in A$,

$$g_i(x, p) = \begin{cases} \pi_c(|x|, p) & \text{if} \quad x_i = 1, \\ \pi_f(p) & \text{if} \quad x_i = 0. \end{cases} \tag{11}$$

For a fringe firm, only quoted price p matters; it does not matter who are the members of the current cartel nor how many firms are in the cartel.

Let us define a set of outcomes where a cartel charges the optimal price, denoted by $A^{\text{OP}} = \{(x, p) \in A \,|\, p = p^*(|x|)\}$. For two distinct outcomes $(x, p), (y, w) \in A$, we say that "(y, w) Pareto-dominates (x, p)" if $g_i(y, w) \geqq g_i(x, p)$ for all $i \in N$ and $g_i(y, w) > g_i(x, p)$ for some $i \in N$. The set of Pareto-efficient outcomes, denoted by A^{PE}, is a subset of outcomes that are not Pareto-dominated. Let us define another subset of A, denoted by B, which will turn out to be a subset of A^{PE}:

$$B \equiv \left\{ (x, p^*(|x|)) \in A \,\middle|\, x = x^c \text{ or } \pi_f^*(|x|) > \pi_c^*(n) \right\}. \tag{12}$$

Because the largest-cartel optimal-pricing outcome $(x^c, p^*(|x^c|))$ always exists, B is nonempty.

Lemma 1. *B coincides with the intersection of A^{OP} and A^{PE}.*

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[4]That is, $|x| = |C(x)|$.

[5]If the price p is high enough, the demand $d(p)$ becomes zero and the supply by the fringe firms becomes strictly positive. Therefore, in an outcome (x, p) with a sufficiently high price, the market clearing condition can be violated; in this sense, such an outcome is not feasible. Although we can redefine the set of possible outcomes so that it only includes *feasible* outcomes, this will make the model unnecessarily complicated.

Proof is relegated to the Appendix.

Next, we define the inducement relation on A. We assume that each individual firm can enter or exit from the existing cartel freely and, thereby, change the current market structure to another. In the course of entry-exit, only individual moves are allowed, while coalitional (simultaneous) moves are not. Furthermore, we assume that the cartel members can change the current price to another through a unanimous agreement. By changing the price, the cartel can induce another market structure from the current market structure. In general, when a nonempty subset S of N changes a given market structure (x, p) to another (y, w), we write $(x, p) \overset{S}{\to} (y, w)$. The relation $\{\overset{S}{\to}\}_{S \subset N}$ is formally defined as follows:

Definition 1 (Inducement relation). *For outcomes $(x, p), (y, w) \in A$ and nonempty $S \subset N$, we have $(x, p) \overset{S}{\to} (y, w)$ if either one of the following conditions is satisfied:*

(i) $S = C(x)$ *and* $x = y$,

(ii) $S = \{i\} \neq C(x)$, $x_j = y_j$ *for all* $j \neq i$, *and* $p = w$,

(iii) $S = \{i\} = C(x)$ *and* $(y, w) = (x^f, p^{\text{comp}})$.

Part (i) means that cartel $C(x)$ can change the current price p to another w through a unanimous agreement by the members. Part (ii) means that a single player i can change the current market structure to another by entry-exit from the cartel without affecting the current price. Part (iii) means that if a single player i is the last one member of the current cartel, it can change the current outcome to the competitive equilibrium outcome by exiting from the cartel.

The indirect dominance relation is defined as follows.

Definition 2 (Indirect domination). *For outcomes $(x, p), (y, w) \in A$, we say that "(y, w) indirectly dominates (x, p)" or "(x, p) is indirectly dominated by (y, w)," which we shall write $(y, w) \gg (x, p)$ or $(x, p) \ll (y, w)$, if and only if there exists a sequence of outcomes and nonempty coalitions*

$$(x, p) = (x^0, p^0) \xrightarrow{S^1} (x^1, p^1) \xrightarrow{S^2} \cdots \xrightarrow{S^M} (x^M, p^M) = (y, w) \qquad (13)$$

such that for each $m = 1, \ldots, M$, (a) $(x^{m-1}, p^{m-1}) \xrightarrow{S^m} (x^m, p^m)$ *and* (b) $g_i(x^{m-1}, p^{m-1}) < g_i(x^M, p^M) = g_i(y, w)$ *for all $i \in S^m$.*

A pair $(A, \gg)$ is called the abstract system associated with the price leadership model. Now we define the solution concept of our model, that is, the farsighted stable set.

Definition 3 (Farsighted stable set). *A subset K of A is said to be a farsighted stable set for the abstract system $(A, \gg)$ associated with the price leadership model if it satisfies the following two conditions:*

(i) *For any $(x,p) \in K$, there does not exist any outcome $(y,w) \in K$ such that $(y,w) \gg (x,p)$;*

(ii) *For any $(x,p) \in A \setminus K$, there exists an outcome $(y,w) \in K$ such that $(y,w) \gg (x,p)$.*

Conditions (i) and (ii) are called *internal stability* and *external stability* of K, respectively.

3 Results

The following lemmas characterize the indirect domination.

Lemma 2. *The largest-cartel optimal-pricing outcome $(x^c, p^*(|x^c|))$ indirectly dominates any other outcome.*

Lemma 3. *Take distinct outcomes $(x,p), (y,w) \in A$. Then, (x,p) indirectly dominates (y,w) if either one of the following conditions is satisfied:*

(i) $C(x) \cap C(y) = \emptyset$, $\pi_f(p) > 0$ *and* $\pi_f(p) > \pi_c(|y|, w)$;

(ii) $C(x) \cap C(y) \neq \emptyset$, $\pi_f(p) > \pi_c(|y|, w)$, $\pi_c(|x|, p) > 0$, *and* $\pi_c(|x|, p) > \pi_c(|x \wedge y|, w)$;

(iii) $C(x) \cap C(y) \neq \emptyset$, $C(x) \not\subset C(y)$, $C(x) \not\supset C(y)$, $p \neq p^{\mathrm{comp}}$, $\pi_c(|x|, p) > 0$, *and* $\pi_c(|x|, p) \geqq \pi_c(|y|, w)$.

With Lemmas 2 and 3, we can show the existence of the farsighted stable sets:

Theorem 1. *For any outcome $(x,p) \in B$, the singleton set $\{(x,p)\}$ constitutes a farsighted stable set.*

Proof. Because the internal stability is satisfied automatically, it suffices to show the external stability. If $(x,p) = (x^c, p^*(|x^c|))$, then the external stability follows from Lemma 2 immediately. Then, let us assume $x \neq x^c$ and $p = p^*(|x|)$. Take an arbitrary $(y,w) \in A$ with $(x,p) \neq (y,w)$. We distinguish three cases: (i) $x = y$; (ii) $x \neq y$ and $|x| \geqq |y|$; (iii) $x \neq y$ and $|x| < |y|$.

Let us consider case (i). By the definition of the inducement relation, we have $(y, w) = (x, w) \xrightarrow{C(y)} (x, p^*(|x|)) = (x, p)$. Further, by the definition of p^*, we can show the following relation: for all $i \in C(y)$,

$$g_i(y, w) = \pi_c(|y|, w) < \pi_c(|y|, p^*(|y|)) = \pi_c(|x|, p^*(|x|)) = g_i(x, p). \qquad (14)$$

Then, we obtain $(x, p) \gg (y, p)$.

Next, let us consider case (ii). By the size-monotonicity of π_c, the definition of p^*, and Proposition 2-(iii), we have the following relation:

$$\pi_c(|y|, w) \leqq \pi_c(|x|, w) \leqq \pi_c(|x|, p^*(|x|)) < \pi_f(p^*(|x|)) = \pi_f(p). \qquad (15)$$

This relation and the fact $\pi_c(|x|, p^*(|x|)) > 0$ imply both $\pi_f(p) > \pi_c(|y|, w)$ and $\pi_f(p) > 0$. Therefore, if $C(x) \cap C(y) = \emptyset$, then the conditions in Lemma 3-(i) are satisfied. On the other hand, if $C(x) \cap C(y) \neq \emptyset$, we have $\pi_c(|x \wedge y|, w) \leqq \pi_c^*(|x \wedge y|) < \pi_c^*(|x|) = \pi_c(|x|, p^*(|x|)) = \pi_c(|x|, p)$ by the size-monotonicity of π_c^*. Then, the conditions in Lemma 3-(ii) are satisfied. Thus, we obtain the desired result.

Lastly, let us consider case (iii). Since $(x, p) \in B$ and $(x, p) \neq (x^c, p^*(|x^c|))$, we have $0 < \pi_c^*(|x^c|) < \pi_f^*(|x|) = \pi_f(p^*(|x|)) = \pi_f(p)$. By the definition and the size-monotonicity of π_c^*, we have $\pi_c(|y|, w) \leqq \pi_c^*(|y|) \leqq \pi_c^*(|x^c|) = \pi_c^*(n)$. Combining these inequalities, we obtain $\pi_f(p) > \pi_c(|y|, w)$ and $\pi_f(p) > 0$. If $C(x) \cap C(y) = \emptyset$, then the conditions in Lemma 3-(i) are satisfied. If $C(x) \cap C(y) \neq \emptyset$, then we have $\pi_c(|x \wedge y|, w) \leqq \pi_c(|x|, w) < \pi_c(|x|, p^*(|x|)) = \pi_c(|x|, p)$. Then, the conditions in Lemma 3-(ii) are satisfied. □

As shown in Lemma 1, any outcome in B is Pareto-efficient. Then, our Theorem 1 shows that an efficient outcome can be attained as an *ultimate* outcome in an essentially *noncooperative* circumstance through the solution concept of the farsighted stable set . A similar result as our Theorem 1 has been obtained by Kamijo and Muto [10].[6] In their model, however, it is assumed that even firms who are not the members of the current cartel can make joint deviations and that the current cartel sets the price at the optimal, joint-profit-maximizing level automatically. Because the cooperative actions by the firms are embedded in their model at the very outset, it is natural to have the efficiency result. On the other hand, because in our model it is assumed that joint entry or exit by a group of firms are not allowed and only the members of the current cartel can make joint moves (of changing price) through a unanimous agreement, it is somewhat surprising to obtain the efficiency result.

[6]Suzuki and Muto [15] have also shown a similar result as Kamijo and Muto [10] in an n-person prisoners' dilemma.

The key in our model is the endogeneity of the price. Let us consider, for example, the largest-cartel optimal-pricing outcome $(x^c, p^*(|x^c|))$, which constitutes a (singleton) farsighted stable set, and another nonstable outcome (x, p) with a smaller size cartel $C(x)$. Even if $(x^c, p^*(|x^c|))$ is better than (x, p) for the members of $C(x)$, the members of $C(x)$ can do nothing except for waiting entry by other firms when the price in (x, p) is determined automatically through the optimal pricing rule as in Kamijo and Muto [10]. On the other hand, if $C(x)$ can control the price, it can force the remaining fringe firms to enter the cartel by decreasing the price to zero and, thereby, form the largest cartel $C(x^c)$. Once the largest cartel $C(x^c)$ has been formed, it can choose the optimal monopoly price and make its members (i.e., all firms) better-off.

We have to prepare the additional lemma to show the uniqueness of our farsighted stable set mentioned in Theorem 1.

Lemma 4. *Let K be a farsighted stable set. Then, for any $(x, p) \in K$ with $(x, p) \neq (x^f, p^{\text{comp}})$, we have $\pi_c(x, p) > 0$.*

The next theorem shows that there is no other type of farsighted stable set.

Theorem 2. *There is no other type of farsighted stable sets than the one described in Theorem 1.*

Proof. Let K be a farsighted stable set. If $K \cap B \neq \emptyset$, then K must be a singleton, otherwise it violates the internal stability. In this case, K is of the type just described in Theorem 1. Then, we can assume $K \cap B = \emptyset$. In the following, we prove by contradiction that this cannot be the case. Specifically, we show that, under the condition $K \cap B = \emptyset$, there is an *infinite* sequence (x^1, p^1), $(x^2, p^2), \ldots$ of outcomes in K such that $|x^1| > |x^2| > \cdots$. This contradicts the finiteness of the number of the firms.

The fact $(x^c, p^*(|x^c|)) \in B$ implies $(x^c, p^*(|x^c|)) \notin K$. By the external stability of K, there must exist an outcome $(x^1, p^1) \in K$ that indirectly dominates $(x^c, p^*(|x^c|))$. If $x^1 = x^c$, we must have $p^1 \neq p^*(|x^c|)$. Then, by the definition of p^*, we have

$$g_i(x^1, p^1) = \pi_c(|x^c|, p^1) < \pi_c(|x^c|, p^*(|x^c|)) = g_i(x^c, p^*(|x^c|)) \qquad (16)$$

for all $i \in N = C(x^c)$. This implies that (x^1, p^1) cannot indirectly dominate $(x^c, p^*(|x^c|))$. Hence, $x^1 \neq x^c$ must hold and thus, $|x^1| < |x^c|$.

In addition, we can show that $|x^1| \neq 0$, that is, $x^1 \neq x^f$. Suppose, in negation, that $x^1 = x^f$. Because we have $g_i(x^c, p^*(|x^c|)) = \pi_c^*(|x^c|) > \pi_f^*(0) = g_i(x^f, p^{\text{comp}})$ for all $i \in N$ by Proposition 2-(ii), then no player wants to deviate from $(x^c, p^*(|x^c|))$ toward $(x^1, p^1) = (x^f, p^{\text{comp}})$—a contradiction.

Moreover, we can show that $p^1 \neq p^*(|x^1|)$. Let S be the first coalition in a sequence that realizes $(x^c, p^*(|x^c|)) \ll (x^1, p^1)$ and suppose, in negation, that $p^1 = p^*(|x^1|)$. Because $(x^1, p^1) \in K$ implies $(x^1, p^1) \notin B$, we have

$$\pi_c^*(|x^c|) \geqq \pi_f^*(|x^1|) = \pi_f(p^*(|x^1|)) = \pi_f(p^1). \tag{17}$$

Further, for any firm $i \in N \setminus C(x^1) = C(x^c) \setminus C(x^1)$, we have[7]

$$g_i(x^c, p^*(|x^c|)) = \pi_c^*(|x^c|) \geqq \pi_f(p^1) = g_i(x^1, p^1). \tag{18}$$

Then, $S = C(x^c)$ cannot be true. Therefore, S must be a singleton $\{i_1\}$ for some $i_1 \in C(x^1)$; otherwise the definition of the indirect domination will be violated. For i_1, we have $\pi_c^*(|x^c|) = g_{i_1}(x^c, p^*(|x^c|) < g_{i_1}(x^1, p^1) = \pi_c(|x^1|, p^1) = \pi_c(|x^1|, p^*(|x^1|)) = \pi_c^*(|x^1|)$. But, since $x^c \neq x^1$ implies $|x^1| < |x^c| = n$, the inequality $\pi_c^*(|x^c|) < \pi_c^*(|x^1|)$ contradicts the size-monotonicity of π_c^*. Hence, $p^1 \neq p^*(|x^1|)$.

Let us consider an outcome $(x^1, p^*(|x^1|))$. Because $C(x^1)$ can induce $(x^1, p^*(|x^1|))$ from (x^1, p^1) by changing price and because we have

$$g_i(x^1, p^1) = \pi_c(|x^1|, p^1) < \pi_c(|x^1|, p^*(|x^1|)) = g_i(x^1, p^*(|x^1|)) \tag{19}$$

for all $i \in C(x^1)$, then $(x^1, p^*(|x^1|))$ indirectly dominates (x^1, p^1). By the internal stability of K, $(x^1, p^*(|x^1|))$ cannot be in K. Then, there must exist an outcome $(x^2, p^2) \in K$ that indirectly dominates $(x^1, p^*(|x^1|))$.

We show that there is at least one firm in $C(x^1)$ who becomes worse-off in (x^2, p^2) than in $(x^1, p^*(|x^1|))$. Let $S^1, S^2, \ldots, S^M$ be the sequence of coalitions that appear in a sequence that realizes $(x^1, p^*(|x^1|)) \ll (x^2, p^2)$:

$$\begin{aligned} (x^1, p^*(|x^1|)) = (y^0, w^0) \xrightarrow{S^1} (y^1, w^1) \xrightarrow{S^2} \cdots \\ \cdots \xrightarrow{S^{M-1}} (y^{M-1}, w^{M-1}) \xrightarrow{S^M} (y^M, w^M) = (x^2, p^2). \end{aligned} \tag{20}$$

Suppose, in negation, that every firm in $C(x^1)$ is not worse-off in (x^2, p^2) than in $(x^1, p^*(|x^1|))$. Because $(x^1, p^1) \xrightarrow{C(x^1)} (x^1, p^*(|x^1|))$ and $(x^1, p^1) \ll (x^1, p^*(|x^1|))$, every firm in $C(x^1)$ is strictly better-off in $(x^1, p^*(|x^1|))$ than in (x^1, p^1) and, therefore, is strictly better-off in (x^2, p^2) than in (x^1, p^1). Moreover, we have the following inducement relation:

$$(x^1, p^1) \xrightarrow{C(x^1)} (x^1, p^*(|x^1|)) \xrightarrow{S^1} \cdots \xrightarrow{S^M} (x^2, p^2). \tag{21}$$

Thus, (x^2, p^2) indirectly dominates (x^1, p^1); but, this contradicts the internal stability of K. Hence, there must be at least one firm in $C(x^1)$ who becomes

[7] Firm i is a cartel member at x^c, but a fringe firm at x^1.

worse-off in (x^2, p^2) than in $(x^1, p^*(|x^1|))$. We denote the set of such firms in $C(x^1)$ by T; note that $\emptyset \neq T \subset C(x^1)$.

Next, we show that $|x^1| > |x^2|$. Again, let $S^1, S^2, \ldots, S^M$ be the sequence of coalitions that appear in a sequence that realizes $(x^1, p^*(|x^1|)) \ll (x^2, p^2)$. We have to distinguish two cases: case (a) where $S^m \cap T = \emptyset$ for all $m = 1, \ldots, M$ and case (b) where $S^m \cap T \neq \emptyset$ for some m.

Let us consider case (a). In this case, no firm in T exits from $C(x^1)$. In other words, all firms in T remain inside the cartel all the way along the sequence. Then, if a certain coalition S in the sequence were to change price from $p^*(|x^1|)$ to another one, then S must include T. But, this contradicts the definition of the indirect domination. Therefore, the price remains unchanged along the sequence. Then, for each $i \in T$, we have

$$\pi_c(|x^1|, p^*(|x^1|)) = g_i(x^1, p^*(|x^1|)) > g_i(x^2, p^2) = \pi_c(|x^2|, p^*(|x^1|)). \qquad (22)$$

By the size-monotonicity of π_c, the inequality $|x^1| > |x^2|$ follows.

In turn, let us consider case (b). We first show that $T = C(x^1) \cap C(x^2)$ and, then, we proceed to the proof of $|x^1| > |x^2|$. Let S^{k+1} be the first coalition in the sequence that contains at least one firm in T (that is , $S^{k+1} \cap T \neq \emptyset$ and $S^m \cap T = \emptyset$ for all $m \leqq k$) and let (y^k, w^k) be the outcome from which S^{k+1} deviates. Further, by the same reason just described in the above paragraph, $w^k = p^*(|x^1|)$ must hold. Take an arbitrary firm $i \in S^{k+1} \cap T$; note that $i \in C(x^1)$ and $i \in C(y^k)$. Then, for firm i, we have

$$\begin{aligned}
&g_i(x^1, p^*(|x^1|)) = \pi_c(|x^1|, p^*(|x^1|)),\\
&g_i(y^k, w^k) = \pi_c(|y^k|, w^k) = \pi_c(|y^k|, p^*(|x^1|)),\\
&g_i(x^1, p^*(|x^1|)) > g_i(x^2, p^2) > g_i(y^k, w^k).
\end{aligned}$$

Combining these [in]equalities, we obtain $\pi_c(|x^1|, p^*(|x^1|)) > \pi_c(|y^k|, p^*(|x^1|))$. By the size-monotonicity of π_c, we have $|x^1| > |y^k|$. This implies that some firms in $C(x^1) \setminus T$ have to exit from the cartel before (y^k, w^k) is reached; in other words, we must have $C(x^1) \setminus T \neq \emptyset$.

Consider arbitrary firms i and j such that $i \in C(x^1) \setminus T$ and $j \in T$. By the definition of T, the status of firm i at (x^2, p^2) must be different from that of firm j at (x^2, p^2). There are two cases: one where $i \in C(x^2)$ and $j \in N \setminus C(x^2)$ and the other where $i \in N \setminus C(x^2)$ and $j \in C(x^2)$. In the former, we have

$$\begin{aligned}
&\pi_c(|x^1|, p^*(|x^1|)) = g_i(x^1, p^*(|x^1|)) \leqq g_i(x^2, p^2) = \pi_c(|x^2|, p^2),\\
&\pi_c(|x^1|, p^*(|x^1|)) = g_j(x^1, p^*(|x^1|)) > g_j(x^2, p^2) = \pi_f(p^2).
\end{aligned}$$

Combining these inequalities and taking account of Proposition 1-(iii), we obtain a contradiction:

$$\pi_f(p^2) < \pi_c(|x^1|, p^*(|x^1|)) \leqq \pi_c(|x^2|, p^2) < \pi_f(p^2). \tag{23}$$

Thus, the former case is not possible and the latter case must hold. The latter case produces two implications: one is that $j \in T$ implies $j \in C(x^2)$ and the other is that $i \in C(x^2)$ implies $i \in T$ *or* $i \notin C(x^1)$. From the former implication, we obtain $T \subset C(x^1) \cap C(x^2)$. Similarly, from the latter implication, we obtain $T \supset C(x^1) \cap C(x^2)$. Hence, $T = C(x^1) \cap C(x^2)$.

Now, we prove $|x^1| > |x^2|$ for case (b). Suppose, to the contrary, that $|x^1| \leqq |x^2|$. The facts $C(x^1) \setminus T \neq \emptyset$, $T = C(x^1) \cap C(x^2)$, and $|x^1| \leqq |x^2|$ imply $C(x^2) \setminus T \neq \emptyset$. Then, we have both $C(x^1) \not\subset C(x^2)$ and $C(x^1) \not\supset C(x^2)$. Further, by Lemma 4 we have $\pi_c(|x^1|, p^1) > 0$ and $\pi_c(|x^2|, p^2) > 0$. Therefore, by Lemma 3-(iii), one of (x^1, p^1) and (x^2, p^2) indirectly dominates the other outcome. This contradicts the internal stability of K. Hence, $|x^1| > |x^2|$.

Similar to the case of (x^1, p^1), it is easy to show that $|x^2| \neq 0$, that is $(x^2, p^2) \neq (x^f, p^{\text{comp}})$. Moreover, we can show that $p^2 \neq p^*(|x^2|)$; the proof is slightly different from the case of (x^1, p^1). Assume, in negation, that $p^2 = p^*(|x^2|)$. By the very definition of (x^2, p^2), there is a dominance sequence from $(x^1, p^*(|x^1|))$ to $[(x^2, p^2) = (x^2, p^*(|x^2|))]$. Let S be the first coalition that appears in the dominance sequence. By the fact $|x^1| > |x^2|$, the size monotonicity of π_c^*, and Proposition 2-(iii), we have

$$\begin{aligned}\pi_c(|x^2|, p^*(|x^2|) = \pi_c^*(|x^2|) &< \pi_c^*(|x^1|) \\ &= \pi_c(|x^1|, p^*(|x^1|)) \\ &< \pi_f^*(|x^1|) = \pi_f(|x^1|, p^*(|x^1|)).\end{aligned} \tag{24}$$

Thus, any player i in S must be in the fringe position at the final outcome $(x^2, p^*(|x^2|))$. When the firm belongs to the cartel at $(x^1, p^*(|x^1|))$, $\pi_f(p^*(|x^2|)) > \pi_c(|x^1|, p^*(|x^1|))$ must hold by the incentive of deviation, and when the firm belongs to the fringe at $(|x^1|, p^*(|x^1|))$, $\pi_f(p^*(|x^2|)) > \pi_f(p^*(|x^1|)) = \pi_f^*(|x^1|) > \pi_c^*(|x^1|) = \pi_c(|x^1|, p^*(|x^1|))$. By Proposition 2-(iii) and the definition of p^*, in both cases, we have

$$\pi_f(p^*(|x^2|)) > \pi_c(|x^1|, p^*(|x^1|)) \geqq \pi_c(|x^1|, p^1)). \tag{25}$$

In the case of $C(x^1) \cap C(x^2) \neq \emptyset$,

$$\pi_c(|x^2|, p^*(|x^2|)) = \pi_c^*(|x^2|) > \pi_c^*(|x^1 \wedge x^2|) \geqq \pi_c(|x^1 \wedge x^2|, p^1), \tag{26}$$

where the second inequality is by the size monotonicity of π_c^* and the third is by the definition of p^*. Hence, by Lemmas 3-(i) and 3-(ii), $(x^2, p^*(|x^2|))$

indirectly dominates (x^1, p^1). This contradicts the internal stability of K and thus $p^2 \neq p^*(|x^2|)$.

Then, the outcome $(x^2, p^*(|x^2|))$ indirectly dominates (x^2, p^2) and it is not in K; therefore, there must exist $(x^3, p^3) \in K$ that indirectly dominates $(x^2, p^*(|x^2|))$. In addition, (x^3, p^3) must satisfy $|x^2| > |x^3|$ and $x^3 \neq x^f$. Repeatedly applying the same argument, we obtain an infinite sequence of outcomes $(x^1, p^1), (x^2, p^2), \ldots$ such that $|x^1| > |x^2| > \cdots$. This contradicts the finiteness of the number of the firms. Hence, finally, we obtain the desired result: $K \cap B \neq \emptyset$. □

4 Remarks

In this paper, we considered the stability of price leadership cartel when each firm has an ability to foresee the future, only the individual moves are allowed to the firms and the cartel can choose any non-negative price. As mentioned in the Introduction, Kamijo and Muto [10] have considered a similar problem without endogeneity of the price and with coalitional deviations. To abstract the pure effect of endogeneity of the price of the cartel, we have to analyze the farsighted stability of the cartel with only individual deviations and without price endogeneity. We are working now on this point in a companion paper; it can be shown that, similar to Nakanishi [12], the shape of the stable set is quite complicated; the existence of the stable set is assured, but the uniqueness cannot be established.

On the other hand, if both coalitional moves and price endogeneity are allowed, the result is obvious. As our Theorems 2 and 3 show, only the one-point stable sets are admitted and, thus, the internal stability does not have concern. Because one situation is more likely to be dominated by another if we allow coalitional deviations, the outcomes described in Theorem 1 are the only stable set in the coalitional move cases.

Appendix

A Proof of Lemma 1

We first show $B \supseteq A^{\mathrm{OP}} \cap A^{\mathrm{PE}}$. Because B is a subset of A^{OP}, it suffices to show that any outcome $(x, p^*(|x|)) \in A^{\mathrm{OP}}$ with $\pi_f^*(|x|) \leqq \pi_c^*(n)$ is Pareto-dominated by another outcome. Fortunately, it is obvious that such an outcome $(x, p^*(|x|))$ is Pareto-dominated by $(x^c, p^*(n))$.

Next, we show $B \subseteq A^{\mathrm{OP}} \cap A^{\mathrm{PE}}$. Because B is a subset of A^{OP}, we will show that B is a subset of A^{PE}. Take an arbitrary $(x, p) \in B$. We have to show that (x, p) cannot be Pareto-dominated. We distinguish two cases: case 1 where $(x, p) = (x^c, p^*(|x^c|))$ and case 2 where $(x, p) \neq (x^c, p^*(|x^c|))$.

Let us consider case 1. Take an arbitrary $(y, w) \in A$ other than (x, p). If $C(y) = \emptyset$ or, equivalently, $y = x^f$, then we have $g_i(x, p) = \pi_c^*(n) > \pi_f^*(0) = \pi_f(p^{\mathrm{comp}}) = g_i(y, w)$ for all $i \in N$ by Proposition 2-(ii). On the other hand, if $C(y) \neq \emptyset$, we have $g_i(x, p) = \pi_c^*(n) > \pi_c^*(|y|) = \pi_c(|y|, p^*(|y|)) \geqq \pi_c(|y|, w) = g_i(y, w)$ for all $i \in C(y)$ by the size-monotonicity of π_c^* and the definition of p^*. That is, (y, w) cannot Pareto-dominate $(x^c, p^*(|x^c|))$.

Next, let us consider case 2. By the inequality $\pi_f^*(|x|) > \pi_c^*(n)$, neither $|x| = 0$ nor $|x| = n$ can be true. Therefore, we have both $C(x) \neq \emptyset$ and $N \setminus C(x) \neq \emptyset$. Suppose, in negation, that there exists an outcome $(y, w) \in A$ that Pareto-dominates (x, p).

If there is a player i such that $i \in N \setminus C(x)$ and $i \in C(y)$, then, by the definition of the Pareto-domination, we have

$$\pi_c(|y|, w) = g_i(y, w) \geqq g_i(x, p) = \pi_f^*(|x|). \tag{27}$$

On the other hand, by the definitions of π_c^* and B, we have

$$\pi_c^*(|y|) \geqq \pi_c(|y|, w) \quad \text{and} \quad \pi_f^*(|x|) > \pi_c^*(n). \tag{28}$$

Combining the above inequalities, we obtain $\pi_c^*(|y|) > \pi_c^*(n)$. This contradicts the size-monotonicity of π_c^*. Such player i cannot exist. Hence, $i \in N \setminus C(x)$ implies $i \in N \setminus C(y)$; equivalently, $C(y) \subset C(x)$.

In turn, if there is a player j such that $j \in C(x)$ and $j \in C(y)$, then, similar to the above paragraph, we obtain the following inequalities:

$$\pi_c^*(|y|) \geqq \pi_c(|y|, w) = g_j(y, w) \geqq g_j(x, p) = \pi_c^*(|x|). \tag{29}$$

By the size-monotonicity of π_c^*, the fact $\pi_c^*(|y|) \geqq \pi_c^*(|x|)$ implies $|y| \geqq |x|$. This, together with $C(y) \subset C(x)$, implies $C(y) = C(x)$ or, equivalently, $x = y$. Then, by the definition of p^*, we obtain

$$g_j(y, w) = g_j(x, w) = \pi_c(|x|, w) < \pi_c(|x|, p^*(|x|)) = \pi_c^*(|x|) = g_j(x, p). \tag{30}$$

This contradicts the definition of the Pareto-domination. Such player j cannot exist. Hence, $j \in C(x)$ implies $j \in N \setminus C(y)$; equivalently, $C(x) \subset N \setminus C(y)$. Therefore, we have $C(y) \subset C(x) \subset N \setminus C(y)$. This can be possible only if $C(y) = \emptyset$, but, as already shown, $C(y) = C(x) \neq \emptyset$—a contradiction. No outcome can Pareto-dominate $(x, p) \in B$. □

B Proof of Lemma 2

Take an arbitrary outcome (y, w) other than $(x^c, p^*(|x^c|))$. We distinguish three cases: case 1 where $y = x^c$, case 2 where $y = x^f$, and case 3 where $y \neq x^c$ and $y \neq x^f$.

First, let us consider case 1: $y = x^c$. Clearly, the cartel $C(y) = C(x^c)$ can change the current price w to the optimal price $p^*(|y|)$, that is, $(y, w) \xrightarrow{C(y)} (y, p^*(|y|)) = (x^c, p^*(|x^c|))$. Further, by the definition of p^*, we have

$$g_i(y, w) = \pi_c(|y|, w) = \pi_c(|x^c|, w) < \pi_c(|x^c|, p^*(|x^c|)) = g_i(x^c, p^*(|x^c|)) \quad (31)$$

for all $i \in C(y)$. The desired result obtains.

Next, let us consider case 2: $y = x^f$. Consider a sequence of deviations in which (a) each player enters the cartel one by one and (b) after all the players enter the cartel, the largest cartel changes the price to $p^*(|x^c|)$:

$$\begin{aligned} (y, w) = (x^0, p^{\text{comp}}) &\xrightarrow{\{i_1\}} (x^1, p^{\text{comp}}) \xrightarrow{\{i_2\}} \cdots \\ \cdots &\xrightarrow{\{i_n\}} [(x^n, p^{\text{comp}}) = (x^c, p^{\text{comp}})] \xrightarrow{C(x^c)} (x^c, p^*(|x^c|)). \end{aligned} \quad (32)$$

For each i_k in the above sequence, we have $g_{i_k}(x^{k-1}, p^{\text{comp}}) = \pi_f(p^{\text{comp}}) = \pi^*_f(0) < \pi^*_c(n) = \pi^*_c(|x^c|) = g_{i_k}(x^c, p^*(|x^c|))$ by Proposition 2-(ii). Further, in the last step, we have $g_i(x^n, p^{\text{comp}}) = g_i(x^c, p^{\text{comp}}) < \pi_c(|x^c|, p^*(|x^c|)) = \pi^*_c(|x^c|) = g_i(x^c, p^{\text{comp}})$ for all $i \in C(x^c)$. Again, the desired result obtains.

Lastly, let us consider case 3: $y \neq x^c$ and $y \neq x^f$. It immediately follows that $0 < |y| < n$. Now, consider a sequence of deviations in which (a) cartel $C(y)$ decreases the price down to zero, (b) each player in $N \setminus C(y)$ enter the cartel one by one until all the players enter the cartel, and (c) after establishing the largest cartel, the cartel $C(x^c)$ changes the price to $p^*(|x^c|)$:

$$\begin{aligned} (y, w) &\xrightarrow{C(y)} \big[(y, 0) = (x^0, 0)\big] \xrightarrow{\{j_1\}} (x^1, 0) \xrightarrow{\{j_2\}} \cdots \\ &\cdots \xrightarrow{\{j_r\}} [(x^r, 0) = (x^c, 0)] \xrightarrow{C(x^c)} (x^c, p^*(|x^c|)), \end{aligned} \quad (33)$$

where $N \setminus C(y) \equiv \{j_1, j_2, \ldots, j_r\}$. In the first (price-cutting) step, we have

$$g_i(y, w) = \pi_c(|y|, w) \leqq \pi^*_c(|y|) < \pi^*_c(n) = \pi^*_c(|x^c|) = g_i(x^c, p^*(|x^c|)) \quad (34)$$

for all $i \in C(y)$ by the size-monotonicity of π_c^*. In each of the intermediate (entry) steps, we have

$$g_{j_k}(x^{k-1}, 0) = \pi_f(0) = 0 < \pi_c^*(|x^c|) = g_{j_k}(x^c, p^*(|x^c|)) \tag{35}$$

for j_k $(k = 1, 2, \ldots, r)$ by Proposition 2-(ii). In the last (price-increasing) step, we have

$$g_i(x^c, 0) = \pi_c(|x^c|, 0) < \pi_c(|x^c|, p^*(|x^c|)) = g_i(x^c, p^*(|x^c|)) \tag{36}$$

for all $i \in C(x^c)$ by the definition of p^*. Hence, the desired result obtains. □

C Proof of Lemma 3

We first prove case (ii) and, then, turn to case (i) and case (iii).

C.1 Part (ii)

Let $\hat{p} \in \mathbb{R}_+$ be a price level that satisfies

$$\pi_f(\hat{p}) < \pi_c(|x|, p). \tag{37}$$

Such a price level $\hat{p}$ exists since $\pi_c(|x|, p) > 0$, $\lim_{p \to +0} \pi_f(p) = 0$ by $0 < d(0) < +\infty$, and π_f is a continuous function of p.

Consider the following steps that form an appropriate sequence of deviations from (y, w) to (x, p):

Step 1 If $C(y) \setminus C(x) = \emptyset$, then go to the next step. Otherwise, consider a sequence in which each firm in $C(y) \setminus C(x)$ exits from the cartel in turn. Let $C(y) \setminus C(x) = \{i_1, i_2, \ldots, i_r\}$. Then,

$$(y, w) = (x^0, w) \xrightarrow{\{i_1\}} (x^1, w) \xrightarrow{\{i_2\}} (x^2, w) \xrightarrow{\{i_3\}} \ldots \xrightarrow{\{i_r\}} (x^r, w), \tag{38}$$

where $x^k \in X$ is defined to satisfy $C(x^k) = C(y) \setminus \{i_1, \ldots, i_k\}$ for each $k = 1, \ldots, r$. Note that $x^r = x \wedge y$.

Step 2 Cartel $C(x \wedge y)$ changes the price from w to $\hat{p}$. (Note that $C(x \wedge y) \neq \emptyset$ by assumption.) Thus,

$$(x \wedge y, w) \xrightarrow{C(x \wedge y)} (x \wedge y, \hat{p}). \tag{39}$$

Step 3 If $C(x) \setminus C(y) = \emptyset$, then go to the next step. Otherwise, consider a sequence in which each firm in $C(x) \setminus C(y)$ enters the cartel in turn. Let $C(x) \setminus C(y) = \{j_1, j_2, \ldots, j_{r'}\}$. Then,

$$(x^r, \hat{p}) \xrightarrow{\{j_1\}} (x^{r+1}, \hat{p}) \xrightarrow{\{j_2\}} (x^{r+2}, \hat{p}) \xrightarrow{\{j_3\}} \ldots \xrightarrow{\{j_{r'}\}} (x^{r+r'}, \hat{p}), \tag{40}$$

where $x^{r+k} \in X$ is defined to satisfy $C(x^{r+k}) = C(x \wedge y) \cup \{j_1, \ldots, j_k\}$ for each $k = 1, \ldots, r'$. Note that $x^{r+r'} = x$.

Step 4 Cartel $C(x)$ changes the price from $\hat{p}$ to p.

$$(x, \hat{p}) \xrightarrow{C(x)} (x, p). \tag{41}$$

Now we show that each firm in the sequence defined above has an incentive to deviate actually. For each i_k in Step 1, we have

$$g_{i_k}(x^{k-1}, w) = \pi_c(|x^{k-1}|, w) \leqq \pi_c(|y|, w) < \pi_f(p) = g_{i_k}(x, p), \tag{42}$$

where the second inequality follows from the size-monotonicity of π_c ((i) in Proposition 1) and the penultimate strict inequality is due to the condition given in this lemma. Thus, all the deviating firms in Step 1 have incentives to deviate toward the ultimate outcome (x, p).

In Step 2, we have

$$g_i(x \wedge y, w) = \pi_c(|x \wedge y|, w) < \pi_c(|x|, p) = g_i(x, p) \tag{43}$$

for all $i \in C(x \wedge y)$. (Note that $C(x \wedge y) \subset C(x)$.) The above inequality follows from the condition given in the lemma. Therefore, cartel $C(x \wedge y)$ has an incentive to change the price as in Step 2.

Moreover, for each deviating firm j_k in Step 3, we have

$$g_{j_k}(x^{r+k-1}, \hat{p}) = \pi_f(\hat{p}) < \pi_c(|x|, p) = g_{j_k}(x, p) \tag{44}$$

by the definition of $\hat{p}$. Thus, j_k is better off in (x, p) than in $(x^{r+k-1}, \hat{p})$.

For $\hat{p}$, we have $\pi_c(|x|, \hat{p}) \leqq \pi_f(\hat{p}) < \pi_c(|x|, p)$ by the definition of $\hat{p}$ and Proposition 1-(iii). Then, in Step 4, we have

$$g_i(x, \hat{p}) = \pi_c(|x|, \hat{p}) < \pi_c(|x|, p) = g_i(x, p) \tag{45}$$

for all $i \in C(x)$. Cartel $C(x)$ has an incentive to change their price to p. Hence, $(x, p) \gg (y, w)$ holds through this sequence of deviations.

C.2 Part (i)

Let $\hat{p} \in \mathbb{R}_+$ be the price such that

$$\pi_f(\hat{p}) < \pi_f(p) \quad \text{and} \quad \pi_c(|z|, \hat{p}) < \pi_c(|x|, p) \tag{46}$$

for any $z \in X$. Such a price $\hat{p}$ exists since $0 < d(0) < +\infty$.

Take an arbitrary $i \in C(y)$. Consider the following finite sequence of deviations:

Step 1 Cartel $C(y)$ changes its price from p to $\hat{p}$.

Step 2 Firms in $C(y) \setminus \{i\}$ exit from the cartel in turn.

Step 3 Firms in $C(x)$ enter the cartel in turn.

Step 4 Firm i exits from the cartel.

Step 5 Cartel $C(x)$ changes its price from $\hat{p}$ to p.

Applying almost the same argument as the proof of "Part (ii)," we can show the incentives of the deviating players in each step.

C.3 Part (iii)

Let $z = x \wedge y$; then $C(z) = C(x) \cap C(y)$. By the conditions given in the lemma, we have both $C(x) \setminus C(z) \neq \emptyset$ and $C(y) \setminus C(z) \neq \emptyset$. Consider the following sequence of deviations:

Step 1 Firms in $C(y) \setminus C(z)$ exit from $C(y)$ one by one until the cartel $C(z)$ is realized:

$$(y, w) = (x^0, w) \xrightarrow{\{i_1\}} (x^1, w) \xrightarrow{\{i_2\}} \cdots \xrightarrow{\{i_r\}} (x^r, w) = (z, w), \tag{47}$$

where $r \equiv |y| - |z|$.

Step 2 $C(z)$ decreases the price down to zero: $(z, p) \xrightarrow{C(z)} (z, 0)$.

Step 3 Firms in $C(x) \setminus C(z)$ enter the cartel until $C(x)$ is established:

$$\begin{aligned}(z, 0) = (x^{r+1}, 0) \xrightarrow{\{j_1\}} (x^{r+2}, 0) \xrightarrow{\{j_2\}} \cdots \\ \cdots \xrightarrow{\{j_{r'}\}} (x^{r+1+r'}, 0) = (x, 0),\end{aligned} \tag{48}$$

where $r' = |x| - |z|$.

Step 4 $C(x)$ increases the price up to p: $(x,0) \xrightarrow{C(x)} (x,p)$.

In Step 1, we have

$$\begin{aligned} g_{i_k}(x^{k-1},w) = \pi_c(|y|-k+1,w) &\leqq \pi_c(|y|,w) \\ &\leqq \pi_c(|x|,p) \\ &< \pi_f(p) = g_{i_k}(x,p) \end{aligned} \tag{49}$$

for all $k = 1,\ldots,r$, where the first inequality follows from the size-monotonicity of π_c, the second from the condition given in the lemma, and the third from Proposition 1-(iii). In Step 2, we have, if $\pi_c(|z|,w) = 0$,

$$g_i(z,w) = \pi_c(|z|,w) < \pi_c(|x|,p) = g_i(x,p) \tag{50}$$

and if $\pi_c(|z|,w) > 0$,

$$g_i(z,w) = \pi_c(|z|,w) < \pi_c(|y|,w) \leqq \pi_c(|x|,p) = g_i(x,p) \tag{51}$$

for all $i \in C(z)$. Thus, in each case, $g_i(x,p) > g_i(z,w)$ for all $i \in C(z)$. In Step 3, we have

$$g_{j_k}(x^{r+k},0) = \pi_f(0) = 0 < \pi_c(|x|,p) = g_{j_k}(x,p) \tag{52}$$

for all $k = 1,\ldots,r'$. And, in Step 4, we have

$$g_i(x,0) = \pi_c(|x|,0) < \pi_c(|x|,p) = g_i(x,p) \tag{53}$$

for all $i \in C(x)$. Hence, (x,p) indirectly dominates (y,w). □

D Proof of Lemma 4

We distinguish two cases: case 1 where $(x^f, p^{\text{comp}}) \in K$ and case 2 where $(x^f, p^{\text{comp}}) \notin K$. Note that $g_i(x^f,p^{\text{comp}}) = \pi_f(p^{\text{comp}}) = \pi_f^*(0) > 0$ for all $i \in N$.

First we consider case 1 where $(x^f, p^{\text{comp}}) \in K$. Take any $(y,w) \in K$ such that $\pi_c(y,w) \leqq 0$. Consider the following finite sequence of deviations from (y,w) to (x^f, p^{comp}):

Step 1 Cartel $C(y)$ changes its price from w to 0.

Step 2 Firms in $C(y)$ exit from the cartel in turn.

In Step 1, we have

$$g_i(y, w) = \pi_c(y, w) \leqq 0 < \pi_f(p^{\text{comp}}) = g_i(x^f, p^{\text{comp}}) \tag{54}$$

for all $i \in C(y)$.

In Step 2, for each deviant firm, the current profit of the deviant firm is 0 and thus is less than the profit of the firm in the final outcome (x^f, p^{comp}).

Therefore, (x^f, p^{comp}) indirectly dominates (y, w) through above sequence of deviations and this contradicts the internal stability of K.

Next we consider case 2 where $(x^f, p^{\text{comp}}) \notin K$. In this case, there must exist $(x, p) \in K$ such that $(x, p) \gg (x^f, p^{\text{comp}})$ to assure the external stability of K. We show that in the outcome (x, p), the firms in the cartel obtain a positive profit. In a dominance sequence that realizes $(x, p) \gg (x^f, p^{\text{comp}})$, there must be at least one firm, say firm i, who joins in the cartel at some step of the sequence and remains in the cartel at the final outcome because the initial outcome has no actual cartel. By the definition of the indirect dominance, we have

$$0 \leqq \pi_f(p^k) = g_i(x^k, p^k) < g_i(x, p) = \pi_c(x, p). \tag{55}$$

The first weak inequality follows from the definition of π_f. Thus, we obtain $0 \leqq \pi_f(p^k) < \pi_c(x, p) \leqq 0$.

Note that $\pi_c(x, p) > 0$ implies $\pi_f(p) > 0$. Then, we have $g_i(x, p) > 0$ for all $i \in N$. Finally, we show that for any $(y, w) \in K$ such that $\pi_c(y, w) \leqq 0$, (x, p) indirectly dominates (y, w). Consider a sequence of deviations from (y, w) to (x, p) such that the dominance sequence from (x^f, p^{comp}) to (x, p) follows the sequence from (y, w) to (x^f, p^{comp}) described as Step 1 and Step 2 in the first part of this proof. Then, it is easily verified that this sequence realizes $(x, p) \gg (y, w)$. This contradicts the internal stability of K. So we have the desired result. □

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