Fibre-reinforced materials with fibres resistant in bending

Kostas P. Soldatos

University of Nottingham

Seminar Outline

1. Introduction – Historical background
2. Perfectly flexible fibres – Plane kinematics
3. Fibres resistant in bending – Creeping fluid flow
4. Fibres resistant in bending – Hyper-elasticity
5. Plane deformations & equilibrium
6. Application I : Solids
7. Application II: Fluids
8. Closure

1. Introduction: fibre-reinforced materials

- Historical background (perfectly flexible fibres):

[1] J.E. Adkins, R.S. Rivlin (1955) *Philos. Trans. R. Soc. London A* **248**, 201.

[2] J.E. Adkins (1956) *J. Ration. Mech. Anal.* **5**, 189.

[3] J.E. Adkins (1955) *Philos. Trans. R. Soc. London A* **249**, 125.

[4] J.E. Adkins (1958) Q. *J. Mech. Appl. Math.* **11**, 88.

[5] A.J.M. Spencer (1972) *Deformations of Fibre-reinforced Materials*, Oxford U.P.

- In ideal fibre-reinforced materials, and regardless of material constitution, constitutive behaviour obeys:

$$\boldsymbol{\sigma} = -p\left(\mathbf{I} - \mathbf{a}\otimes\mathbf{a}\right) + T\mathbf{a}\otimes\mathbf{a} + \mathbf{s}$$

$\boldsymbol{\sigma},\ \mathbf{I}:$ Cauchy stress and unit tensors; $\mathbf{a}:$ Fibres direction vector;

$p,\ T:$ Reactions to incompressibility and inextensibility constraints;

$\mathbf{s}:$ Extra stress tensor, dependent on material constitution.

[6] T.G. Rogers (1992) *Proc. IUTAM Symp.*, Troy, N.Y., Springer, G.J. Dvorak, ed.

In the case of linear visco-elasticity:

$$\mathbf{s} = 2\int_{-\infty}^{t} \mu_T\left(t-\tau\right)\mathbf{d}(\mathbf{x},\tau)d\tau + 2\int_{-\infty}^{t}\left[\mu_L\left(t-\tau\right)-\mu_T\left(t-\tau\right)\right]\left[\mathbf{a}\otimes\mathbf{a}\,\mathbf{d}(\mathbf{x},\tau)+\mathbf{d}(\mathbf{x},\tau)\mathbf{a}\otimes\mathbf{a}\right]d\tau$$

t: time; $\boldsymbol{x}$: position vector;

$\boldsymbol{d} = d\boldsymbol{e}/dt$: rate-of-strain tensor ; $\boldsymbol{e}$: infinitesimal strain tensor;

μ_T and μ_L: appropriate, time-dependent relaxation functions.

Special case 1 (Linearly elastic behaviour): both μ_T and μ_L are given constants.

Special case 2 (Fluid behaviour): $\mu_T\left(t-\tau\right)=\eta_T\delta(t)$, $\mu_L\left(t-\tau\right)=\eta_L\delta(t)$,

$\delta(t)$: Dirac delta function;

η_T and η_L: appropriate viscosity parameters.

- Conventional kinematics (perfectly flexible fibres):
  (single family of fibres defined by a unit vector field)

Solids

$$x_i = x_i\left(X_Q\right),\; F_{iQ} = \frac{\partial x_i}{\partial X_Q},$$

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F},\quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$$

Fluids

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

Incompressibility constraint:

$$\det\left(\mathbf{F}\right) = 1$$

$$d_{ii} = \frac{\partial v_i}{\partial x_i} = 0$$

Fibre inextensibility constraint:

$$a_i = F_{iQ}A_Q \text{ or } A_R A_S C_{RS} = 1$$

$$a_i a_j d_{ij} = a_i a_j \frac{\partial v_i}{\partial x_j} = 0$$

or

$$\frac{Da_i}{Dt} = \frac{\partial a_i}{\partial t} + v_k \frac{\partial a_i}{\partial x_k} = a_k \frac{\partial v_i}{\partial x_k}$$

2. Conventional plane kinematics (x_1x_2-plane) Perfectly flexible fibres

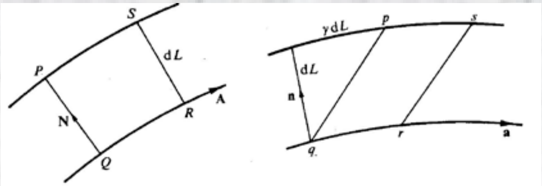
[5] A.J.M. Spencer (1972) *Deformations of Fibre-reinforced Materials*, Oxford U.P.

[7] A.C. Pipkin, T.G. Rogers (1971) *ASME J. Appl. Mech.* **38**, 634-640.

Fibres form a family of plane material curves (the *a*-curves); arch length: l_a

The orthogonal trajectories of the *a*-curves are called the *n*-curves;
arch length l_n and unit vector: $\boxed{\mathbf{n} = (n_1, n_2)^T = (-a_2, a_1)^T}$

The *a*- and the *n*-curves define a rectangular, curvilinear, local co-ordinate system:



Deformation of an initially rectangular element showing the amount of shear γ.

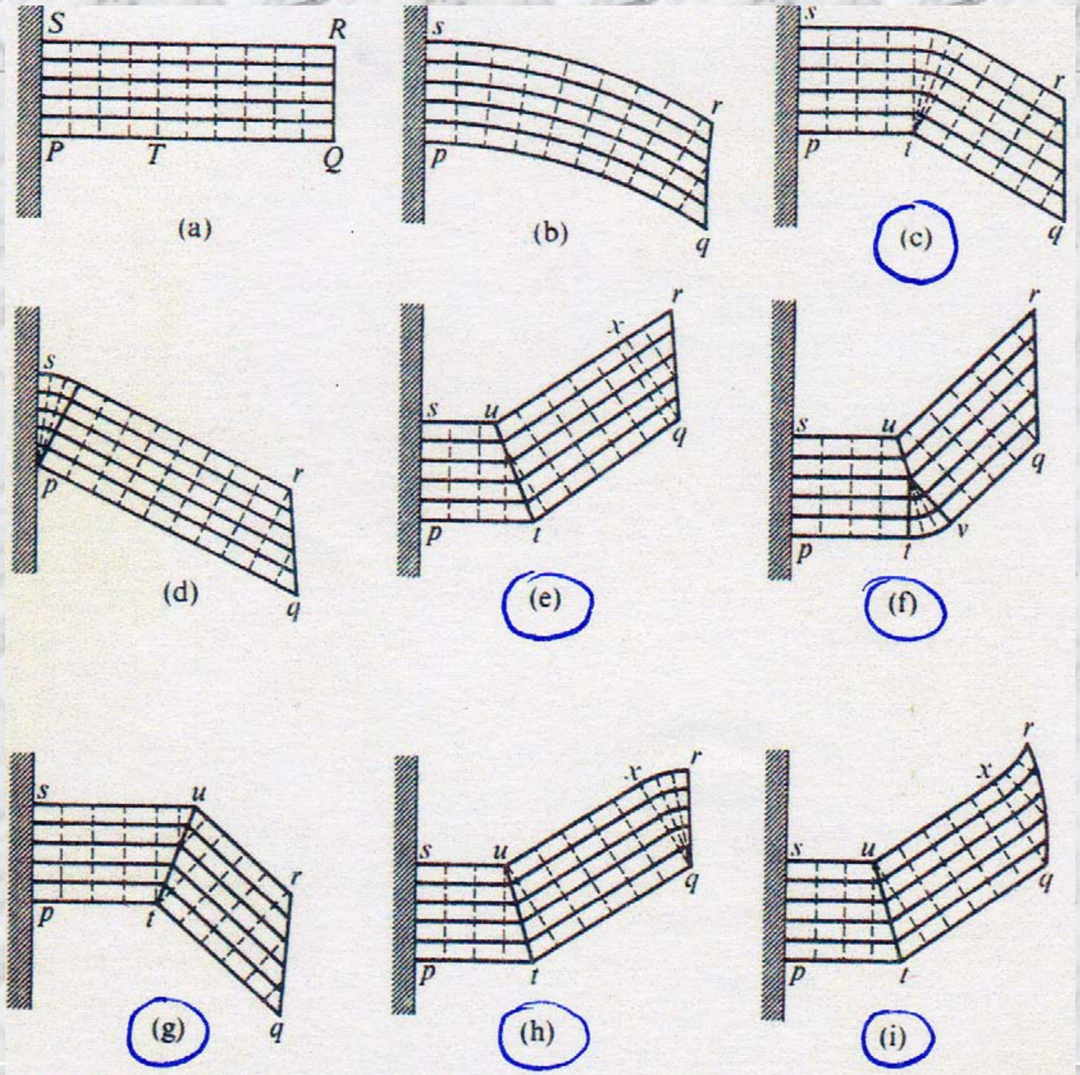
Material incompressibility proves further that the constant normal distance between two parallel *a*-curves (distance along n-curves) is conserved during deformation.

Hence, deformation of any material element is simple shear along inextensible fibres.

Plane kinematics are so restrictive that enable non-unique determination of kinematically admissible deformations without the use of equilibrium equations.

Example 1: Bending of a rectangular block

[5] A.J.M. Spencer (1972) *Deformations of Fibre-reinforced Materials*, Oxford U.P.



. 3.2. Kinematically admissible deformations of a rectangular block. Solid lines denote
a-curves, broken lines denote n-curves

Example 2: Isothermal forming flow of fibre-resin systems

[8] T.G. Rogers and J.M. O'Neil (1991) *Comp. Manuf.* **2**(3-4), 153-160.

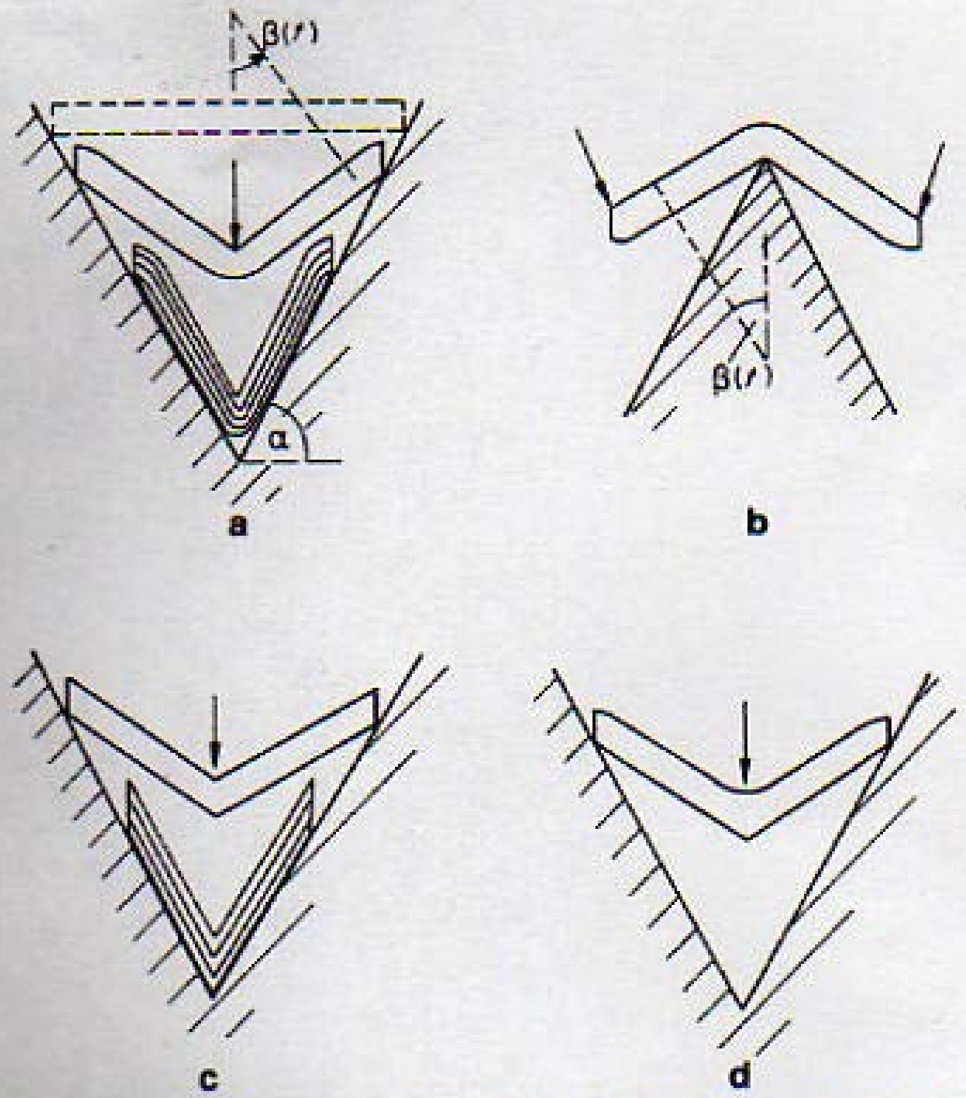
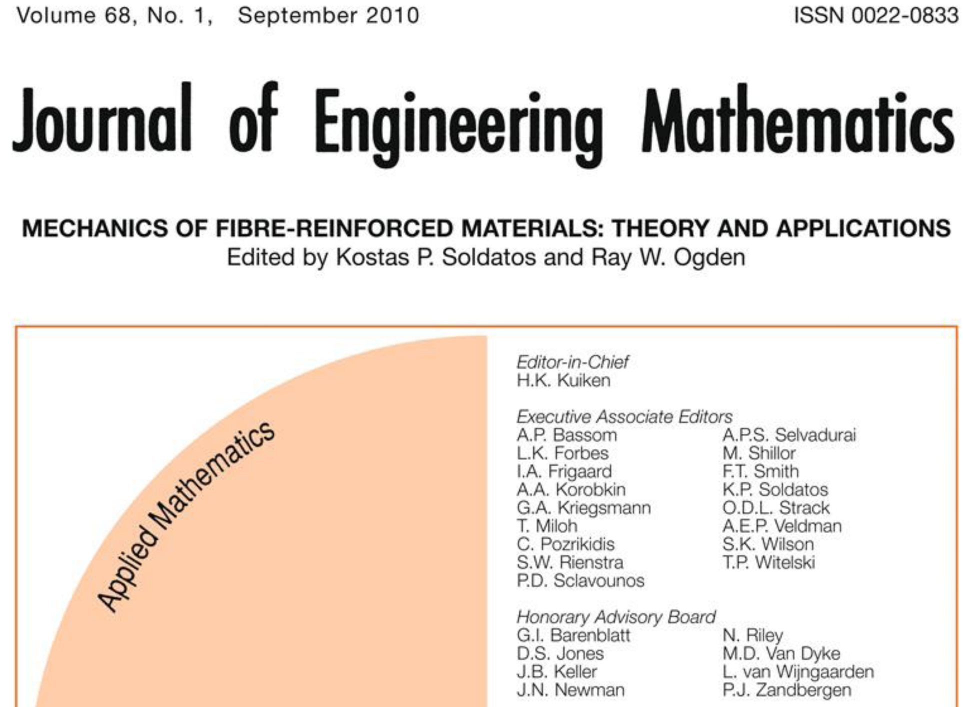


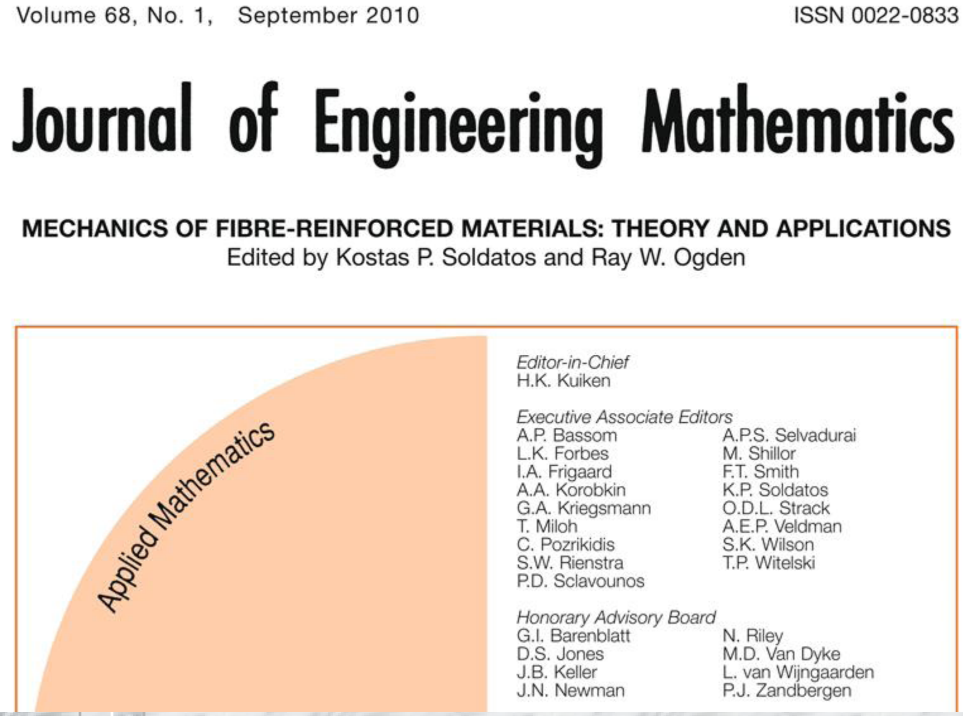
Figure 2 Deformations under concentrated line loads: (a), (b), admissible deformations; (c), kinematically admissible but statically inadmissible; (d), kinematically inadmissible



Kostas P. Soldatos:

Second-gradient plane deformations of ideal fibre-reinforced materials:
Implications of hyper-elasticity theory

J Eng Math (2010) 68:99–127
DOI 10.1007/s10665-009-9353-4



Volume 68, No. 1, September 2010 ISSN 0022-0833

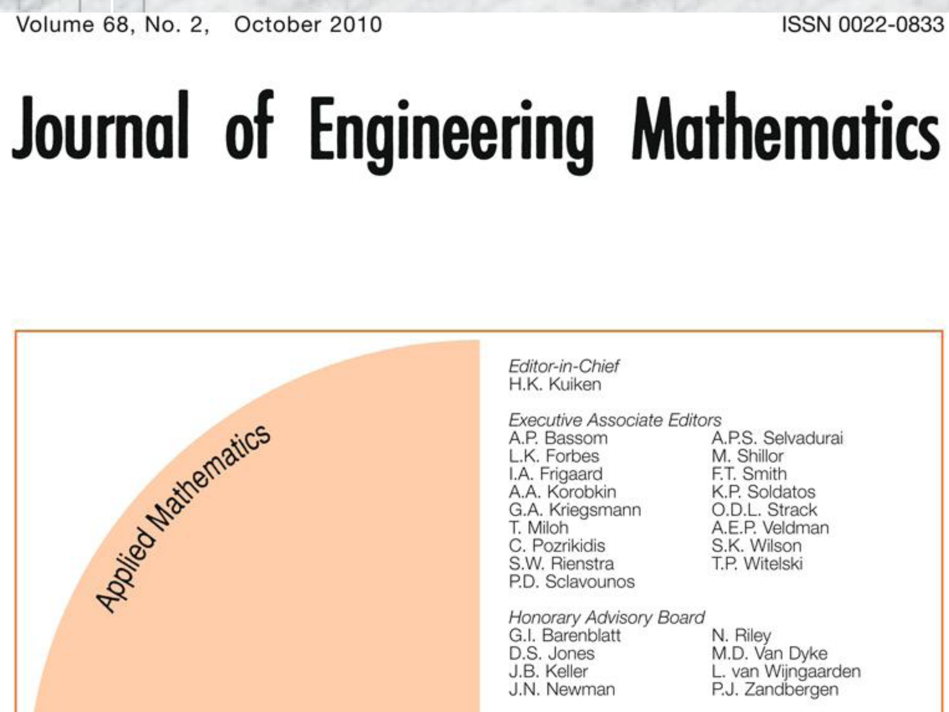
Journal of Engineering Mathematics

MECHANICS OF FIBRE-REINFORCED MATERIALS: THEORY AND APPLICATIONS
Edited by Kostas P. Soldatos and Ray W. Ogden

Kostas P. Soldatos:

Second-gradient plane deformations of ideal fibre-reinforced materials:
Implications of hyper-elasticity theory

J Eng Math (2010) 68:99–127
DOI 10.1007/s10665-009-9353-4



Volume 68, No. 2, October 2010 ISSN 0022-0833

Journal of Engineering Mathematics

Kostas P. Soldatos:

Second-gradient plane deformations of ideal fibre-reinforced materials II:
Forming flows of fibre–resin systems when fibres resist bending

J Eng Math (2010) 68:179–196
DOI 10.1007/s10665-010-9366-z

3. Fibres resistant in bending – Fluids

- Conventional kinematics (perfectly flexible fibres):

Rate of deformation tensor: $d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$

Spin tensor: $\Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$

Constraints:

$$d_{ii} = \frac{\partial v_i}{\partial x_i} = 0$$

$$\frac{Da_i}{Dt} = \frac{\partial a_i}{\partial t} + v_k \frac{\partial a_i}{\partial x_k} = a_k \frac{\partial v_i}{\partial x_k}$$

- Advanced kinematics - fibre bending resistance:

Fibres gradient tensor: $g_{ij} = \frac{\partial a_i}{\partial x_j}$ (full theory)

Curvature vector: $\kappa_i = g_{ij} a_j = \frac{\partial a_i}{\partial x_j} a_j$ (restricted version)

There exist couple-stresses (stress is tensor non-symmetric)

Couple-stress tensor: $\boldsymbol{m}$

- Equilibrium for slow (creeping) flows:

$$\frac{\partial \sigma_{ji}}{\partial x_j} + b_i = \rho f_i, \quad \frac{\partial m_{ji}}{\partial x_j} + e_{ijk}\sigma_{jk} = 0$$

Plane creeping flow: $\frac{\partial \sigma_{\beta\alpha}}{\partial x_\beta} = 0, \quad \sigma_{[21]} = (\sigma_{21} - \sigma_{21}) = \frac{\partial m_{\beta 3}}{\partial x_\beta}$

- Constitutive equations:

Postulate that $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{d}, \mathbf{a}, \boldsymbol{\kappa}, \rho, \theta)$ and $\mathbf{m} = \mathbf{m}(\mathbf{d}, \mathbf{a}, \boldsymbol{\kappa}, \rho, \theta)$

and ([9] Zheng, Q.-S. (1994) Appl. Mech. Rev., Vol. 47, 554-587, 1994)

use standard results of the theory of tensor representations. Then:

$$\sigma_{(\alpha\beta)} = -p(\delta_{\alpha\beta} - a_\alpha a_\beta) + T a_\alpha a_\beta + s_{(\alpha\beta)}, \qquad \sigma_{33} = -p + s_{33},$$

$$s_{(\alpha\beta)} = \alpha_1 d_{\alpha\beta} + \alpha_2 \kappa_\alpha \kappa_\beta,$$

$$m_{\alpha 3} = \beta_1 \kappa_\alpha + \beta_2 d_{\alpha\beta}\kappa_\beta .$$

The visco-metric functions α_1, α_2 and β_1, β_2 depend on θ and the invariants:

$$I_1 = tr\mathbf{d}^2, \quad I_2 = \boldsymbol{\kappa}\cdot\boldsymbol{\kappa}, \quad I_3 = \boldsymbol{\kappa}\cdot\mathbf{d}\boldsymbol{\kappa}$$

4. Fibres resistant in bending – Hyper-elasticity

- Conventional kinematics (perfectly flexible fibres):

$$\boxed{x_i = x_i\left(X_Q\right)}$$

Deformation gradient tensor: $F_{iQ} = \dfrac{\partial x_i}{\partial X_Q}$

Cauchy-Green deformation tensor: $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$

Constraints:

$$\det\left(\mathbf{F}\right) = 1$$

$$a_i = F_{iQ}A_Q \text{ or } A_R A_S C_{RS} = 1$$

- Advanced kinematics - fibres bending resistance:

Fibres gradient tensor: $\boxed{G_{iQ} = \dfrac{\partial a_i}{\partial X_Q}} = g_{ij}F_{jQ} = \dfrac{\partial}{\partial X_Q}\left(A_S \dfrac{\partial x_i}{\partial X_S}\right)$ (full theory)

Curvature vector: $\kappa_i = A_R G_{iR} = A_R \dfrac{\partial a_i}{\partial X_R}$ (restricted version)

- The strain energy density, W, is an isotropic invariant of:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \mathbf{A}, \mathbf{\Lambda} = \mathbf{F}\ \mathbf{G}^{\mathrm{T}}$$

through some appropriate number of strain measure invariants

Invariants involved in the full version of 3D theory:

[10] A.J.M. Spencer, K.P. Soldatos (2007) *Int. J. Non-lin. Mech.* **42**, 355.

$$I_1 = tr\mathbf{C}, \quad J_2 = I_2 = \tfrac{1}{2}\left[\left(tr\mathbf{C}\right)^2 - tr\mathbf{C}^2\right], \quad \left(I_3 = \det\mathbf{C}, \quad I_4 = \mathbf{ACA}\right) \quad I_5 = \mathbf{AC}^2\mathbf{A},$$

$$I_6 = tr\mathbf{\Lambda}^s = tr\mathbf{\Lambda}, \quad I_7 = tr\left(\mathbf{\Lambda}^s\right)^2, \quad I_8 = tr\left(\mathbf{\Lambda}^a\right)^2, \quad I_9 = tr\left(\mathbf{\Lambda}^s\right)^3, \quad I_{10} = tr\mathbf{C\Lambda}^s, \quad I_{11} = tr\mathbf{C}^2\mathbf{\Lambda}^s = tr\mathbf{C}^2\mathbf{\Lambda},$$

$$I_{12} = tr\mathbf{C}\left(\mathbf{\Lambda}^s\right)^2, \quad I_{13} = tr\mathbf{C}^2\left(\mathbf{\Lambda}^s\right)^2, \quad I_{14} = tr\mathbf{C}\left(\mathbf{\Lambda}^a\right)^2, \quad I_{15} = tr\mathbf{C}^2\left(\mathbf{\Lambda}^a\right)^2, \quad I_{16} = tr\mathbf{C}^2\left(\mathbf{\Lambda}^a\right)^2\mathbf{C\Lambda}^a,$$

$$I_{17} = tr\mathbf{\Lambda}^s\left(\mathbf{\Lambda}^a\right)^2, \quad I_{18} = tr\left(\mathbf{\Lambda}^s\right)^2\left(\mathbf{\Lambda}^a\right)^2, \quad I_{19} = tr\left(\mathbf{\Lambda}^s\right)^2\left(\mathbf{\Lambda}^a\right)^2\mathbf{\Lambda}^s\mathbf{\Lambda}^a, \quad I_{20} = \mathbf{A\Lambda}^s\mathbf{A}, \quad I_{21} = \mathbf{A}\left(\mathbf{\Lambda}^s\right)^2\mathbf{A},$$

$$I_{22} = \mathbf{A}\left(\mathbf{\Lambda}^a\right)^2\mathbf{A}, \quad I_{23} = \mathbf{AC\Lambda}^s\mathbf{A}, \quad I_{24} = \mathbf{AC\Lambda}^a\mathbf{A}, \quad I_{25} = \mathbf{AC}^2\mathbf{\Lambda}^a\mathbf{A}, \quad I_{26} = \mathbf{A\Lambda}^a\mathbf{C}\left(\mathbf{\Lambda}^a\right)^2\mathbf{A},$$

$$I_{27} = \mathbf{A\Lambda}^s\mathbf{\Lambda}^a\mathbf{A}, \quad I_{28} = \mathbf{A}\left(\mathbf{\Lambda}^s\right)^2\mathbf{\Lambda}^a\mathbf{A}, \quad I_{29} = \mathbf{A\Lambda}^a\mathbf{\Lambda}^s\left(\mathbf{\Lambda}^a\right)^2\mathbf{A}, \quad I_{30} = tr\mathbf{C\Lambda}^s\mathbf{\Lambda}^a, \quad I_{31} = tr\mathbf{C}^2\mathbf{\Lambda}^s\mathbf{\Lambda}^a,$$

$$I_{32} = tr\mathbf{C}\left(\mathbf{\Lambda}^s\right)^2\mathbf{\Lambda}^a, \quad I_{33} = tr\mathbf{C}\left(\mathbf{\Lambda}^a\right)^2\mathbf{\Lambda}^s\mathbf{\Lambda}^a.$$

The resulting constitutive equations look very complicated

$$\sigma_{(ij)} \equiv \tfrac{1}{2}\left(\sigma_{ij} + \sigma_{ji}\right) = \frac{\rho}{\rho_0}\sum_{\alpha=1}^{33}\frac{\partial W}{\partial I_\alpha}\left\{F_{iQ}F_{jS}\left(\frac{\partial I_\alpha}{\partial C_{QS}} + \frac{\partial I_\alpha}{\partial C_{SQ}}\right) + \left(G_{iQ}F_{jS} + G_{jQ}F_{iS}\right)\frac{\partial I_\alpha}{\partial \Lambda_{SQ}}\right\},$$

$$\bar{m}_{ji} = \frac{2}{3}e_{ikm}\frac{\rho}{\rho_0}\sum_{\alpha=1}^{33}\frac{\partial W}{\partial I_\alpha}\frac{\partial I_\alpha}{\partial \Lambda_{PQ}}F_{mP}\left(F_{jQ}a_k + F_{kQ}a_j\right), \qquad \sigma_{[ij]} \equiv \tfrac{1}{2}\left(\sigma_{ij} - \sigma_{ji}\right) = \frac{1}{2}e_{jik}\bar{m}_{\ell k,\ell}.$$

- Plane strain of hyper-elastic solids:

Restricted 2D theory [$x_\alpha = x_\alpha(X_\Gamma)$]

$$\hat{J}_1 = I_1 = tr\hat{\mathbf{C}}, \quad \hat{J}_2 = \hat{\boldsymbol{\Lambda}}^T \hat{\boldsymbol{\Lambda}}, \quad \hat{J}_3 = \hat{\boldsymbol{\Lambda}}\hat{\mathbf{C}}\hat{\boldsymbol{\Lambda}} \quad : \quad \hat{W} = \hat{W}\left(\hat{J}_1, \hat{J}_2, \hat{J}_3\right)$$

Resulting constitutive equations:

$$\sigma_{(\alpha\beta)} \equiv \tfrac{1}{2}\left(\sigma_{\alpha\beta} + \sigma_{\beta\alpha}\right) = -p\delta_{\alpha\beta} + Ta_\alpha a_\beta + \overline{s}_{(\alpha\beta)}, \qquad \sigma_{33} = -p + \overline{s}_{33},$$

$$\overline{s}_{(\alpha\beta)} = \sum_{\ell=1}^{3} \frac{\partial \hat{W}}{\partial \hat{J}_\ell} \left\{ \hat{F}_{\alpha\Gamma}\hat{F}_{\beta\Delta}\left(\frac{\partial \hat{J}_\ell}{\partial \hat{C}_{\Gamma\Delta}} + \frac{\partial \hat{J}_\ell}{\partial \hat{C}_{\Delta\Gamma}} \right) + \left(\hat{F}_{\alpha\Gamma}\hat{G}_{\beta\Delta} + \hat{F}_{\alpha\Delta}\hat{G}_{\beta\Gamma} \right) \frac{\partial \hat{J}_\ell}{\partial \hat{\Lambda}_{\Gamma\Delta}} \right\},$$

$$\overline{s}_{33} = 2\sum_{\ell=1}^{3} \frac{\partial W}{\partial \hat{J}_\ell} \frac{\partial \hat{J}_\ell}{\partial C_{33}} \Big|_{C_{33}=1},$$

$$\sigma_{[21]} = s_{[21]} = \frac{1}{2} m_{\alpha 3,\alpha}, \quad m_{\alpha 3} = \frac{2}{3} e_{3\beta\gamma} \sum_{\ell=1}^{3} \frac{\partial \hat{W}}{\partial \hat{J}_\ell} \frac{\partial \hat{J}_\ell}{\partial \hat{\Lambda}_{\Delta\Gamma}} \hat{F}_{\gamma\Delta}\left(\hat{F}_{\alpha\Gamma} a_\beta + \hat{F}_{\beta\Gamma} a_\alpha \right).$$

- Customary modification of extra stress:

$$s_{ii} \equiv s_{\alpha\alpha} + s_{33} = 0, \qquad s_{(\alpha\beta)} a_\alpha a_\beta = 0$$

Final form of constitutive equations ($\hat{W}_k = \dfrac{\partial \hat{W}}{\partial \hat{J}_k}$):

$$\boxed{\sigma_{(\alpha\beta)} = -p\delta_{\alpha\beta} + T a_\alpha a_\beta + s_{(\alpha\beta)}, \quad \sigma_{33} = -p + s_{33},}$$

$$s_{(\alpha\beta)} = \boxed{2\hat{W}_1 \hat{F}_{\alpha\Gamma} \hat{F}_{\beta\Gamma}} + A_\Delta \left(\hat{F}_{\alpha\Gamma} \hat{G}_{\beta\Delta} + \hat{F}_{\beta\Gamma} \hat{G}_{\alpha\Delta} \right) \left(2\hat{W}_2 \hat{\Lambda}_{\Gamma\Omega} A_\Omega + \hat{W}_3 C_{\Gamma\Omega} A_\Omega \right)$$
$$+ 2\hat{W}_3 \hat{F}_{\alpha\Gamma} A_\Gamma \hat{F}_{\beta\Delta} \hat{\Lambda}_{\Gamma\Omega} A_\Omega - \left\{ \boxed{2\hat{W}_1} + 2\hat{J}_2 \hat{W}_2 + \hat{J}_3 \hat{W}_4 \right\} \delta_{\alpha\beta}$$
$$- \left\{ \boxed{2\left(\hat{J}_1 - 2 \right) \hat{W}_1} - 2\hat{J}_2 \hat{W}_2 + \hat{J}_3 \hat{W}_3 \right\} a_\alpha a_\beta,$$

$$\boxed{s_{33} = 2\hat{W}_1},$$

$$\sigma_{[21]} = s_{[21]} \equiv \tfrac{1}{2} \left(\sigma_{21} - \sigma_{12} \right) = \frac{1}{2} m_{\alpha 3,\alpha},$$

$$m_{\alpha 3} = \frac{2}{3} e_{3\beta\gamma} \hat{F}_{\gamma\Delta} \left(\hat{F}_{\alpha\Gamma} \hat{F}_{\beta\Lambda} + \hat{F}_{\beta\Gamma} \hat{F}_{\alpha\Lambda} \right) A_\Lambda \left(2\hat{W}_2 A_\Gamma \hat{\Lambda}_{\Delta\Sigma} A_\Sigma + \hat{W}_3 C_{\Gamma\Sigma} A_\Sigma A_\Delta \right).$$

5. Plane deformations (x_1x_2-plane)

[5] A.J.M. Spencer (1972) *Deformations of Fibre-reinforced Materials*, Oxford U.P.

[7] A.C. Pipkin, T.G. Rogers (1971) *ASME J. Appl. Mech.* **38**, 634-640.

Deformed fibres form a family of plane material curves (the *a*-curves);
curvature κ_α, arch length l_a and unit vector: $\mathbf{a} = \left(a_1, a_2\right)^T$ with $\boxed{a_\alpha = \hat{F}_{\alpha\Gamma} A_\Gamma}$

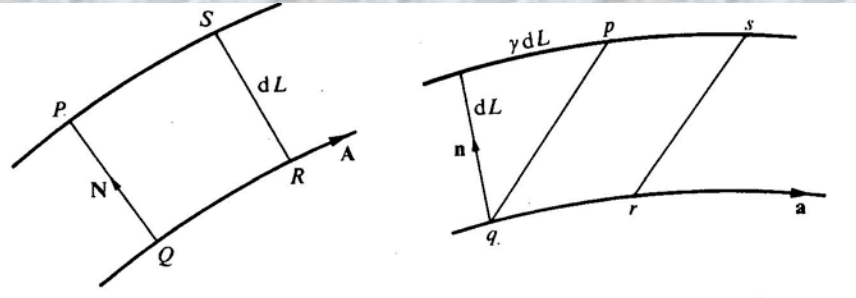
The orthogonal trajectories of the *a*-curves are called the *n*-curves;
curvature κ_n, arch length l_n and unit vector: $\boxed{\mathbf{n} = \left(n_1, n_2\right)^T = \left(-a_2, a_1\right)^T}$

Corresponding families of curves in the un-deformed configuration are noted as *A*-curves and *N*-curves; curvatures κ_A and κ_N, respectively.

Material incompressibility requires: $\boxed{\kappa_n = \kappa_N}$

Curvature definitions: $\kappa_n = a_{\alpha,\alpha}, \quad \kappa_a = -n_{\alpha,\alpha}.$

Plane strain kinematics are so restrictive that enable non-unique determination of kinematically admissible deformations without use of equilibrium equations.



Deformation of an initially rectangular element showing the amount of shear γ.

The *a*- and the *n*-curves define a rectangular, curvilinear, local co-ordinate system:

- Symmetric part of the extra stress: $s_{(\alpha\beta)} = s_a a_\alpha a_\beta + s_n n_\alpha n_\beta + \tau^s \left(a_\alpha n_\beta + a_\beta n_\alpha \right)$
- Anti-symmetric part: $s_{[\alpha\beta]} = -\tau^a \left(a_\alpha n_\beta - a_\beta n_\alpha \right)$
- Constraint equations: $\left\{ \begin{array}{c} s_{ii} = s_a + s_n + s_{33} = 0 \\ \boxed{s_\alpha = 0} \end{array} \right\} \Rightarrow \boxed{s_n = -s_{33}}$
- Multiplication of these equations by $a_\alpha n_\beta$ yields: $\boxed{\begin{array}{l} \tau^s = s_{(\alpha\beta)} a_\alpha n_\beta , \\ \tau^a = s_{[21]} = \frac{1}{2} m_{\alpha 3,\alpha} . \end{array}}$

Hence, all extra stress components are completely determined with use of kinematic relations and constitutive equations only (no use of equilibrium equations yet).

5.1 Static/quasi-static equilibrium: $\sigma_{\beta\alpha,\beta} = 0$

With *p* and *T* functions of x_1 and x_2 only, use of the Serret-Frenet relations yields

$$\frac{\partial}{\partial l_a}\left(-p+T\right)+\kappa_n T + a_\alpha s_{\beta\alpha,\beta} = 0, \qquad -\frac{\partial p}{\partial l_n}+\kappa_a T + n_\alpha s_{\beta\alpha,\beta} = 0. \qquad (\S)$$

Further development of (§) is possible and yields

$$\frac{\partial t_a}{\partial l_a}+\kappa_n\left(t_a-t_n\right)=2\kappa_a\tau^s-\frac{\partial\left(\tau^s-\tau^a\right)}{\partial l_n}, \qquad \frac{\partial t_n}{\partial l_n}+\kappa_a\left(t_a-t_n\right)=-2\kappa_n\tau^s-\frac{\partial\left(\tau^s+\tau^a\right)}{\partial l_a}, \quad (\S\S)$$

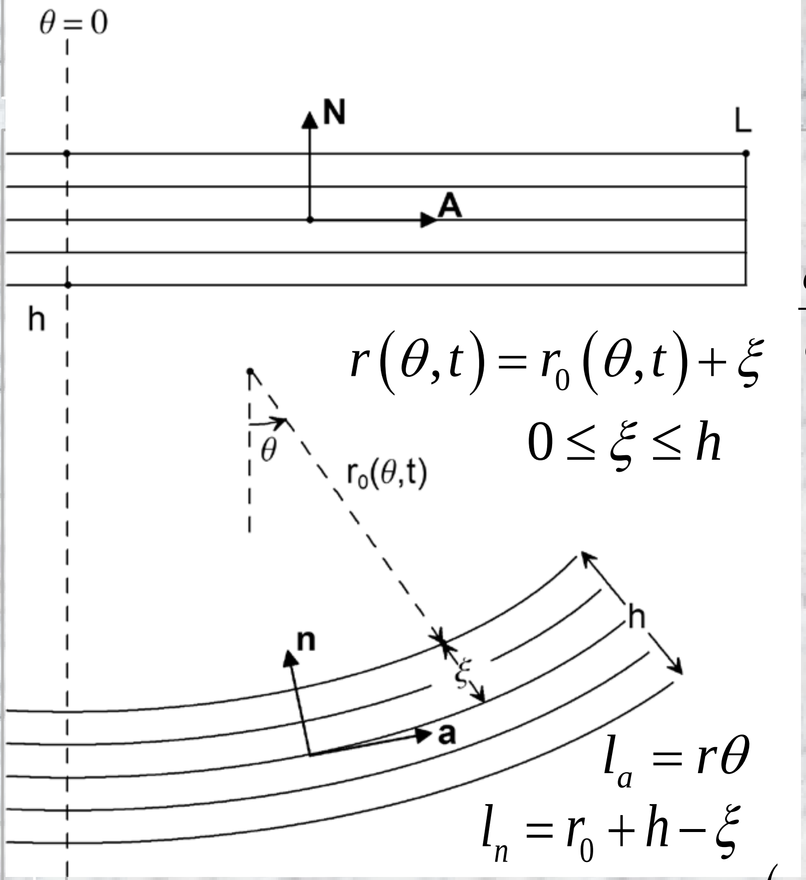
where the transformed unknowns ("tensions" along the *a*- and *n*-curves) are

$$\boxed{t_a = -p+T, \quad t_n = -p - s_{33}}$$

Solution of (§§) will finally yield the stresses as follows:

$$\sigma_{\alpha\beta} = t_a a_\alpha a_\beta + t_n n_\alpha n_\beta + \left(\tau^s-\tau^a\right)a_\alpha n_\beta + \left(\tau^s+\tau^a\right)a_\beta n_\alpha, \qquad \sigma_{33} = t_n + 2s_{33}.$$

5.2 Parallel fibres($\kappa_n = 0$): The stress resultants technique



Integrate through the thickness both of the simplified equations of equilibrium:

$$\frac{\partial t_\theta}{\partial \theta} = 2\tau^s + r\frac{\partial(\tau^s - \tau^a)}{\partial \xi}, \quad r\frac{\partial t_r}{\partial \xi} + t_r = t_\theta + \frac{\partial(\tau^s + \tau^a)}{\partial \theta}.$$

and introduce the stress resultants:

$$\hat{T} = \int_0^h t_\theta d\xi, \quad \hat{S} = \int_0^h (\tau^s + \tau^a) d\xi$$

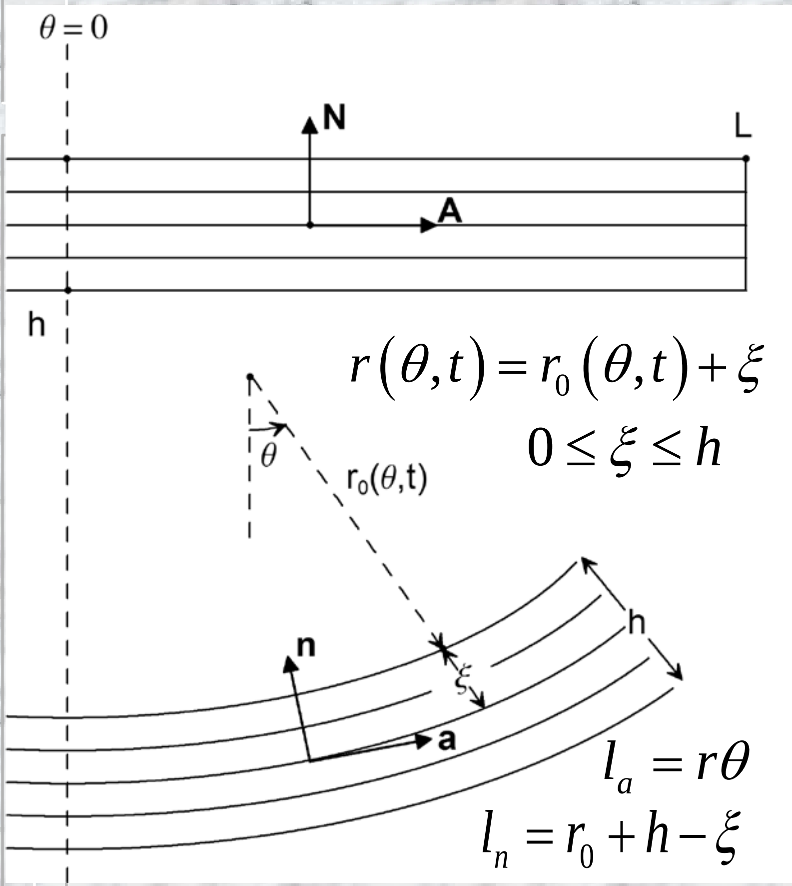
Then, eliminate $\hat{T}$ in order to obtain:

$$r_0(\tau_0^s - \tau_0^a) - r_1(\tau_1^s - \tau_1^a) + \frac{\partial}{\partial \theta}(r_1 t_{r1} - r_0 t_{r0}) = \frac{\partial^2 \hat{S}}{\partial \theta^2} + \hat{S}$$

This is an integro-differential equation for the principal unknown: $r_0(\theta,t)$. In most fluid mechanics applications, several boundary terms in the LHS are zero.

.

Fibre-reinforced fluids containing parallel fibres:



Basic integro-differential equation:

$$r_0\left(\tau_0^s - \tau_0^a\right) - r_1\left(\tau_1^s - \tau_1^a\right) + \frac{\partial}{\partial\theta}\left(r_1 t_{r1} - r_0 t_{r0}\right) = \frac{\partial^2 \hat{S}}{\partial\theta^2} + \hat{S}$$

Stress resultants:

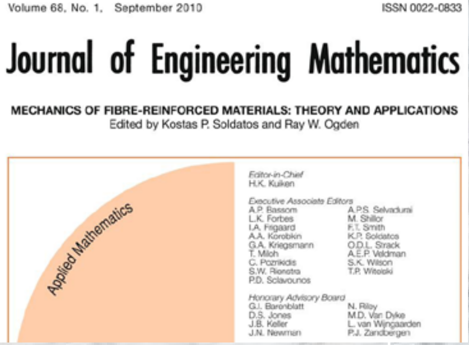
$$\hat{T} = \int_0^h t_\theta \, d\xi, \quad \hat{S} = \int_0^h \left(\tau^s + \tau^a\right) d\xi$$

where

$$\tau^s = \alpha_1 \dot{\gamma}/2, \quad \tau^a = \frac{1}{2r}\left[\frac{\partial}{\partial r}\left(\beta_2 \dot{\gamma}/2\right) - \frac{\partial \beta_1}{\partial\theta}\right],$$

with visco-metric parameters dependent on

$$I_1 = \mathrm{tr}\,\mathbf{d}^2 = 2\dot{\gamma}^2, \quad I_2 = \boldsymbol{\kappa}\cdot\boldsymbol{\kappa} = r^{-2} = \left(r_0 + \xi\right)^{-2}$$



6. Application I:

"Area-preserving" azimuthal shear strain

Undeformed Tube's Geometry (Polar co-ordinates)

$$B_0 \leq R \leq B_1, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L$$

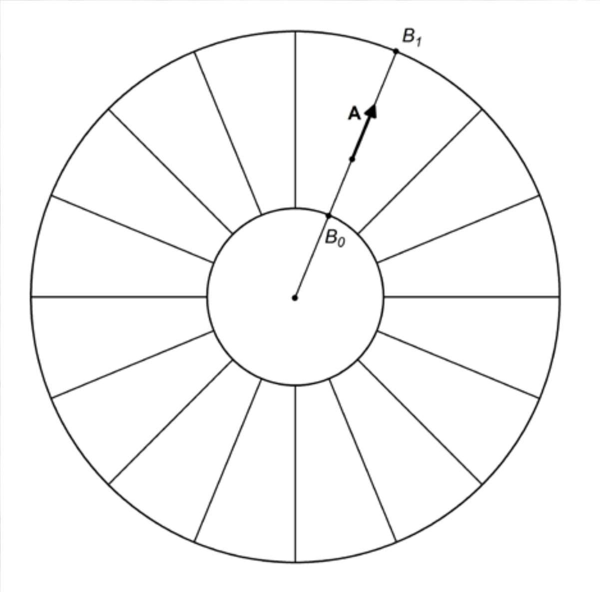
Fibres direction: $A_R = 1, \quad A_\Theta = 0$

Curvatures: $\kappa_A = 0, \quad \kappa_N = 1/R$

Deformation pattern (plane strain)

$$r = \left(R^2 + c\right)^{1/2}, \quad \theta = \Theta + g\left(R\right), \quad z = Z \quad \left(c > -B_0^2\right)$$

Undeformed cross-section



Unknowns to be determined

c and $g\left(R\right) = \theta - \Theta$

Boundary conditions

at $R = B_1$: $\theta - \Theta = g\left(B_1\right) = \psi, \quad t_{rr} = 0$

at $R = B_0$: $\theta - \Theta = g\left(B_0\right) = 0, \quad t_{rr} = 0$

Special case: Pure azimuthal shear strain (c = 0)

[11] R.S. Rivlin (1949) *Phil. Trans. R. Soc. London A*, **242**, 173-195.

[12] F. Kassianidis, R.W. Ogden, J. Merodio, T.J. Pence (2008) *Math. Mech. Solids*, **13**, 690-724.

Boundary conditions

at $R = B_1: \quad \theta - \Theta = g\left(B_1\right) = \psi, \quad t_{rr} = 0$

at $R = B_0: \quad \theta - \Theta = g\left(B_0\right) = 0, \quad t_{rr} = 0$

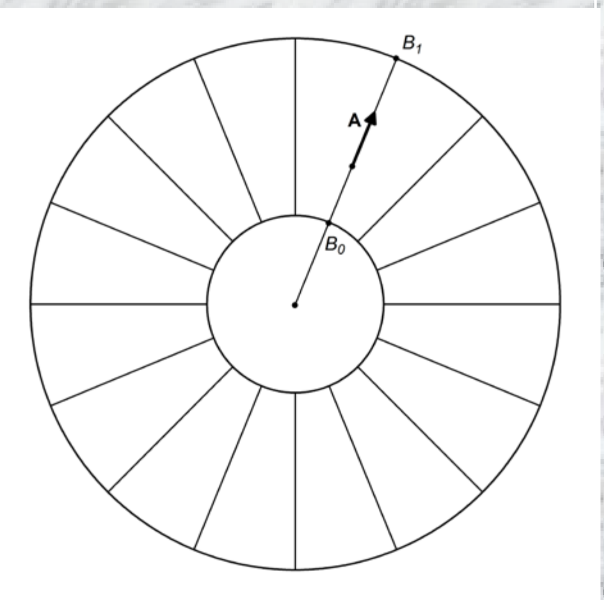
Deformation pattern (plane strain)

$$r = R, \quad \theta = \Theta + g\left(R\right), \quad z = Z$$

Unknown to be determined

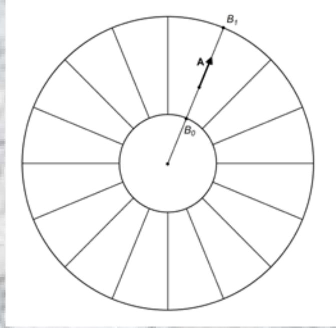
$$g\left(R\right) = \theta - \Theta$$

Undeformed cross-section



Not possible when fibres are inextensible

,



Deformation Gradient and Cauchy-Green Deformation Tensors

$$\mathbf{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{\partial r}{R\partial\Theta} \\ \dfrac{r\partial\theta}{\partial R} & \dfrac{r\partial\theta}{R\partial\Theta} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ \gamma & \lambda^{-1} \end{bmatrix}, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} \lambda^2+\gamma^2 & \gamma\lambda^{-1} \\ \gamma\lambda^{-1} & \lambda^{-2} \end{bmatrix}$$

$\lambda = R/r$: radial stretch $\qquad \gamma = \lambda^{-1}Rg'(R)$: amount of shear

$det(\boldsymbol{F}) = 1$: material incompressibility constraint is satisfied

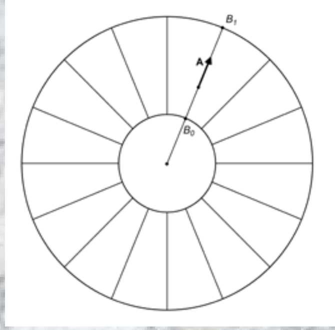
Unit tangent of deformed fibres: $\mathbf{a} = \mathbf{FA} \Rightarrow a_r = \lambda, \quad a_\theta = \gamma$

Fibres inextensibility constraint: $\mathbf{a}\cdot\mathbf{a} = 1 \Rightarrow g(R) = \pm\left[\tan^{-1}\left(R/\sqrt{c}\right) - \psi_0(c)\right]$

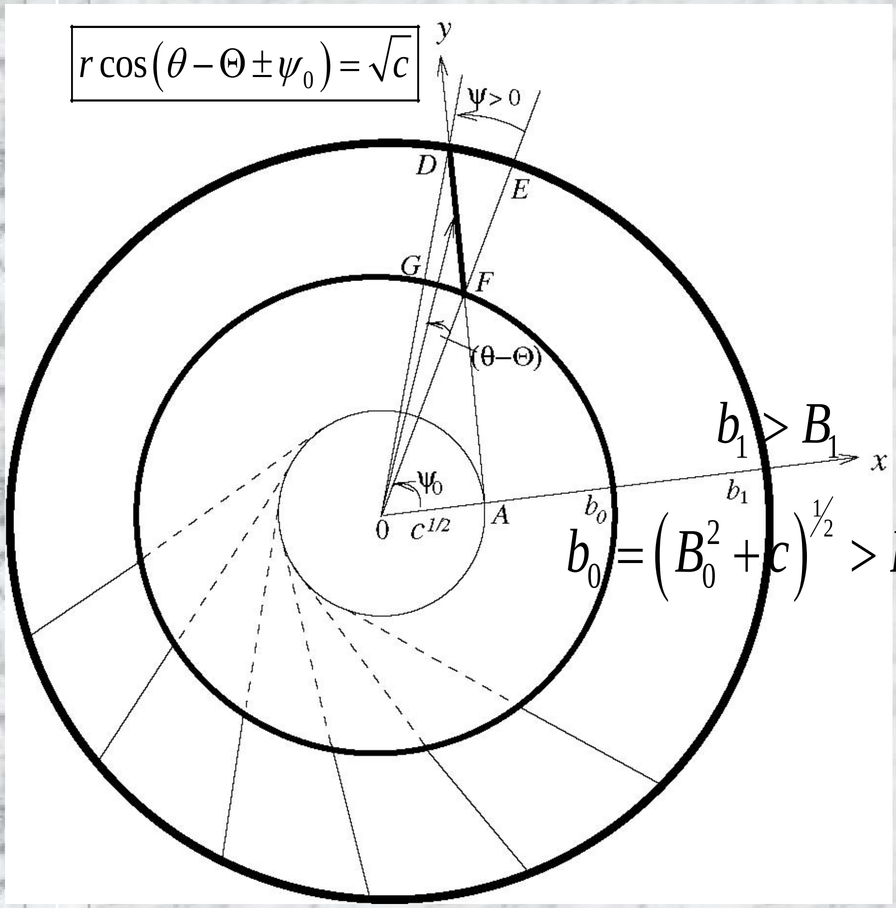
Boundary condition at $R = B_0$: $\theta - \Theta = g\left(B_0\right) = 0 \Rightarrow \psi_0(c) = \tan^{-1}\left(B_0/\sqrt{c}\right)$.

Deformation: $\theta = \Theta + g\left(R\right) \Rightarrow \boxed{r\cos\left(\theta - \Theta \pm \psi_0\right) = \sqrt{c}}$ (straight line)

Deformed cross-section (Polars)



$$b_0 \le r = \left(R^2 + c\right)^{\frac{1}{2}} \le b_1, \quad 0 \le \theta < 2\pi$$



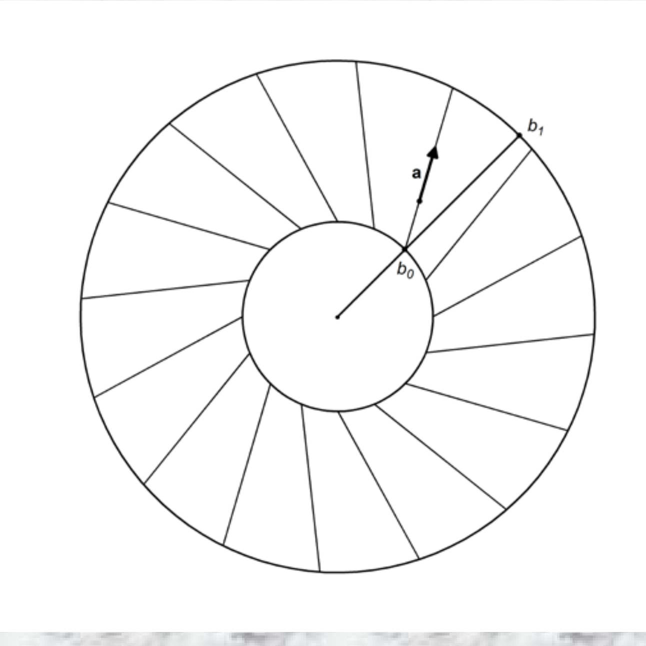
$$b_1 > B_1$$

$$b_0 = \left(B_0^2 + c\right)^{\frac{1}{2}} > B_0$$

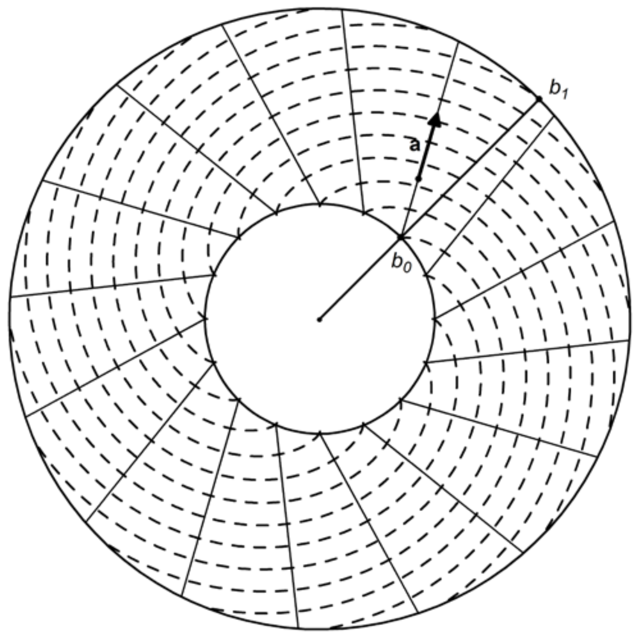
Boundary condition at $R = B_1$

$$\theta - \Theta = g\left(B_1\right) = \psi$$

$$\boxed{\sqrt{c} = \frac{\left(B_1 - B_0\right) \pm \sqrt{\left(B_1 - B_0\right)^2 - 4B_0 B_1 \tan^2\psi}}{2\left|\tan\psi\right|}}$$



The local co-ordinate system



Unit tangent of *n*-curves:

$$\mathbf{n} = (n_r, n_\theta)^T = (-a_\theta, a_r)^T = (-\gamma, \lambda)^T$$

Equation of the *n*-curves:

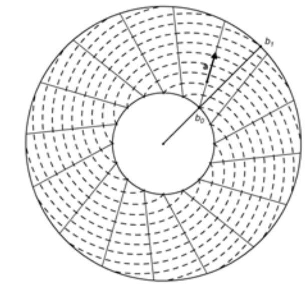
$$\theta - \Theta = \pm\left[\left(\frac{r^2}{c} - 1\right)^{1/2} - \sec^{-1}\left(\frac{r}{\sqrt{c}}\right)\right]$$

Curvature of the *a*- and the *n*-curves: $\kappa_a = 0, \quad \kappa_n = 1/R = \left(r^2 - c\right)^{-1/2} = 1/y$

Second gradient deformation tensor: $\mathbf{\Lambda} = \begin{bmatrix} 0 & 0 \\ -\gamma^2 c^{-1/2} & R^{-1} \end{bmatrix}$

Strain measure invariants: $\boxed{\tilde{J}_1 = 1 - \lambda^{-2}}, \quad \tilde{J}_2 = \gamma^4 c^{-1}, \; \tilde{J}_3 = -\gamma^3 \lambda^{-1} c^{-1/2}$

Equilibrium equations in local co-ordinates



$$\frac{\partial t_y}{\partial y} + y^{-1} t_y = y^{-1} t_n - \frac{\partial\left(\tau^s - \tau^a\right)}{\partial l_n}, \qquad \frac{\partial t_n}{\partial l_n} = -2\left(r^2 - c\right)^{-1/2} \tau^s - \frac{\partial\left(\tau^s + \tau^a\right)}{\partial y}$$

Solution along the n-curves: $$t_n = -\int_{b_0}^{r} \left(\frac{\lambda}{r} \frac{\partial\theta}{\partial r} - \gamma \right)^{-1} \left[\frac{2\tau^s}{\left(r^2 - c\right)^{1/2}} + \frac{\partial\left(\tau^s + \tau^a\right)}{\partial y} \right] dr + f_1(y)$$

where $\dfrac{\partial\theta}{\partial r} = \left(c^{-1} - r^{-2}\right)^{1/2}, \quad b_0 \le r \le b_1$ and $B_0 \le y \le B_1.$

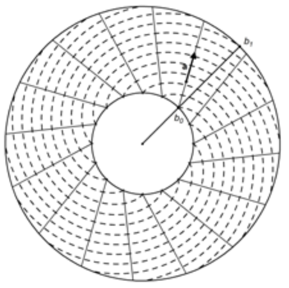
Solution along the n-curves: $$t_y = -y^{-1} \int_{B_0}^{y} \left(t_n - y \frac{\partial\left(\tau^s - \tau^a\right)}{\partial l_n} \right) dy + y^{-1} f_2\left(l_n\right)$$

Remaining boundary conditions:

at $r = b_k$ $(k = 0, 1)$: $t_{rr} = 0 \quad \Rightarrow \quad t_y\Big|_{y=B_k} + \dfrac{c}{B_k^2} t_n\Big|_{r=b_k} = -\dfrac{2\sqrt{c}}{B_k} \tau^s\Big|_{r=b_k}$

Hence

$$t_n = t(r) - \frac{B_0 B_1}{(B_1 - B_0)(c + B_0 B_1)} \int_{b_0}^{b_1} t(r)\, dr$$



$$t_y = -y^{-1} \int_{B_0}^{y} \left(t_n - y \frac{\partial(\tau^s - \tau^a)}{\partial l_n} \right) dy + \frac{B_1}{(B_1 - B_0)} y^{-1} \int_{B_0}^{B_1} \left(t_n - y \frac{\partial(\tau^s - \tau^a)}{\partial l_n} \right) dy$$
$$+ \frac{c}{(B_1 - B_0)} y^{-1} \int_{b_0}^{b_1} \left(\frac{\lambda}{r} \frac{\partial \theta}{\partial r} - \gamma \right)^{-1} \left[\frac{2\tau^s}{(r^2 - c)^{1/2}} + \frac{\partial(\tau^s + \tau^a)}{\partial y} \right] dr$$
$$- 2\sqrt{c}\, y^{-1} \tau^s \Big|_{r=b_0} - \frac{2\sqrt{c} B_1 y^{-1}}{(B_1 - B_0)} \left(\tau^s \Big|_{r=b_1} - \tau^s \Big|_{r=b_0} \right),$$

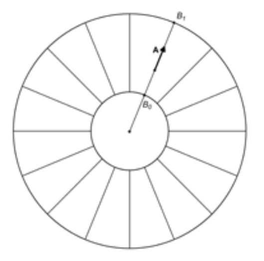
where

$$t(r) = \int_{r}^{b_0} \left(\frac{\lambda}{r} \frac{\partial \theta}{\partial r} - \gamma \right)^{-1} \left[\frac{2\tau^s}{(r^2 - c)^{1/2}} + \frac{\partial(\tau^s + \tau^a)}{\partial y} \right] dr - \frac{B_0}{\sqrt{c}\,(B_1 - B_0)} \int_{b_0}^{b_1} \left(\frac{\lambda}{r} \frac{\partial \theta}{\partial r} - \gamma \right)^{-1} \left[\frac{2\tau^s}{(r^2 - c)^{1/2}} + \frac{\partial(\tau^s + \tau^a)}{\partial y} \right] dr$$
$$+ \frac{B_0 B_1}{c(B_1 - B_0)} \int_{B_0}^{B_1} \frac{\partial(\tau^s - \tau^a)}{\partial l_n} y\, dy + \frac{2 B_0 B_1}{\sqrt{c}\,(B_1 - B_0)} \left(\tau^s \Big|_{r=b_1} - \tau^s \Big|_{r=b_0} \right).$$

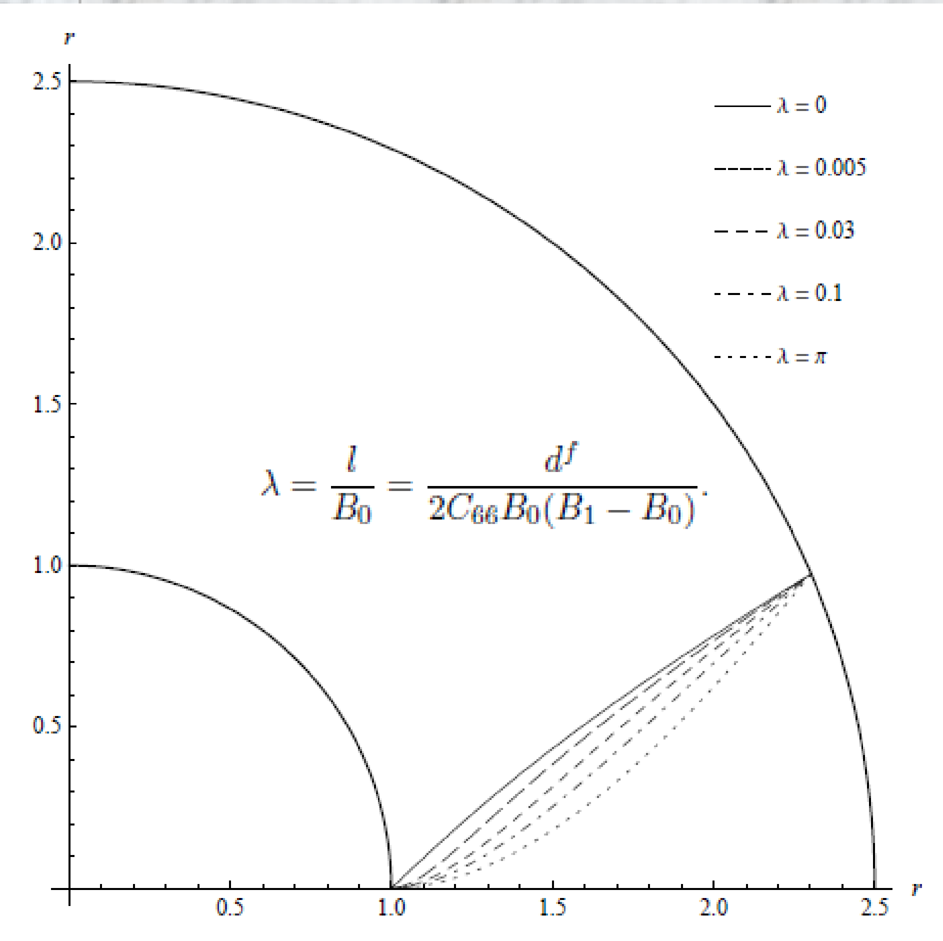
Polar form of stresses:

$$\sigma_{\alpha\beta} = t_y a_\alpha a_\beta + t_n n_\alpha n_\beta + (\tau^s - \tau^a) a_\alpha n_\beta + (\tau^s + \tau^a) a_\beta n_\alpha,$$
$$\sigma_{33} = t_n + 2 s_{33}.$$

6.1 The linear problem – Small azimuthal shear strain



[13] M.A. Dagher, K.P. Soldatos (2011) *J. Mech.Mater. Struct.* (accepted)



Governing equation:

$$\frac{d^f}{2C_{66}}\left(r^2 v'''' + 2v'''\right) - \left(r^2 v'' + rv' - v\right) = 0$$

Boundary conditions:

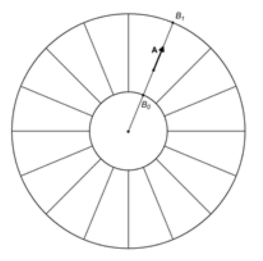
Outer: $v(B_1) = \psi = 1, \quad v''(B_1) = 0$

Inner: $v(B_0) = 0, \qquad v'(B_1) = 0$ (clamped)

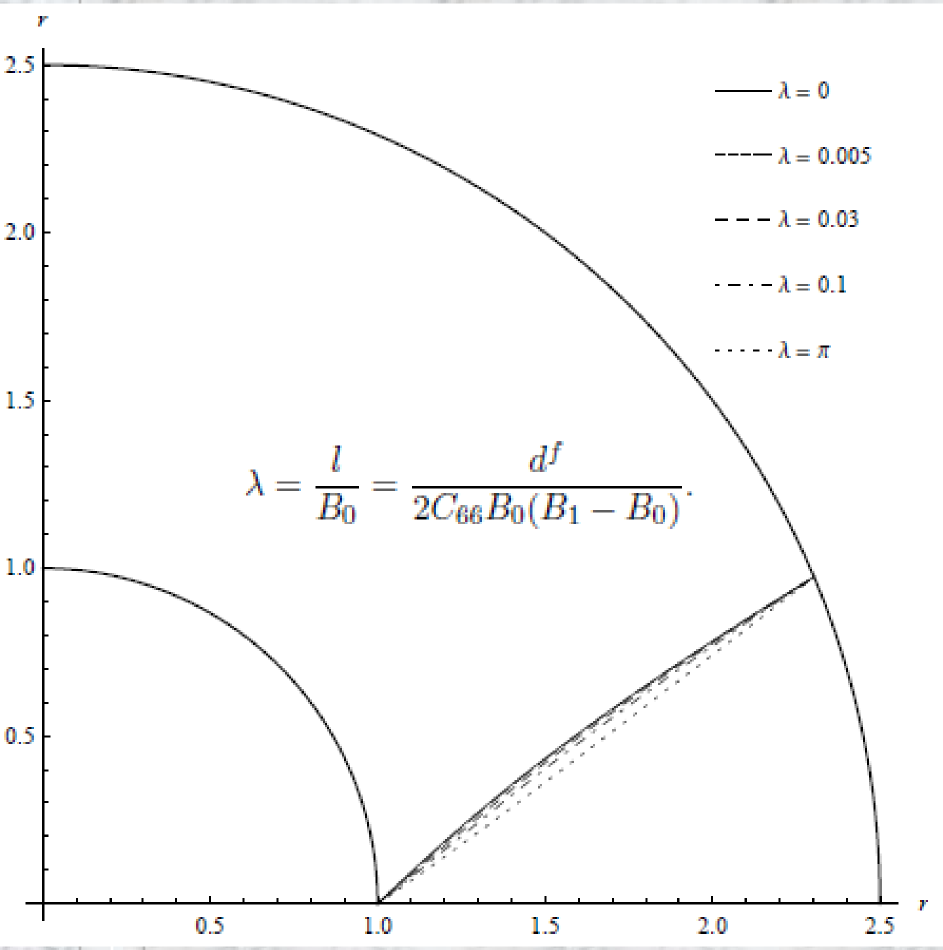
or $v''(B_1) = 0$ (simple support)

Figure: Deformation pattern of the horizontal fibre (clamped at the inner tube boundary)

The linear problem – Small azimuthal shear strain



[13] M.A. Dagher, K.P. Soldatos (2011) *J. Mech.Mater. Struct.* (accepted)



Governing equation:

$$\frac{d^f}{2C_{66}}\left(r^2 v'''' + 2v'''\right) - \left(r^2 v'' + rv' - v\right) = 0$$

Boundary conditions:

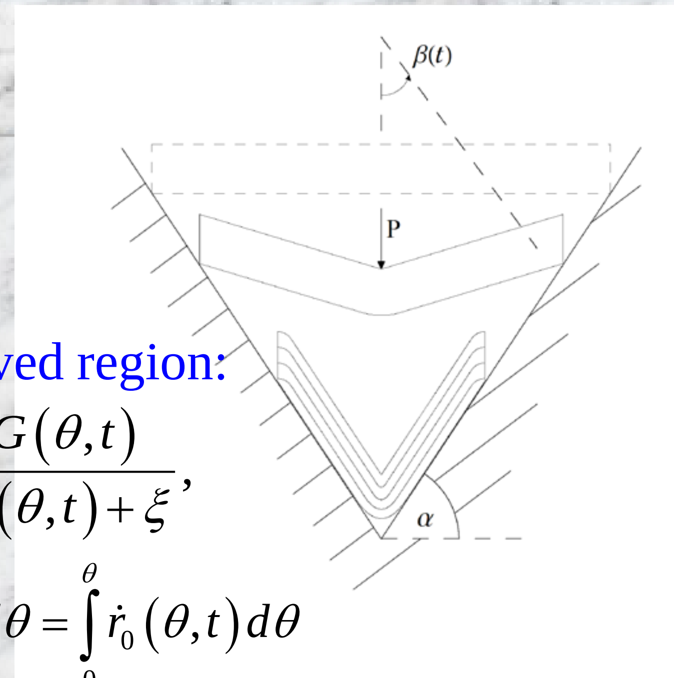
Outer: $v(B_1) = \psi = 1, \quad v''(B_1) = 0$

Inner: $v(B_0) = 0, \qquad v'(B_1) = 0$ (clamped)

or $v''(B_1) = 0$ (simple support)

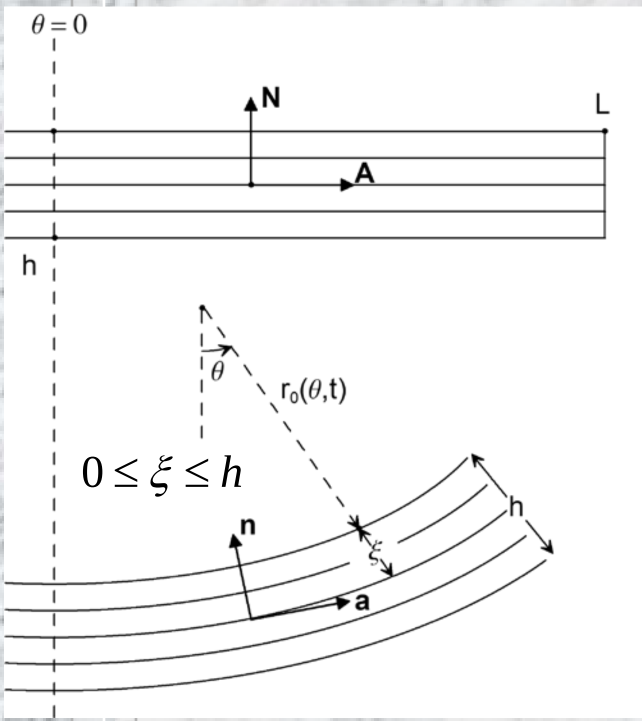
Figure: Deformation pattern of the horizontal fibre (simply supported at inner boundary)

7. Application II: Flow of an initially flat element moulded into a sharp corner



Conventional Theory Solution:

[8] T.G. Rogers and J.M. O'Neil (1991)



Rate of shear in a curved region:

$$\dot{\gamma}_c = -\frac{G(\theta,t)}{r(\theta,t)} \equiv -\frac{G(\theta,t)}{r_0(\theta,t)+\xi},$$

$$G(\theta,t) = \int_0^\theta \frac{\partial}{\partial t} r_0(\theta,t)\,d\theta = \int_0^\theta \dot{r}_0(\theta,t)\,d\theta$$

Rate of shear in flat regions: $\dot{\gamma}_f = \dot{\beta} = d\beta/dt.$

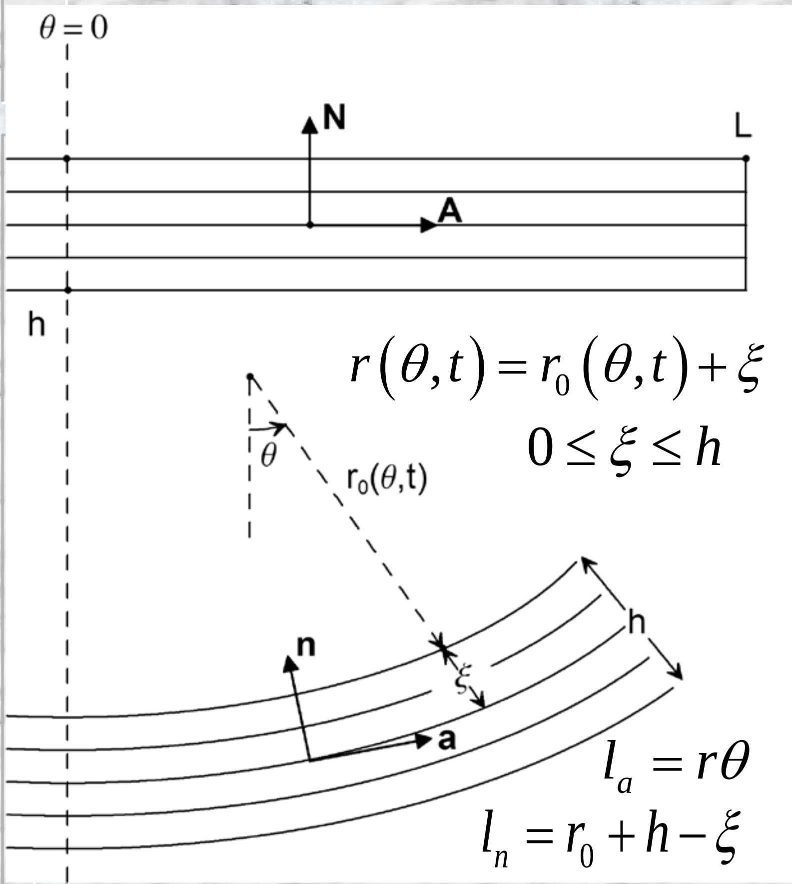
If shear viscosity remains constant during the flow:

$$\gamma_f = \beta = \alpha - \sin^{-1}\left\{\frac{\sin\alpha - \tanh kt}{1-\sin\alpha\tanh kt}\right\}, \quad k = \frac{P}{2}\sec\alpha$$

Forming process is completed at: $t = t_\alpha = \dfrac{1}{k}\tanh^{-1}(\sin\alpha).$

.

Fibre-reinforced fluids containing parallel fibres:



$$r(\theta,t) = r_0(\theta,t) + \xi$$

$$0 \le \xi \le h$$

$$l_a = r\theta$$

$$l_n = r_0 + h - \xi$$

Basic integro-differential equation:

$$r_0\left(\tau_0^s - \tau_0^a\right) - r_1\left(\tau_1^s - \tau_1^a\right) + \frac{\partial}{\partial\theta}\left(r_1 t_{r1} - r_0 t_{r0}\right) = \frac{\partial^2 \hat{S}}{\partial\theta^2} + \hat{S}$$

Stress resultants:

$$\hat{T} = \int_0^h t_\theta \, d\xi, \quad \hat{S} = \int_0^h \left(\tau^s + \tau^a\right) d\xi$$

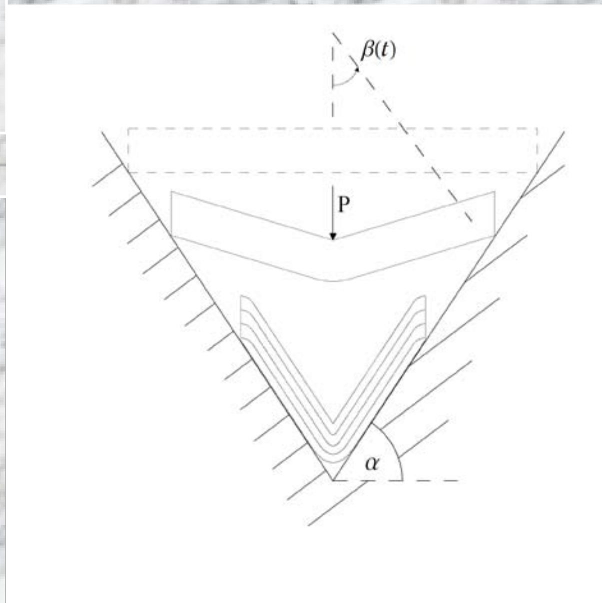
where

$$\tau^s = \alpha_1 \dot{\gamma}/2, \quad \tau^a = \frac{1}{2r}\left[\frac{\partial}{\partial r}\left(\beta_2 \dot{\gamma}/2\right) - \frac{\partial \beta_1}{\partial \theta}\right],$$

with visco-metric parameters dependent on

$$I_1 = tr\,\mathbf{d}^2 = 2\dot{\gamma}^2, \quad I_2 = \mathbf{k}\cdot\mathbf{k} = r^{-2} = \left(r_0 + \xi\right)^{-2}$$

Fibre-reinforced fluid containing parallel fibres:



Basic integro-differential equation:

$$\frac{\partial^2 \hat{S}}{\partial \theta^2} + \hat{S} = 0$$

Stress resultants: $\hat{T} = \int_0^h t_\theta \, d\xi, \quad \hat{S} = \int_0^h \left(\tau^s + \tau^a\right) d\xi$

Assuming that fibres bend into circular arcs and employing the modified invariants

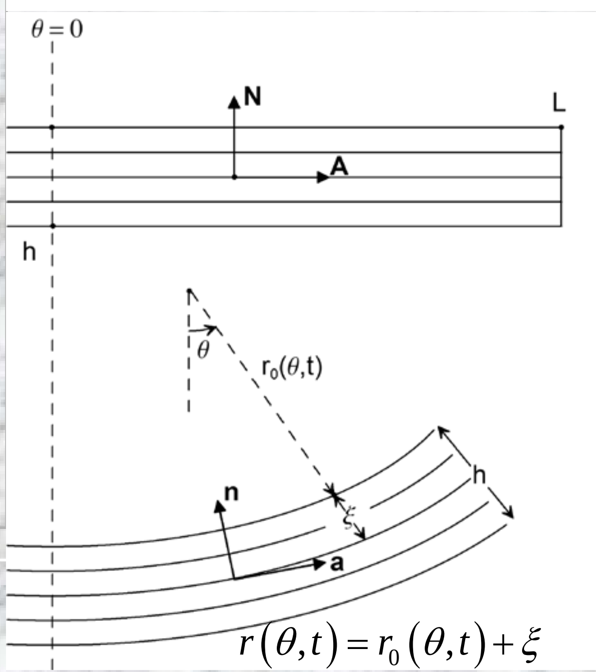
$$J_1 = I_1 I_2 / 2 = \left(\theta \dot{r}_0\right)^2, \quad J_2 = I_2 = r^{-2}$$

it is found that

$$\hat{S} = -\frac{\theta \dot{r}_0}{2} \int_0^h \left\{ \frac{1}{r} \left[\alpha_1 - \frac{1}{r^3} \left(\frac{1}{2} \beta_2 + \frac{1}{r} \frac{\partial \beta_2}{\partial J_2} \right) \right] + 2\dot{r}_0 \frac{\partial \beta_1}{\partial J_1} \right\} d\xi$$

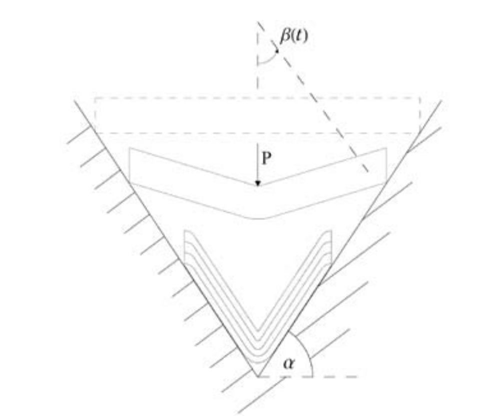
Also, for connection with conventional solution:

$$\alpha_1 = \eta$$



- Visco-metric parameters dependent on J_2 only

Basic integro-differential equation: $\hat{S} = 0$



$r(\theta,t) = r_0(\theta,t) + \xi$

Stress resultant: $\hat{S} = -\dfrac{\theta \dot{r}_0}{2} \int_0^h \dfrac{1}{r} \left\{ \alpha_1 - \dfrac{1}{r^3} \left(\dfrac{1}{2} \beta_2 + \dfrac{1}{r} \dfrac{\partial \beta_2}{\partial J_2} \right) \right\} d\xi$

For connection with conventional consider that $\alpha_1 = \eta$ = constant

Consider β_2 proportional to the n-th power of J_2, namely

$$\beta_2(J_1, J_2) = \eta_2 J_2^n = \eta_2 r^{-2n} \quad (n = 0, 1, 2, \ldots)$$

Then

$$\hat{S} = -\frac{\eta \theta \dot{r}_0}{2} \left\{ \ln(1 + h/r_0) - \hat{\eta}_2 \frac{(h/r_0)^{2(n+1)} \left[(1 + h/r_0)^{2(n+1)} - 1 \right]}{(1 + h/r_0)^{2(n+1)}} \right\},$$

where

$$\hat{\eta}_2 = \frac{\eta_2 (2n+1)}{4\eta (n+1) h^{2(n+1)}}$$

$\hat{\eta}_2$

Table: The value of h/r_0 for different values of and n

$\hat{\eta}_2$	h/r_0			
	n = 0	n = 1	n = 2	n = 3
0.001	64.7	6.73	3.38	2.43
0.01	17.0	3.50	2.21	1.78
0.1	4.12	1.80	1.44	1.30
1.0	0.952	0.915	0.936	0.951
2.0	0.625	0.750	0.822	0.865
5.0	0.367	0.574	0.692	0.763
10.0	0.249	0.471	0.608	0.693
20.0	0.171	0.387	0.535	0.630
50.0	0.105	0.299	0.449	0.556
100.0	0.0732	0.247	0.396	0.505
200.0	0.0513	0.205	0.349	0.459
500.0	0.0321	0.165	0.295	0.405
1000.0	0.0226	0.134	0.260	0.369
2000.0	0.0159	0.111	0.229	0.355

8. Closure

We considered deformation of incompressible materials, reinforced by a single family of inextensible fibres which can resist bending:

1. We formulated a relevant theory suitable for modelling (i) plane, creeping flow of fluids, and (ii) plane strain deformations of hyper-elastic solids.

2. We outlined a complete form of solution for relevant boundary value problems.

3. The applications verified that strong fibres resist local bending.

4. Future theoretical progress may consider possibilities of dismissing one or both of the material constraints involved.

5. However, the concept of ideal fibre-reinforced materials offers a valuable, first approximation to the solution of problems of interest.

6. The outlined theoretical concepts can be extended and influence further development in this and other subjects of Theoretical & Applied Mechanics.

Nottingham University (1968)

Staff and Students of the Department of Theoretical Mechanics